Rubberlike Membranes at Inner Pressure

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Abstract

The methods of building experiments of stretching of round membranes by normal pressure has been developed. The experimental data are compared with results of the theoretical research. The nonlinear theory of shells is offered for calculation of elastic membranes.

Keywords: rubber, membrane, bifurcation, numerical method

1 Introduction

Rubber and rubberlike materials as elastic having nonlinear mechanical properties found a wide using in industry. Elastic are used for manufacture of membranes, shells, shock absorbers, compress elements. It’s articles can be expose with large deformation (till a few hundreds percents) without destruction. For describing of pressure-deformation condition of such elements nonlinear theory of elasticity is used. It’s possible to mark out two kinds of non-linearity for thin-skinned membranes and shells from elastic. The physical non-linearity is described with nonlinear dependence «pressure - deformation» with an elastic potential [1-7] and geometrical non-linearity is characterized by large deflections of average membrane surface from starting position [8-11]. Mathematical models of these membranes are presented as edge-problem for systems of nonlinear differential equations [12-16]. These problems could have non-unique solution.
Therefore different numerical algorithms are developed and are developing for decision [17-20]. In work the theoretical and experimental results of single-axial homogeneous stretching of rubber samples and stretching of round membrane by normal pressure has been shown.

2 Elastic potential

The relation between pressure and deformation in nonlinear theory of elasticity is described with the elastic potential. The elastic potential is described as a function of main invariants or main principal stretches. In the theory of thin shells with static Kirchhoff hypothesis, law of elasticity for incompressible material take on aspect [9]:

$$\sigma_i = \lambda_i \frac{\partial \Phi}{\partial \lambda_i} - \lambda_i \frac{\partial \Phi}{\partial \lambda_3} \quad (i = 1, 2),$$

where $\Phi = \Phi(\lambda_1, \lambda_2, \lambda_3)$ is an elastic potential which is a function of the principal stretches $\lambda_1$, $\lambda_2$, $\lambda_3$ for isotropic material; $\sigma_1$ and $\sigma_2$ are the stresses on average surface. For incompressible material it must to be observed the condition: $\lambda_1\lambda_2\lambda_3 = 1$. Nowadays, the potentials offering in literature are based empiristic on the experimental data, not on the statistic theory. The Mooney’s potentials, Ogden’s potentials and others [4, 5, 21 - 27] just like this. Several of these are used to solve concrete task often enough. Below potential will be used [2, 9]:

$$\Phi = \frac{2\mu}{n} \left( \lambda_1^n + \lambda_2^n + \lambda_3^n - 3 \right),$$

(1)

where $\mu$ is initial shear modules and $n$ is parameter.

3 Single-axial stretching of samples

To define mechanical characteristics of elastics different methods, as a rule based on single-axial and bi-axial stretching of samples, are used [28]. The authors of research have made the experiments of single-axial stretching of rubber samples in kind of plane membrane. By way of testing samples strips of sheet rubber of different grade was taken; width from 55 mm to 120 mm, length from 150 mm to 400 mm, thickness from 0.5 mm to 13 mm. Especially manufactured installation technically gave the opportunity to stretch sample with length of 200 mm to six times. The state close to homogenous has been realized in the vicinity of central part of stretched sample. The 200 mm length was taken as based sample length. The rectangular net (with square cell 5x5 mm) had been depicted on the undeformed sample with the stamp, which had been especially manufactured for these purpose to visual control on the net distortion on the surface of the deformed sample.

The sample was fixed into clamps on the edges, one edge was fastened hardly, and on another one the pressure stretching the sample was being applied.
The distance between “control” marks was being measured at fixed pressure. The “control” marks was plotted on the surface of undeformed sample with the distance 100 mm one from another. The pressure was being measured with the especially manufactured dynamometer. Altogether, the experiments of single-axial stretching has been made with 8 sorts of rubber. Typical dependence «pressure-deformation» for testing sorts of rubber is shown on Figure 1 - the experimental points are marked with symbol 0.

