Soliton Perturbation Theory for the Generalized Fifth-Order Nonlinear Equation

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Abstract

The adiabatic parameter dynamics of 1-soliton solution of the generalized fifth-order nonlinear equation is obtained by virtue of the soliton perturbation theory. The adiabatic change of soliton velocity is also obtained in this paper.

1 INTRODUCTION

The dimensionless form of the generalized fifth-order nonlinear equation (gfNE) that is going to be studied in this paper is given by

\[ q_t + aqq_{xxx} + bq_xq_{xx} + cq^2q_x + q_{xxxx} = 0 \] (1)
where \( a \), \( b \) and \( c \) are real constants parameters. This equation is a generalized version of many equations. Some of these are given in the following table for specific values of \( a \), \( b \) and \( c \).

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>−15</td>
<td>−75/2</td>
<td>45</td>
<td>Kaup-Kupershmidt</td>
</tr>
<tr>
<td>−15</td>
<td>−15</td>
<td>45</td>
<td>Sawada-Kotera-I</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>Sawada-Kotera-II</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>30</td>
<td>Lax</td>
</tr>
</tbody>
</table>

These cases have already been studied for the past two decades and are therefore well understood [1-10]. The 1-soliton solution of (1) is given by [6]

\[
q(x, t) = \frac{A}{\cosh^2 B(x - \bar{x})} \tag{2}
\]

where

\[
A = \frac{3a\sqrt{v}}{2c} \tag{3}
\]

\[
B = \frac{\sqrt{v}}{2} \tag{4}
\]

and the velocity of the soliton is defined as

\[
v = \frac{d\bar{x}}{dt} \tag{5}
\]

so that from (3) and (4)

\[
B^2 = \frac{cA}{6a} \tag{6}
\]

It is to be noted that this 1-soliton solution, for the generalized case, exists for

\[
b = \frac{10c - a^2}{a} \tag{7}
\]
2 MATHEMATICAL PROPERTIES

Equation (1) has at least two integrals of motion [9] that are known as linear momentum \(M\) and energy \(E\). These are respectively given by

\[ M = \int_{-\infty}^{\infty} q \, dx = \frac{2A}{B} \tag{8} \]

and

\[ E = \int_{-\infty}^{\infty} q^2 \, dx = \frac{4A^2}{3B} \tag{9} \]

These conserved quantities are calculated by using the 1-soliton solution given by (2). The center position of the soliton \(\bar{x}\) is defined as

\[ \bar{x} = \frac{\int_{-\infty}^{\infty} x q \, dx}{\int_{-\infty}^{\infty} q \, dx} = \frac{\int_{-\infty}^{\infty} x q \, dx}{M} \tag{10} \]

where \(M\) is defined in (8). Thus, the velocity of the soliton is given by

\[ v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} \bar{x} q \, dx}{\int_{-\infty}^{\infty} q \, dx} = \frac{\int_{-\infty}^{\infty} x q \, dx}{M} \tag{11} \]

On using (1) and (8), the velocity of the soliton reduces to

\[ v = \frac{4(b - 3a)AB^2}{15} + \frac{8cA^2}{45} \tag{12} \]

3 PERTURBATION TERMS

The perturbed gfKdV equation that is going to be studied in this paper is given by

\[ q_t + aqq_{xxx} + bq_xq_{xx} + cq^2 q_x + q_{xxxx} = \epsilon R \tag{13} \]

where, in (13), \(\epsilon\) is the perturbation parameter and \(0 < \epsilon \ll 1\) [1, 3], while \(R\) gives the perturbation terms. In presence of perturbation terms, the momentum and the energy of the soliton do not stay conserved. Instead, they undergo adiabatic changes that lead to the adiabatic deformation of the soliton.
amplitude, width and a slow change in the velocity [1, 2]. Using (9), the law of adiabatic deformation of the soliton energy is given by [1, 2, 5]

\[ \frac{dE}{dt} = 2\epsilon \int_{-\infty}^{\infty} qRdx \] (14)

while the adiabatic law of change of the velocity of the soliton is given by [1, 2, 5]

\[ v = \frac{4(b - 3a)AB^2}{15} + \frac{8cA^2}{45} + \frac{\epsilon}{M} \int_{-\infty}^{\infty} xRdx \] (15)

In this paper, the perturbation terms that are going to be considered are

\[ R = \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q_{xxx} + \nu qq_xq_{xx} + \sigma q_x^3 + \xi q_x q_{xxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxx} + \kappa q_{xxxx} \] (16)

In (16) for fluids, dissipation gives rise to the first two terms and so \( \alpha \) and \( \beta \) are small dissipative coefficients [1, 2]. Also, \( \delta \) or \( \psi \) represent the coefficient of higher order nonlinear dispersive term [4, 8] and \( m \) is a positive integer with \( 1 \leq m \leq 4 \) [4, 8]. The coefficient of \( \rho \) provide a higher order stabilizing term and must therefore be taken into account [4, 8]. The perturbation term given by coefficient of \( \eta \) was recently considered [1, 2] while the remaining perturbation terms arise in the context of extended version of integrable equations [8].

### 3.1 APPLICATIONS

In presence of these perturbation terms, the adiabatic variation of the energy of the soliton is given by

\[ \frac{dE}{dt} = \frac{16\epsilon A^2}{105} (35\alpha - 7\beta + 5\rho) \] (17)

Using (9), one can integrate equation (17) to yield

\[ A(t) = \left[ \frac{105\sqrt{6aA_0}}{105\sqrt{6a - 4et\sqrt{cA_0}}(35\alpha - 7\beta + 5\rho)} \right]^2 \] (18)
where $A_0$ is the initial amplitude of the soliton. This leads to the long term behaviour of the soliton amplitude as

$$\lim_{t \to \infty} A(t) = \begin{cases} A_0, & 7\beta = 35\alpha + 5\rho \\ 0, & 7\beta \neq 35\alpha + 5\rho \end{cases}$$

(19)

The law of the change of velocity for the given perturbation terms in (16) is given by

$$v = \frac{4(b - 3a)AB^2}{15} + \frac{8cA^2}{45} - \epsilon \left[ \frac{m\delta A^m}{(m+1)(2m+1)} B\left(m, \frac{1}{2}\right) ight.$$

$$+ \frac{A}{315} \left\{ 3(7\gamma - 14\lambda - 15\xi + 5\eta + 25\kappa) + 2\nu A \right\} \right]$$

(20)

4 CONCLUSIONS

In this paper, soliton perturbation theory is used to study the peturbed gfNE. This theory is used to establish the adiabatic parameter dynamics of the soliton energy. This relation is then further exploited to obtain the adiabatic variation of the soliton amplitude. The constraints on the perturbation parameters are then determined under which the soliton amplitude vanishes or stays constant. Also, it is shown that the velocity undergoes a slow change due to these perturbation terms. The quasi-stationary aspects of the perturbed soliton in presence of such perturbation terms will be studied and reported in future publications.

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