Modeling of Wave Characteristics Parameters in the Shoaling Zone with Saint-Venant Equations

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Abstract

In coastal areas, even in the absence of any disturbance, ocean waves feel the effect of the seabed. This perturbation, function of the local water depth $d$, causes the variation of almost all their characteristic parameters. Shallow water, is divided into two areas: the intermediate zone (zone lift or area shoaling : area before the bathymetric breaking) and Swash and Surf areas (area after breaking). In the shoaling area, the wave amplitude will gradually increase and their profile is hereby amended while their wavelength decreases.

To take into account nonlinear effects generated by the action of the seabed on the deformation of the free surface of the ocean, we used the Saint-Venant equations with topography term (Green Naghdi equations) to model the variations of these parameters according to the local water depth from the intermediate zone to the point of breaking depth.

The results show that the amplification of the height is proportional to $d^{-1/4}$ while the remaining wavelength decreases proportional to the square root of the local water depth $d$. This seabed action also induces a phase delay in the wave propagation.

Keywords: swell; Saint-Venant equations; wave height; wavelength; wave shoaling; breaking point
1 Introduction

The swells are surface waves (oscillations of ocean-atmosphere interface) that are maintained by the continuous oscillation between kinetic and potential gravity energies: they are gravity waves. They are generally created by the wind action on the ocean surface. As for swells, they are adults waves and propagate following an direction almost straight. The coastal zone, in which the swell feels the perturbation effect of the seabed, is divided into two areas: the intermediate zone (which carries the lifting swell before bathymetric deforation) called shoaling area and the area after bathymetric breaking zones consisting of Surf and Swash. In fact in this area, the waves lose their nature and take a sinusoidal shape almost sawtooth.

In previous work, Phillipes Bonneton, Vijay G. Panchang and other authors have shown that the basic parameters that characterize waves are their wavelength ($L$), their period ($T$), the vertical elevation of sea level ($\eta$) or peak to valley height ($H$), their phase velocity ($C_\phi$) and group velocity $C_g$. They were described by presenting their forms of variation and analytical expressions of some from the linear theory which the dispersion relation can be written $\omega^2 = gk \tanh(kd)$. This dispersion relation, established for the Airy wave (sinusoidal waves which have a small amplitude) can not better describe the evolution of certain parameters such as the peak-to-trough which amplifies and even the length of height wave which contracts.

To better describe analytically the variations of these parameters during propagation in the area shoaling (including height and wavelength), we combined the Saint-Venant equations and those of Green Naghdi $^{11,19,24}$. To this end, a system of equations that will provide analytical expressions that justify the numerical and experimental results $^{16,23,28}$ obtained by Phillipes Bonneton, Vijay G. Panchang and others.

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2 Materials and Methods

2.1 Variations of parameters according to linear theory

The only parameter which remains practically constant along the wave propagation is the time period \( T \approx Cste \) \[^{20}\]

The dispersion relation of regular waves is \( \omega^2 = gk \tanh(kd) \). From this relationship, it follows the expression of the wavelength \( L \), depending on the local depth of water is \( L = \frac{gT^2}{2\pi} \tanh(kd) \). In deep waters: \( d \geq \frac{L_o}{2} \), \[^{18,20}\]

\[
\tanh(kd) \approx 1, \text{ then } L = L_o = \frac{gT^2}{2\pi} \quad \forall \mu = \frac{d}{L_o} \geq \frac{1}{2} \quad \text{(1)}
\]

In the area shoaling, using the dispersion relation of the waves Airy (linear theory), we have:

\[
L = L_o \tanh(kd) \quad \text{if } \mu \leq \frac{1}{2} \quad \text{(2)}
\]

Let \( \delta' = \frac{L}{L_o} \) the coefficient of variation of the wavelength according to the linear theory, posing

\[
k d = 2\pi \frac{d}{L_o} \frac{L_o}{L} = \frac{2\mu}{\delta}, \text{ we have: } \delta' = \frac{L}{L_o} = \left\{ \begin{array}{ll}
1 & \text{if } \mu \geq \frac{1}{2} \\
\tanh\left(\frac{2\pi\mu}{\delta'}\right) & \text{if } \mu \leq \frac{1}{2}
\end{array} \right. \quad \text{(3)}
\]