For elastic potential (1) the dependence «pressure-deformation» in case of single-axial stretching is defined with formula:

\[ P = \frac{2\mu}{n} \left( \lambda^n - \lambda^{-n/2} \right), \]

where \( \lambda \) is relative lengthening of sample. The experimental data were being approximated with this dependence with the «smallest squares» technic. In case Figure 1 the following meanings of parameters has been received: \( \mu = 9 \text{ N/cm}^2 \) and \( n = 1.5 \). The continuous line corresponds with estimated dependence on Figure 1. For all testing samples (8 sorts of rubber) parameter \( n \) was in diapason from 1.2 to 1.8, and parameter \( \mu \) from 1.5 to 2. These results are consistent with the results obtained in [28].

![Graph showing the pressure-deformation relationship](image)

**Figure 1** The dependence «pressure-deformation» in case of single-axial stretching

### 4 Round membrane

To carry out the experiments of stretching of plane membranes the technic of its stretching by normal pressure has been developed. The stretching of membranes was being carried out with following scheme. Sideways view on the installation, which is used to stretching of round membrane by normal pressure. In the hard leaf of metal the round opening had being cut out. Manometer and entering
valve has been assembled into the hard platform. Sheet rubber had being put on this platform. The leaf with round opening was being put at the top. The platform and the leaf with an opening had being joined to etch other. This construction was being installed against a background of the wall with a grids. The small parts of air, which had been making pressure on the membrane, were giving by means of pump. Manometer was being used for pressure measurement. In the process of membrane loading by pressure photos was being made with numerical camera; the form of membrane after computing was being built at certain pressure. The offered technic of making of experimental research of round membrane stretching by normal pressure allows to compare results of single-axial stretching with those of membrane stretching by normal pressure for rubbers from one and the same sort and from one and the same “sheet” material.

The round membranes were undergoing of stretching by normal pressure. These membranes were manufactured from “sheet” rubber (leaf width are 1.2, 1.5, 2 mm), one and the same rubber, that in the experiments of single-axial stretching of strips. The diameters of interior contour are 75 mm and 100 mm. The sideways aspect on the deformed membrane at different meanings of pressure is shown on the Figure 2. These meanings are marked on Figure 3 with numerals (0.24, 0.40, 0.32).

The typical “dependence pressure - displacement in the membrane center” is shown on Figure 2. The experimental points are marked with symbol 0. By way of “without-size displacement” the meanings of formula $z(0)/R$ were used. In this one $z(0)$ - displacement in the membrane center, and $R$ - contour radius. By way of “without-sizing pressure” the value of formula $qR/h\mu = Q$ were used, where $q$ is pressure, $h$ is the width of undeformed membrane. In all experiments the dependence “pressure-deformation” had got the maximum point. The maximum point (Figure 3) had become achievable almost straight away after membrane way out of contour boarding in the vicinity of fixed-zone (Figure 2, $Q = 0.32$). These results are correspond with results in works [29, 30].

Figure 2 Pressure versus vertical displacement at the central point
Figure 3 The form of round membrane at pressures $Q = 0.24, 0.40, 0.32$

5 Mathematical model of membrane

By way of models of membranes and shells non-linear without-moment theory of shells had been used. The balance equations of axisymmetric deformation of rotation shell in case of normal pressure are [9]:

$$\frac{d}{ds}\left( s \frac{1}{\lambda_1} \frac{dr}{ds} \right) - T_2 - q r \frac{dz}{ds} = 0,$$

$$\frac{d}{ds}\left( s \frac{1}{\lambda_2} \frac{dz}{ds} \right) + q r \frac{dr}{ds} = 0,$$ \hspace{1cm} (2)

$$\frac{dr}{ds} = \lambda_1 \cos \varphi, \quad \frac{dz}{ds} = -\lambda_1 \sin \varphi,$$

$$\lambda_2 = r/s,$$

$0 \leq s \leq R.$

In these correlation $\varphi$ is an angle between rotation axe and normal axe to average surface in deformed configuration, $T_1$ and $T_2$ are forces, are acting on the average surface in meridian and parallel destinations, $\lambda_1$ and $\lambda_2$ are principal stretches. The dependences between forces and the principal stretches for elastic potential (1) are:

$$T_1 = \frac{2h\mu}{n} \frac{1}{\lambda_1} \left( \lambda_1^e - \lambda_1^n \right), \quad T_2 = \frac{2h\mu}{n} \frac{1}{\lambda_2} \left( \lambda_2^e - \lambda_2^n \right).$$

By way of boundary conditions, the following conditions had being examined:

$$\text{at } s = 0: \ s = 0, \ \frac{dz}{ds} = 0;$$ \hspace{1cm} (3)

$$\text{at } s = R: \ r = R, \ z = 0.$$ \hspace{1cm} (4)

The offered non-linear equations system was solved with applying two algorithms. These algorithms has been developed to solution for non-linear equations in partial derivatives. In one case the discretization of differential opera-
tors by finite differences has been done. The obtaining system of non-linear algebraic equations was solved by simple iteration method. In another case, the boundary problem has been led to solution of Cauchy’s problem for system of ordinary differential equations [31-36]. Theoretical results obtained by both methods gave good coincidence to experimental data. The continuous line on Figure 3 is corresponding with estimated dependence at \( n = 2 \) in formula (1).

