An Iterative Method for Solving
Cubic Nonlinear Schrödinger Equation

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Abstract

In this work we apply an iterative method (Adomian Decomposition Method or ADM) for solving the cubic nonlinear Schrödinger equation. The efficiency of this method is illustrated by investigating the convergence results for this type of models. The numerical result show the reliability and accuracy of the ADM [6].

Keywords: Iterative method, Adomian Decomposition Method, Schrödinger equation
1 Introduction

Many mathematical models can be expressed using nonlinear partial differential equations and those can be found in wide variety engineering applications. We know that the most general form of nonlinear PDEs is given by

$$ F(u, u_t, u_x) $$ \hspace{1cm} (1)

Therefore, the Adomian Decomposition Method (ADM) is applied here. ADM is a semi-numerical-analytic method for solving ordinary and partial differential equations. G. Adomian first introduced the concept of ADM in [1,4]. This technique constructs an analytical solution in the form of a polynomial [2]. This method is an alternative procedure for obtaining analytical series solution of the differential equations. The series often coincides with the Taylor expansion of the true solution at point $x_0 = 0$, in the value case, although the series can be rapidly convergent in a small region.

2 Description of ADM

The Adomian decomposition method is applied to a general nonlinear equation in the form [3]

$$ Lu + Ru + Nu = g $$ \hspace{1cm} (2)

Here, the linear terms are decomposed into $L + R$ and the nonlinear terms are represented by $Nu$. Here, $L$ is the operator of the highest-ordered derivatives with respect to $t$ and $R$ is the remainder of the linear operator. Thus we get

$$ Lu = -Ru - Nu + g $$ \hspace{1cm} (3)

$L^{-1}$ is regarded as the inverse operator of $L$ and is defined by

$$ L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt \, dt $$ \hspace{1cm} (4)

If $L$ is a second order operator, then $L^{-1}$ is defined by a two-fold indefinite integral

$$ L^{-1}Lu = u(x, t) - u(x, 0) - t \frac{\partial u(x, 0)}{\partial t} $$ \hspace{1cm} (5)

Now, operating on both sides of Eq.(2) by using $L^{-1}$ we obtain

$$ L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu $$ \hspace{1cm} (6)
Therefore we have

\[ u(x, t) = u(x, 0) + t \frac{\partial u(x, 0)}{\partial t} + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{7} \]

The ADM represents the solution of Eq.(7) as a series

\[ u(x, t) = \sum_{n=0}^{+\infty} u_n(x, t) \tag{8} \]

Here, the operator \( Nu \) (nonlinear) is decomposed as

\[ Nu = \sum_{n=0}^{+\infty} A_n \tag{9} \]

Now, substituting (8) and (9) into (7) we obtain

\[ \sum_{n=0}^{+\infty} u_n(x, t) = u_0 - L^{-1}R \sum_{n=0}^{+\infty} u_n(x, t) - L^{-1} \sum_{n=0}^{+\infty} A_n \tag{10} \]

where

\[ u_0 = u(x, 0) + t \frac{\partial u(x, 0)}{\partial t} + L^{-1}g \tag{11} \]

Then, consequently we can obtain

\[ u_1 = -L^{-1}Ru_0 - L^{-1}A_0 \]
\[ u_2 = -L^{-1}Ru_1 - L^{-1}A_1 \]
\[ \vdots \]
\[ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n \tag{12} \]

where \( u_n(x, t) \) will be determined recurrently, and \( A_n \) are the so-called polynomials (Adomian) of \( u_0, u_1, \ldots, u_n \) defined by

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ F \left( \sum_{i=0}^{\infty} \lambda_i u_i \right) \right], \quad n = 0, 1, 2, \ldots \tag{13} \]

In this case we obtain

\[ A_0 = f(u_0) \]
\[ A_1 = u_1 f'(u_0) \]
\[ A_2 = u_2 f''(u_0) + \frac{1}{2!} u_1^2 f''(u_0) \]
\[ \vdots \tag{14} \]
Now, if we introduce the parameter $\lambda$ conveniently, we can obtain that

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$$

(15)

where

$$N(u(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n$$

(16)

Therefore, expanding by Taylor’s series at $\lambda = 0$ we have

$$N(u(\lambda)) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N(u(\lambda)) \right] \lambda^n$$

(17)

The Adomian’s polynomials $A_n$ can be calculated using the recurrence equation

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{n=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}$$

(18)

If we are working with systems of differential equations (or algebraic type alike), the nonlinear terms $N$ can be of the form

$$N = N(u_1, u_2, \ldots, u_k, \ldots)$$

where

$$u_k = \sum_{n=0}^{\infty} u_{k,i}$$

### 3 Application and results

For an illustration of this method, we use two test problems in this section. We will show that how the Adomian Decomposition Method is computationally efficient and we shall also consider the performance of ADM with theoretical solution [5].

**Test problem: Cubic nonlinear Schrödinger equation.** In this section we consider the nonlinear Schrödinger (NLS) equation with cubic nonlinearity:

$$\begin{cases}
  u_t - \frac{1}{2}i u_{xx} + (\cos^2 x)iu + i|u|^2u = 0 \\
  u(x, 0) = \sin(x)
\end{cases}$$

(19)
We consider this problem on \([-L, L]\) subject to periodic boundary conditions. The exact solution for this problem is \(u(x, t) = \frac{\sin(x)}{e^{3it/2}}\).