According to this theory, the expressions of the phase velocity and group velocity are respectively

\[
C_\varphi = C_{\varphi_o} \sqrt{\delta' \tanh\left(\frac{2\pi\mu}{\delta'}\right)} \quad \text{with } C_{\varphi_o} = \sqrt{\frac{gL_o}{2\pi}}
\]

\[
C_g = C_{g_o} \left(1 + \frac{2kd}{\sinh(2kd)}\right) \sqrt{\frac{L}{L_o} \tanh(kd)} \quad \text{with } C_{g_o} = \frac{1}{2} \sqrt{\frac{gL_o}{2\pi}} \quad \text{(4)}
\]

If \( \delta'_1 = \frac{C_g}{C_{\varphi_o}} \) and \( \delta'_2 = \frac{C_g}{C_{g_o}} \) are coefficients of variations of the phase and group velocities, then:

\[
\delta'_1 = \left\{ \begin{array}{ll}
1 & \text{if } \mu \geq \frac{1}{2} \\
\sqrt{\delta' \tanh\left(\frac{2\pi\mu}{\delta'}\right)} & \text{if } \mu \leq \frac{1}{2}
\end{array} \right.
\]

and \( \delta'_2 = \left\{ \begin{array}{ll}
1 & \text{if } \mu \geq \frac{1}{2} \\
\left(1 + \frac{\delta'}{\sinh(4\pi\mu)}\right) \sqrt{\delta' \tanh\left(\frac{2\pi\mu}{\delta'}\right)} & \text{if } \mu \leq \frac{1}{2}
\end{array} \right. \quad \text{(5)}
\]

Taking \( \delta'_3 \) as the ratio between group velocity \( C_g \) and phase velocity \( C_\varphi \), we have:

\[
\delta'_3 = \frac{C_g}{C_\varphi} = \left\{ \begin{array}{ll}
\frac{1}{2} & \text{if } \mu \geq \frac{1}{2} \\
\frac{1}{2} \left(1 + \frac{\delta'}{\sinh(4\pi\mu)}\right) & \text{if } \mu \leq \frac{1}{2}
\end{array} \right.
\]
Assuming that the total energy carried by wave is preserved in the shoaling area, according Peronno G. (2003), we have:

\[ H = H_o \left[ \tanh(kd) + \frac{kd}{\cosh^2(kd)} \right]^{-1/2} \tag{6} \]

If \( \kappa' = \frac{H}{H_o} \) is the coefficient of variation of height along the linear theory (coefficient shoaling):

\[ \kappa' = \begin{cases} 
1 & \text{if } \mu \geq \frac{1}{2} \\
\tanh \left( \frac{2\pi\mu}{\delta'} \right) + \frac{(2\pi\mu)}{\cosh^2\left( \frac{2\pi\mu}{\delta'} \right)} \end{cases}^{-1/2} \quad \text{if } \mu \leq \frac{1}{2} \tag{7} \]

### 2.2 Breaking criterion

Kaminsky and Kraus [1993] conducted a comparative study based on an analysis of previous studies (409 cases) covering a wide range of curvatures of wave and beach slopes. They show a positive relationship between the breaking index and slope (with a good representation of \( \gamma = 0.78 \) for the slopes \( \tan \beta \leq 0, 1 \)). They advise to use the following formula where \( H_b \) is the wave height and \( d_b \) to local water depth at breaking point:

\[ \gamma = \frac{H_b}{d_b} = 1, 2 \left( \sqrt{\frac{L_o}{H_o \tan \beta}} \right)^{0.27} \tag{8} \]

### 2.3 Boundary Conditions

The vertical elevation of the free surface of the sea, in its complex form at a deep water, obtained by solving the Laplace equation in the Airy’s theory is \( \eta_o = \frac{H_o}{2} e^{i(k_o x - \omega t)} \). In the area shoaling where \( \phi \) is the phase shift induced by the seabed, let \( \eta = \frac{H_o}{2} e^{i(k x - \omega t + \phi)} \) \[6\]. From these expressions, we deduce the conditions at the limit of deep water and the intermediate zone:

\[ L \left( \frac{L_o}{2} \right) = L_o = \frac{gT^2}{2\pi}; \quad H \left( \frac{L_o}{2} \right) = H_o \quad \text{and} \quad \phi \left( \frac{L_o}{2} \right) = 0 \tag{9} \]