The dependence \( z = z(r) \) for Neo-Hookean potential at meanings \( Q = 0.17, 1.20, 1.84, 1.80 \) is shown on Figure 4. The same result is obtained in [25, 37].

![Figure 4](image-url)

Figure 4 The dependence \( z = z(r) \) at meanings \( Q = 0.17, 1.20, 1.84, 1.80 \)

The first two equations in (2) subject to the boundary conditions (3)-(4) can be converted:

\[
T_1 \sin \varphi = \frac{1}{2} qs \lambda_2^2,
\]

(5)

\[
\frac{d(sT_1)}{ds} = T_2 \cos \varphi.
\]

(6)

The solution \( z = z(r) \) of the system of equations (2) for positive values \( q \) must be continuously differentiable function. So there should be inequality \( 0 < \varphi < \pi \). In this case, the function \( T_1 = T_1(\lambda_1, \lambda_2) \) must take positive values on the solutions of the equations (2). The function \( \sin \varphi \) is limited function. Therefore, the inequality \( q < \max_{\lambda_1, \lambda_2} \left( 2T_1(\lambda_1, \lambda_2)/\lambda_2^2 \right) \) follows from (5). At the point \( s = 0 \), as follows from equation (6) and (3), (4) must to be equality \( T_1 = T_2 \); and for an isotropic material, also equality \( \lambda_1 = \lambda_2 = \lambda \) must to be. So the pressure and stress \( T_1 \) are associated with inequality \( q < \max_{\lambda} \left( 2T_1(\lambda, \lambda)/\lambda^2 \right) \). Therefore, if the
function \( T(\lambda, \lambda) / \lambda^2 \) with the growth in will decrease, the dependence of the pressure-strain will have a maximum point. Therefore, if the function with \( \lambda \) increasing will decrease, the dependence “pressure-deformation” will have a maximum point. For potential (1)

\[
\frac{1}{\lambda^2} T(\lambda, \lambda) = \frac{2h \mu}{n \lambda^3} \left( \lambda^n - \frac{1}{\lambda^6} \right)
\]

and at \( n < 3 \) the maximum point will exist. The dependence “pressure-displacement” is shown on Figure 5 at \( n = 1, 2, 2.5, 3, 4 \) in formula of elastic potential (1). For geometrically linear equations the analysis made in [38].

![Figure 5](image)

Figure 5 Pressure versus vertical displacement at the central point for meanings \( n = 1, 2, 2.5, 3, 4 \) in (1)

The maximum point in dependence “pressure-displacement” is examined as a point of loss of stability [9, 25, 39 - 43]. If pressure, which exceed the pressure in maximum point (Figure 3), will be acted on the membrane, then the loading process will stop to be balanced. The process of uninterrupted stretching in time will be started up to destruction or the transition on another balanced trajectory will be started. On the other hand, the physical properties of membrane can be non-uniformed. The inner hard inserts with different sizes, the inner microspores and cracks, surface defects, which density can be increased at acting of aggressive mediums, can appear at membrane manufacturing [44-46]. Hence, its exploitation in vicinity of loading maximum point can be a reason of unexpected stretching because of changing of mechanical properties.

6 Conclusion

As it’s following from analysis of experimental and theoretical results of single-axial and bi-axial stretching of samples, manufactured from one and the
same sheet metal, such dependences as “pressure-deformation” dependence, are describing by different potentials. It can be explained that, inner reorganization of macromolecules in samples progresses differently in single-axial and bi-axial non-uniformed tense state. In addition, it’s possible, that it isn’t correct to build the elastic potential for bi-axial tense state using integral characteristics sort of ‘pressure-deformation’. Apart from that, the theoretical results correspond with experimental results good enough.

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