The Eq.(19) can be written in terms of the differential operator as

\[
i L_t u = -\frac{1}{2} L_{xx} u + (\cos^2 x) u + |u|^2 u
\]

or

\[
L_t u = \frac{1}{2} i L_{xx} u - i (\cos^2 x) u - i |u|^2 u
\]

As before, assuming the existence of the inverse operator

\[
L_t^{-1}(\cdot) = \int_0^t (\cdot) dt
\]

we apply it to Eq.(21) and we get

\[
u = i L_t^{-1}\left(\frac{1}{2} L_{xx} u + (\cos^2 x) u - |u|^2 u\right)
\]

The term non-linear corresponds to \(N(u) = |u|^2 u\), which is equivalent in terms of the Adomian’s polynomials to \(A_n = N(u) = |u|^2 u\). Thus, assuming a solution in the form \(u(x, t) = \sum_{n=0}^{\infty} u_n\) we obtain recurrence relation

\[
\sum_{n=0}^{\infty} u_n = \sin x + i L_t^{-1}\left(\frac{1}{2} L_{xx} \sum_{n=0}^{\infty} u_n - \cos^2 x \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n\right)
\]

\[
= \sin x + \frac{1}{2} i L_t^{-1}\left(L_{xx} \sum_{n=0}^{\infty} u_n\right) - i L_t^{-1}\left(\cos^2 x \sum_{n=0}^{\infty} u_n\right) + i L_t^{-1}\left(- \sum_{n=0}^{\infty} A_n\right)
\]

Now, identifying the first term as \(u_0 = \sin(x)\), we get

\[
u_0 = \sin(x)
\]

\[
u_1 = \frac{1}{2} i L_t^{-1}(L_{xx} u_0) - i L_t^{-1}\left((\cos^2 x) u_0\right) + i L_t^{-1}\left(- A_0\right)
\]

\[
: \quad \vdots
\]

\[
u_{n+1} = \frac{1}{2} i L_t^{-1}(L_{xx} u_n) - i L_t^{-1}\left((\cos^2 x) u_n\right) + i L_t^{-1}\left(- A_n\right)
\]

for all \(n \geq 1\). Then we have that \(A_0 = u_0^2 \bar{u}_0\), which leads to \(A_0 = \sin^3 x\). Therefore, the first term of the recurrence equation \(u_1\) is equivalent to
\[
\begin{align*}
\frac{1}{2}L_1^{-1}(L_{xx}(\sin x)) - iL_1^{-1}\left((\cos^2 x)(\sin x)\right) + iL_1^{-1}\left(-\sin^3 x\right) &= \frac{1}{2}it \sin x - it \sin x + it \sin^3 x - it \sin^3 x \\
&= -\frac{3}{2}it \sin x
\end{align*}
\]

In the same way, we can compute \(u_2\), that is

\[
u_2 = -\frac{9}{8}t^2 \sin x = \left(-\frac{3it}{2}\right)^2 \frac{2!}{2!} \sin x
\]

or similarly

\[
u_3 = \left(-\frac{3it}{2}\right)^3 \frac{3!}{3!} \sin x
\]

Finally, the solution is of the form \(u(x,t) = \sum_{n=0}^{\infty} u_n\). So that,

\[
u(x,t) = \sum_{n=0}^{\infty} u_n
\]

\[
u(x,t) = u_0 + u_1 + u_2 + \cdots
\]

\[
u(x,t) = \sin x - \frac{3it}{2} \sin x + \frac{\left(-\frac{3it}{2}\right)^2}{2!} \sin x + \frac{\left(-\frac{3it}{2}\right)^3}{3!} \sin x + \cdots
\]

\[
u(x,t) = \left(1 - \frac{3it}{2} + \frac{\left(-\frac{3it}{2}\right)^2}{2!} + \frac{\left(-\frac{3it}{2}\right)^3}{3!}\right) \sin x
\]
which corresponds to the exact solution $u(x, t) = e^{-\frac{3it}{2}} \sin x$. Here, in Figures (1) and (2) we can see the first two approximations of the Adomian method for the Schrödinger equation. Figure (1) shows that the approximation $u_0$ decays over time, whereas figure (2) shows that the $u_0 + u_1$ approximation grows gradually over time. Figure (3) presents the absolute errors in the Adomian method for $n = 0, \ldots, 10$, where

$$E_n(t) = \sup_{t \in [0, T]} \left( \sup_{t \in [-L, L]} |u_n - u_{\text{exact}}| \right)$$

We can see that errors decrease when $n$ is incremented for any fixed $t$. 

Figure 2: Approximation $|u_0 + u_1|$ to $x \in [-10, 10]$ and $t \in [0, 1]$.

Figure 3: Absolute error $E_n$ for Adomian with $n = 0, 1, \ldots, 10$. 

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4 Conclusion

In this paper, we calculated the exact solution of the cubic nonlinear CNS model by using ADM. We demonstrated that this iterative method is quite efficient to determine solution in closed form. The new scheme obtained using the ADM yields an analytical solution in the form of a rapidly convergent series. This is why the ADM makes the solution procedure much more attractive.

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