### 2.4 Characterization of the swell in the shoaling area

To describe changes in the wavelength and the peak to valley height in coastal areas during propagation, we combined the Saint-Venant equations and those of Green Naghdi. The system of equations below to establish analytical expressions that justify the numerical and experimental results obtained by P. Bonneton, Vijay G. Panchang and other authors.\[16,23,28\]
\[
\left\{ \begin{array}{c}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \\
\frac{\partial}{\partial t}(hu^2) + \frac{\partial}{\partial x}(gh^2) = gh\frac{\partial d}{\partial x}
\end{array} \right. \quad \text{with} \quad \begin{cases} 
\eta = -\frac{1}{g} \left( \frac{\partial \Phi}{\partial t} \right) \\
h = \eta + d \\
u = \frac{\partial \Phi}{\partial x}
\end{cases} \quad [19,22,23]
\]

As the amplitude of the wave is negligible, all non-linear terms are negligible, and neglecting the convective effect to that induced by the acceleration \( \frac{\partial (hu)}{\partial x} \approx 0 \) and \( u \frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial t} \), the Saint-Venant equations leads to:

\[
\frac{1}{g} \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial d}{\partial x} \frac{\partial \eta}{\partial x} - d \frac{\partial^2 \eta}{\partial x^2} = 0
\]

In the coastal zone, Xiao-Bo Chen, (2006), implies that the effect of seabed on the deformation of the free surface induces a phase shift \( \phi \) on wave propagation\([6]\) and includes \( \eta = \frac{H(x)}{2} e^{i(kx-\omega t+\phi)} = \eta_a(x)e^{-i\omega t} \) with \( \eta_a(x) = \frac{H(x)}{2} e^{i(kx+\phi)} \). Thus, the above equation becomes:

\[
d\frac{\partial^2 \eta_a}{\partial x^2} + \frac{\partial d}{\partial x} \frac{\partial \eta_a}{\partial x} + \frac{\omega^2}{g} \eta_a = 0
\]

I. I. Didenkulova and al. (2008), have chosen the origin of the space at the free surface on the beach. In this condition, the equation of the tangent at almost flat seabed, inclined at an angle \( \beta \) relative to the horizontal, is: \([15,16]\)

\[
z = ax = -d \quad \text{where} \quad a = \tan \beta
\]

By the change of variable \( \eta_o(x) = \eta_o(d) \) and using the equation of the tangent to the seabed local point of water depth \( d \), we obtain:

\[
d\frac{\partial^2 \eta_o}{\partial d^2} + \frac{\partial \eta_o}{\partial d} + \frac{\omega^2}{ga^2} \eta_o = 0
\]

2.5 Solution of the differential equation

The general solution of this equation is any linear combination of Bessel functions of \( \nu \) order, \( J_\nu \) and \( Y_\nu \) the first kind and the second kind or Hankel functions \( \nu \) order \( H^{(1)}_\nu \) and \( H^{(2)}_\nu \) the first kind and the second kind,\([5]\)

\[
\eta_o(\alpha) = A J_\nu(\alpha) + B Y_\nu(\alpha) \quad \text{or}
\]
\[ \eta_a(\alpha) = A.H^{(1)}_\nu(\alpha) + B.H^{(2)}_\nu(\alpha) \text{ with } \alpha = 2a\omega \sqrt{\frac{d}{g}} \] (15)

And since \( \eta_a \) is a complex function whose real part must be a decreasing function of \( d \) before the breaking, the general solution can be written:

\[ \eta(x) = \eta(d) = A.\sqrt{\frac{2}{\pi\alpha}} e^{i(\alpha - \frac{\pi(2\nu+1)}{4} - \omega t)} \text{ with } \alpha = 2a\omega \sqrt{\frac{d}{g}} \] (16)

