Central Collision of Two Rolling Balls:

Theory and Examples

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Abstract

Paper is focused to central collision of two rolling rigid and heavy smooth balls and using elements of mathematical phenomenology and phenomenological mapping obtain corresponding post collision and outgoing angular velocities of the balls and applied these results for investigation vibro-impact dynamics of two rolling balls along circle trace. This task is fully solved and obtained results are original and new! Original plans of component impact velocities and angular velocity of each of two different rolling balls in central collision and corresponding outgoing angular velocities are presented. Use Petrović’s elements of mathematical phenomenology, especially mathematical analogy between kinetic parameters of collision of two bodies in translator motion and collision of two rolling different size balls, new original expressions of two outgoing angular velocities for each of rolling balls after collision are defined. Using this new and original result vibro-impact dynamics of two rolling different heavy balls on the circle trace in vertical plane in period of series collisions is investigated. Use series of the elliptic integrals, new nonlinear equations for obtaining angles of balls positions at positions of collisions are defined. Branches of phase trajectories of the balls in vibro-impact dynamics are theoretically presented.

Keywords: Theory, rolling balls; collision; pre-impact; post-impact, impulse; moment of impulse; impact forces; impact couple; rolling trace; arrival angular velocity; impact angular velocity; outgoing angular velocity; theorems; collision of rolling balls in circle line; phase trajectory; angular velocity discontinuity
I. Introduction

"In connection with the game of billiards ..., there are various dynamic tasks, whose solutions contain in this event. I think that people who know Theoretical mechanics, and even students of polytechnics, with interest familiarize themselves with explanations of all the original phenomenon that can be observed from the time of movement billiard balls."

II. Gaspar-Gistav de Koriolis,

Mathematical theory of billiards game.

I.1. The elements of the dynamics of billiards [1-3], [4] are coupled into complex system, whose dynamics are different phenomena observed dynamics of the system. Starting from the geometric basis for switching to the theory of impact and collisions between two or a few number of the balls, it is possible to see that impacts and collisions are in centre of this dynamics. Shown are the plans of translational and angular velocities of rolling of one ball before and after the impact, and also the two balls collide. Rolling balls are main elements in numerous mechanical engineering systems.

Theory of impact dynamics of system as well as vibro-impact dynamics in now day is important research task. This is reason that motivation for our research is presentation of the theory of the collision of two rolling, rigid, homogeneous and heavy, smooth balls with different radii and different masses.

The theory of the central collision of two bodies in translator moron (mass particles), as known theory, is starting point in our paper, and using Petrović’s theory of Elements of mathematical phenomenology [5], [6] and Phenomenological mappings [7] are basic theories for obtaining kinetic parameters of the central collision of two rolling rigid, homogeneous and heavy, smooth balls along horizontal straight trace, as well as along curvilinear circle trace in vertical plane. (Also, see papers from Special Issue of IJNLM [8], especially References [9-14]).

The results, in present paper, are focused to obtain expressions of the post-collision outgoing angular velocities of two rolling rigid, heavy, smooth balls, as well as definition of the coefficient of restitution of collision of two rolling rigid smooth balls with different size (radii), and investigation of its vibro-impact nonlinear dynamics on a circle trace in vertical plane using in phase plane branch of phase trajectory between successive between two collisions.
Classical approach is used, as Newton theory and mathematical analogy between kinetic elements of the collision of two rigid body in translator motion and collision of two rolling, heavy smooth balls, taking into account that both motions before, pre-collision and after, post-collisions are simple with only one type of the motion, translator of two rigid bodies and analogous rolling of two rolling balls. For detail see Theory of Mihailo Petrovic (one of three doctoral students of French international scientist Julius Henri Poincaye): “Elements of mathematical phenomenology” [5] and “Phenomenological mappings” [6].

The output is by basic hypotheses on classical theory of the collision and impact, founded by classical theory of impact by Newrin. Then, let us, to point out the start by the foundation of the theory of impacts and history about the competition held in organization of Royal Scientific Society [15]. The Royal Scientific Society, in London in 1668 announced a competition for the solution of problems of the dynamics of impact. And, on this competition their works are submitted, both by, now known scientists, Vilis (John Wallis, (1616-1703), Mechanica sive de mote-1688) and Hajgens (Christiaan Huygens (14 April 1629 – 8 July 1695) – De motu corporum ex percusione). Using the results of the collision submitted by the Royal Scientific Society learned Willis and Huygens, and giving their generalizations, Isaac Newton founded the fundamental basics of the theory of impacts. And before Newton and Huygens and Willis, was exploring the dynamics of impacts. Thus, for example, collision problems are dealt with Galileo Galilei, who came to the conclusion that the impact force in relation to the pressure force infinitely large, but it not came to the knowledge of the relationship of impact impulse and linear momentum.

It will be shown Karnoova teorema (Lazare Carnot 1753-1824., Principes fondamentaux de l’équilibre et de movement - 1803), who says that "In a collision, the system inelastic material bodies loss their kinetic energy and that is equal to the kinetic energy lost speed."

Aim of paper is not to investigate different coefficients of restitution, influence of the friction and properties of contact surfaces of bodies in collision as very important results for applications in engineering sciences and engineering systems with impacts. Also, aim of this paper is not to make a paper to present state-of-art in all area of impact dynamics and citations of all important results obtained in area of the collision and impact dynamics and point out names of the researchers with original advances to scientific knowledge in large and important area of applied science of the collision and impact of rigid and deformable bodies. But accepting some Reviewers’ suggestions in the revised list of the References of this paper, some of the References by Glocker and Pfeiffer [15], Stronge et all. [16-18], Brogliato [19], Klarbring [20], Chatterjee and other [21-24] are included.

