Harmonic Wave Excitation in a Semi Infinite Medium

V. G. Gupta and Kapil Pal*

Department of Mathematics, University of Rajasthan, Jaipur-302004, India
guptavguor@rediffmail.com; *palkapiluor@yahoo.co.in

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Abstract

In the present paper, we obtained the most general solution of the one-dimensional partial differential equation for harmonic wave excitation in a semi infinite medium, by computing the symmetry groups using the general prolongation formula for their infinitesimal generators of a groups of transformations based on the technique given by Olver([4], [5]) in explicit form. In the recent year the authors Bao, Wei and Zhao [1], Kurus [3], Ramahi and Seydou [6], Ranosava [7] Sneddon and Read [8] worked for the solution of Helmholtz equation.

Keywords: Axial Displacement, Wave Number, Commutation-Relation

1. Introduction

1.1 The harmonic wave excitation problem in a semi-infinite medium:

The harmonic wave excitation problem in a semi-infinite medium the governing equation is the Helmholtz equation in the standard form as

\[ u_{xx} + k^2 u = 0 \quad (1.1.1) \]

Where \( u \) is an axial displacement of the rod and \( k \) is the wave number, [2].

2. Main Result

Let us consider one-dimensional Helmholtz equation for harmonic wave excitation in a semi-infinite medium

\[ u_{xx} + k^2 u = 0 \quad (2.1) \]

which is the second order differential equation with one independent variables and one dependent variable.
Lemma 1: Let

\[ v = \xi(x, u) \frac{\partial}{\partial x} + \phi(x, u) \frac{\partial}{\partial u} \]  

(2.2)

be a symmetry of Helmholtz equation (2.1). Then the smooth coefficient functions \( \phi \) and \( \xi \) are given by \( \phi = \beta u + \alpha \) where \( \alpha = \alpha(x) \) and \( \beta = \beta(x) \) are functions and \( \xi \) independent of \( u \).

Proof: Firstly we determine the second prolongation of \( v \) (see Oliver[5]),

\[ pr^{(2)}v = v + \phi^x \left( \frac{\partial}{\partial u_x} \right) + \phi^{xx} \left( \frac{\partial}{\partial u_{xx}} \right) \]  

(2.3)

By using infinitesimal criterion of invariance the equation (2.1) takes the form

\[ \phi^{xx} + k^2 \phi = Q \left( u_{xx} + k^2 u \right) \]  

(2.4)

where \( Q(x, u^{(2)}) \). By substituting the values of \( \phi^{xx} \) and \( \phi \) in equation (2.4) and equating the coefficients of the terms in the first and second order partial derivatives of \( u \), the determining equations for the symmetry group of the one-dimensional Helmholtz equation are found as follows see Table 1

<table>
<thead>
<tr>
<th>Monomial</th>
<th>Coefficient</th>
<th>Equation Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \phi_{xx} + k^2 \phi - k^2 Qu = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>( u_x )</td>
<td>( 2\phi_{xx} - \xi_{xx} = 0 )</td>
<td>2</td>
</tr>
<tr>
<td>( u_x^2 )</td>
<td>( \phi_{xx} - 2\xi_{xx} = 0 )</td>
<td>3</td>
</tr>
<tr>
<td>( u_x^3 )</td>
<td>( -\xi_{xx} = 0 )</td>
<td>4</td>
</tr>
<tr>
<td>( u_{xx} )</td>
<td>( \phi_x - 2\xi_x = Q )</td>
<td>5</td>
</tr>
<tr>
<td>( u_x u_{xx} )</td>
<td>( -3\xi_x = 0 )</td>
<td>6</td>
</tr>
</tbody>
</table>

The requirement for equation (6) is that \( \xi \) independent of \( u \), equation (3) gives \( \phi = \beta u + \alpha \) where \( \alpha = \alpha(x) \) and \( \beta = \beta(x) \) are functions.

Lemma 2: The most general infinitesimal symmetry of the one-dimensional Helmholtz equation in a semi-infinite medium has coefficient function of the form \( \xi = c_1 \) and \( \phi = (c_2 / k^2) u + \alpha \) where \( c_1 \) and \( c_2 \) are arbitrary constant and \( \alpha \) is an arbitrary solution of the Helmholtz equation.