**3 Results and Discussion**

**3.1 Height and wavelength in the shoaling zone**

To general solution \( \eta_a(d) = A.\sqrt{\frac{2}{\pi\alpha}} e^{i(\alpha - \frac{\pi(2\nu+1)}{4})} \), to the previous boundary conditions, and identification with \( \eta_a(d) = \frac{1}{2}H(d)e^{i(kx+\varphi)} \), we deduced for \( \mu_b \leq \mu \leq \frac{1}{2} \) that:

\[
\begin{align*}
H(d) &= H_o \left( \frac{L_o}{2d} \right)^{1/4} = H_o \left( \frac{1}{2} \right)^{1/4} ; \quad L = L_o\sqrt{\frac{2d}{L_o}} = L_o\sqrt{2\mu} \\
\varphi &= \left( 2a\sqrt{\pi} + \frac{\pi}{a} \right) \left( \sqrt{\frac{2d}{L_o}} - 1 \right) = \left( 2a\sqrt{\pi} + \frac{\pi}{a} \right) \left( \sqrt{2\mu} - 1 \right)
\end{align*}
\] (17)

If \( \delta = \frac{L}{L_o} \) and \( \kappa = \frac{H}{H_o} \) are respectively the coefficients of variation of the wavelength and the peak to valley height of the swell before the breaking point, their expressions according to the parameter of the wave train \( \mu = \frac{d}{L_o} \) are:

\[
\kappa = \frac{H}{H_o} = \begin{cases} 1 & \text{if } \mu \geq \frac{1}{2} \\ \left( \frac{1}{2\mu} \right)^{1/4} & \text{if } \mu_b \leq \mu \leq \frac{1}{2} \end{cases}
\]

\[
\delta = \frac{L}{L_o} = \begin{cases} 1 & \text{if } \mu \geq \frac{1}{2} \\ (2\mu)^{1/2} & \text{if } \mu_b \leq \mu \leq \frac{1}{2} \end{cases}
\]

and \( \varphi(d) = \begin{cases} 0 & \text{if } \mu \geq \frac{1}{2} \\ \left( 2\sqrt{\pi}\tan \beta + \frac{\pi}{\tan \beta} \right) \left( \sqrt{2\mu} - 1 \right) & \text{if } \mu_b \leq \mu \leq \frac{1}{2} \end{cases} \) (18)

**3.2 Wave height and water depth at the breaking point**

Using criterion breaking expression proposed by Kaminsky and Kraus since 1993, for low slopes \( \tan \beta < 0.1 \) and the height previously found, we have:

\[
\begin{align*}
&\text{For } \tan \beta < 0, 1 \text{ we have:} \\
&d_b = \left[ \frac{L_oH_o^4}{3,1472} \left( \sqrt{\frac{L_o}{H_o}} \tan \beta \right)^{-1.081} \right]^{1/5} \\
&H_b = H_o \left[ 0.6 \left( \frac{L_o}{H_o} \right)^{1.135} \left( \tan \beta \right)^{0.27} \right]^{1/5}
\end{align*}
\] (19)
3.3 Variations of phase and group velocities

Let $\delta_1 = \frac{C}{C_\phi}$, $\delta_2 = \frac{C}{C_{go}}$, and $\delta_3 = \frac{C}{C_\phi}$ the coefficients of variation of the phase and group velocities and the ratio of these two speeds.

$$\delta_1 = \begin{cases} 
1 & \text{if } \mu \geq \frac{1}{2} \\
\sqrt{\delta \tanh \left( \frac{2\pi \mu}{\delta} \right)} & \text{if } \mu_b \leq \mu \leq \frac{1}{2}
\end{cases}$$

$$\delta_2 = \begin{cases} 
1 & \text{if } \mu \geq \frac{1}{2} \\
\left(1 + \left(\frac{2\pi \mu}{\delta \sinh \left( \frac{4\pi \mu}{\delta} \right)} \right) \right) \sqrt{\delta \tanh \left( \frac{2\pi \mu}{\delta} \right)} & \text{if } \mu_b \leq \mu \leq \frac{1}{2}
\end{cases}$$

and

$$\delta_3 = \begin{cases} 
\frac{1}{2} & \text{if } \mu \geq \frac{1}{2} \\
\frac{1}{2} \left(1 + \left(\frac{2\pi \mu}{\delta \sinh \left( \frac{4\pi \mu}{\delta} \right)} \right) \right) & \text{if } \mu_b \leq \mu \leq \frac{1}{2}
\end{cases}$$

(20)
• Figure 1: a) shows the evolution of the deformation of the free surface in the absence of seabed $\eta_0$ and in the presence of seabed $\eta$. The evolution of these two parameters shows that the perturbation effect of the seabed induces a phase delay of $\eta$ from $\eta_0$. The amplitude remains constant in the deep water and grows for effect of the seabed in the area shoaling.

• Figure 1: b) shows the evolution of coefficients of variations of height and wavelength according with the linear theory one hand, and according to the results obtained from the Saint-Venant equations other.

- According to the results of Saint-Venant equations, the height is amplified to the breaking point, but the linear theory shows that the height decreases to $0.2 \leq \mu \leq 0.5$ before start grow up at the breaking point.

- For the wavelength, it slower further according to our results, but it contracts slower for linear theory.

• Figure 1: c) represents the variation of late phase $\varphi$ induced by a perturbative action of seabed on the incident waves from deep water. This phase varies depending on the local water depth $d$ and depending on the seabed slope.

• These results show that the height increases when the local water depth $d$ decreases to the breaking point and it is proportionally to $d^{-1/4}$. The wavelength decreases and is proportionate to the square root of the local water depth $d$. The perturbative effect of this background also induces a phase delay $\varphi < 0$ $\forall$ $\tan \beta$ during propagation in the coastal zone compared to that of the incident wave comes from deep water. This phase shift $\varphi < 0$ depends on the angle $\beta$ sea breaking relative to the horizontal and the local water depth $d$.

• Variations of $\delta_1$ and $\delta_1'$ show that the phase velocity is constant in deep water but decreases in area shoaling when the local water depth $d$ decreases. But this decrease is more marked in the case of the results obtained with the Saint-Venant equation than the linear theory.

• Variations of $\delta_2$ and $\delta_2'$ show that the group velocity is constant in the deep waters. But in the area shoaling, this speed decreases according to the results obtained while she believes using the linear theory and to the point
where $\mu \approx 0.16$ prior to start decrease. In reality, the speed should not grow when dispersive system is taking up a slope (probable source of energy loss).

- The coefficient $\delta_1$ and $\delta_3'$ are $\frac{1}{2}$ in deep water, they reveal that the phase velocity is twice the group in deep water. These reports increase in shoaling area and tend to 1 when $d$ tends to 0. But growth is more pronounced in the case of the linear than the results obtained by the equations of Saint-Venant theory.

- The curves of figure 2 b) show firstly that the local water depth at the breaking point decreases when the bottom slope increases. This decrease is growing more when the height $H_o$ decreases, but very little when the wavelength decreases. On the other hand, the wave height at breaking point $H_b$ increases when the bottom slope increases. This growth is more pronounced when the height $H_o$ increases, but very little when the wavelength increases.

- The curves in figure: 3 b) and figure: 3 c) show that the bathymetric breaking point of wave, characterized by $(d_b, H_b)$, depends not only on the bottom slope but also the value of $H_o$.

- As for the curves in Figure 3 a), they show that the criterion of a breaking wave is not constant ($0.78 \leq \gamma \leq 1.2$ for $\tan \beta < 0, 1$): it increases when $H_o$ decreases but increases when the bottom slope increases.

## 4 Conclusion

From previous results, we deduce that all the parameters studied are almost constants in deep waters. The period of propagation of a wave is almost constant along its propagation. The wavelength, the phase velocity and the group velocity decreases when the local depth of water decreases in the shoaling zone. This wavelength is proportional to the square root of the local water depth $d$. Unlike the Airy theory gives the impression that the group velocity believes in this area, the group velocity decreases because the system is dispersive.

For the peak to valley height, it is increased in the intermediate zone when the remaining depth decreases and is proportional to $d^{-1/4}$ but reaches its maximum value at the breaking point. The effect of seabed pertunatif induces a phase delay $\varphi < 0$ in the propagation of the wave associated with the swell. This phase shift depends not only on the local depth of water $d$ but also the inclination of the bottom with respect to the horizontal.

Overall, the use of Saint-Venant equations is better to model the variations of the characteristics parameters of the swell in the area shoaling unlike the linear theory which does not provide information, not only on the decrease of the group velocity, but also on the evolution of the phase shift induced by the perturbative action of seabed on the deformation of the free surface of the ocean in this area.
References


Modeling of wave characteristics parameters


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