Proof: Using lemma 1, the equation (1) gives \( \beta = Q \), form the equation (5) we found \( \xi = c_1 \) and from equation (2) we get \( \beta_x = c_1 / k^2 \). Thus most general infinitesimal symmetry of the one-dimensional Helmholtz equation in a semi-infinite medium has coefficient function of the form \( \xi = c_1 \) and \( \phi = (c_2 / k^2) u + \alpha \) where \( c_1 \) and \( c_2 \) are arbitrary constant and \( \alpha \) is an arbitrary solution of (2.1)
Lemma 3: The Lie algebras of infinitesimal symmetries of the Helmholtz equation in a semi-infinite medium is spanned by the two vector fields $v_1 = \partial_x$, $v_2 = (1/k^2) u \partial_u$ and the infinite-dimensional sub-algebra $v_\alpha = \alpha \partial_u$ where $\alpha$ is an arbitrary solution of the Helmholtz equation.

Proof: The proof is evident by using lemma 1 and lemma 2.

Theorem 1: The symmetry Lie algebra $\mathfrak{g}$ of the Helmholtz equation in a semi-infinite medium is spanned by the set of vector field $v_1$, $v_2$ and $v_\alpha$.

Proof: Using lemma 3, the commutation relation between these vector fields are given by the following see Table 2

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0</td>
<td>$v_\alpha$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>0</td>
<td>0</td>
<td>$-\left(1/k^2\right) v_\alpha$</td>
</tr>
<tr>
<td>$v_\alpha$</td>
<td>$-v_\alpha$</td>
<td>$(1/k^2) v_\alpha$</td>
<td>0</td>
</tr>
</tbody>
</table>

by using commutation-relation table it is easy to derive $\mathfrak{g}$ is a Lie algebra with Lie bracket operation.

Lemma 4: The one-parameter groups $G_i$ ($i=1,2,\alpha$) generated by the $v_i$ are given as follows $G_1$: $(x + \varepsilon, u)$, $G_2$: $(x, e^{\varepsilon/k^2} u)$, $G_\alpha$: $(x, u + \varepsilon \alpha)$ where each $G_i$ is a symmetry group.

Proof: The one parameter group generated by $v_i$ is given by $\exp(\varepsilon v_i)(x, u) = (\tilde{x}, \tilde{u})$, so by using lemma 3 it is obvious.

Theorem 2: The solution of Helmholtz equation by using its different symmetry groups are given by $u^{(1)} = f(x - \varepsilon)$, $u^{(2)} = e^{\varepsilon/k^2} f(x)$, $u^{(\alpha)} = f(x) + \varepsilon \alpha$ where $u = f(x)$ be an assume solution to the Helmholtz equation, $\varepsilon$ is any real number and $\alpha$ any other solution to the Helmholtz equation.

Proof: Putting the value of $x$ and $u$ in solution $u = f(x)$ and using $(x, u) = (\tilde{x}, \tilde{u})$ for each $G_i$ and using lemma 4 we get the above function which are the solution to the Helmholtz equation.
3 Conclusion

In our investigation the symmetry group $G_2$ and $G_{\alpha}$ reflects the linearity of the Helmholtz equation. The group $G_1$ is space translation symmetry group. At the end the most general solution that we can obtain from a given solution $u = f(x)$, by group transformations is in the form given below

$$u = e^{(\varepsilon_1/\varepsilon_2)}f(x - \varepsilon_1) + \alpha$$

(3.1)

where $\varepsilon_1$, and $\varepsilon_2$ are real constant and $\alpha$ be an arbitrary solution to the one-dimensional Helmholtz equation for the harmonic wave excitation in a semi-infinite medium. The most general solution (3.1) gives us all possible most general infinitesimal symmetries of Helmholtz equation (2.1).

4 Special Case

If we take $k = 1$ then equation 3.1 reduces to

$$u = e^{\varepsilon_2}f(x - \varepsilon_1) + \alpha$$

(4.1)

where $\varepsilon_1$, and $\varepsilon_2$ are real constant and $\alpha$ be an arbitrary solution to the one-dimensional Helmholtz equation for the harmonic wave excitation in a semi-infinite medium.

References


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Received: June, 2011