Numerous papers and unique important and valuable monographs contain important and original results and models of impacts and collision of two bodies, focused to the investigation of the real kinetic problems to obtain influence of the friction and contact surfaces of real bodies in the collisions to outgoing post-collision kinetic parameters of the bodies.
The unilateral contact and the friction coupled via a contact bi-potential, as well as the application of the augmented Lagrangian method to the contact laws are investigated in the Reference [21]. In this Reference for additional comments is cited the interesting discussion by Klarbring [21]. For dynamic implicit analysis in structural mechanics, the most commonly used integration algorithm is the second order algorithm such as Newmark, Wilson, etc. Work equivalent composite coefficient of restitution is investigated by Coaplen, Stronge and Ravani [16]. Visco-plastic analysis for direct impact of sports balls is subject of research by Ismail and Stronge [17]. Very important results in area of “Oblique frictional impact of a bar: analysis and comparison of different impact laws” is content of the Reference [24] by Payr and Glocker. The results presented in monograph [15] titled: “Multi-body dynamics with unilateral contacts” by Pfeiffer and Glocker are important for theory of dynamic system with impacts.

\[ m \dot{v}_1(t_0 + \tau) \]
\[ m \dot{v}_1(t_0) \]

\[ m \dot{v}_2(t_0 + \tau) \]
\[ m \dot{v}_2(t_0) \]

Figure 1. Central collision between two bodies, with mass \( m_1 \) and \( m_2 \) in translator motion (a* and b*) and with translator pre-impact velocities \( \dot{v}_1(t_0) \) and \( \dot{v}_2(t_0) \) (c*) and with outgoing post-impact velocities \( \dot{v}_1(t_0 + \tau) \) and \( \dot{v}_2(t_0 + \tau) \) (c* and d*).

Also see part of monograph [21] titled by “Analytical Dynamics of Discrete Systems” and authored by Rosenberg. Very useful is content of the monograph [22] titled: “Impact Mechanics” by Stronge. Periodic Motion Induced
by the Painlevé Paradox presented in [19] by Leine, Brogliato and Nijmeijer is also valuable results in the area. Rigid body dynamics with friction and impact is content of [24] written by Stewart.

I.2. Let’s start with large known classical theory [25] of central collision between two bodies, with mass $m_1$ and $m_2$, in translator motion and with translator velocities $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ at the moment before collision between them. These velocities we denote as arrival, or impact or pre-impact velocieties at the moment $t_0$ (see Figure 1.). At this moment $t_0$ of the central collision start between these bodies, contact of these two bodies is at point $P$, in which both bodies posses common tangent plane –plane of contact (touch). In theory of central collision, it is proposed that collision takes very short period time $(t_0,t_0+\tau)$, and that $\tau$ tend to zero. After this short period bodies in collision separate and are in outgoing kinetic state by post-impact-outgoing velocities $\vec{v}_1(t_0 + \tau)$ and $\vec{v}_2(t_0 + \tau)$.

On the basis of theorem of conservation of linear momentum (impulse) of motion the following relation is valid [26]:

$$m_1\vec{v}_1(t_0) + m_2\vec{v}_2(t_0) = m_1\vec{v}_1(t_0 + \tau) + m_2\vec{v}_2(t_0 + \tau) \quad (1)$$

and coefficient of the restitution of body central collision is:

$$k = \frac{v_1(t_0 + \tau)}{v_1(t_0)} = \frac{v_2(t_0 + \tau) - v_1(t_0 + \tau)}{v_1(t_0) - v_2(t_0)} \quad (2)$$

and present ratio between difference of translator velocities in post-collision and pre-collision kinetic states, definre by Newton classical theory of impact. 

Post-central-collision –outgoing body translator velocities are in the form [15]:

$$v_1(t_0 + \tau) = \frac{(m_1 - km_1)v_1(t_0) + (1 + k)m_2v_1(t_0)}{m_1 + m_2} = v_1(t_0) - \frac{1 + k}{m_1 + m_2} (v_1(t_0) - v_2(t_0)) \quad (4)$$

$$v_2(t_0 + \tau) = \frac{(m_2 - km_1)v_2(t_0) + (1 + k)m_1v_1(t_0)}{m_1 + m_2} = v_2(t_0) + \frac{1 + k}{m_2 + m_1} (v_1(t_0) - v_2(t_0)) \quad (5)$$

**Impuls (linear momentum) of collision** in this case is:

$$K_{fad} = m_1(v_1(t_0 + \tau) - v_1(t_0)) = - \frac{m_1m_2}{m_1 + m_2}(1 + k)(v_1(t_0) - v_2(t_0)) \quad (6)$$

As it is known from classical literature [25], coefficient of the restitution of body collision depends of kind of collision: $1^*$ for the pure no elastic (plastic)collision, the coefficient of restitution is equal to zero- $k = 0$; $2^*$ for the
pure ideal elastic collision, the coefficient of restitution is equal to unique, $k = 1$; and $3^*$ for arbitrary case between ideal plastic and ideal elastic collision, the coefficient of restitution is in interval between zero and unique, $0 < k < 1$.

From the comparison between outgoing (post-collision) velocities in no elastic collision of two translator bodies in pre-collision state, we can point out the following conclusions:

* in the case of pure plastic collision of two bodies, $k = 0$, outgoing (post-collision) velocities are equal to each other;

* in the case of ideal elastic collision of two bodies in translator motions, $k = 1$, outgoing (post-collision) velocity of the body with larger pre-collision impact velocity is smaller, and outgoing (post-collision) velocity of the body with smaller pre-collision impact velocity is larger; in this case, ideal elastic impact, $v_1(t_0) = v_2(t_0)$, and opposite direction, then both outgoing velocities of the both bodies are equal intensity $v_1(t_0 + \tau) = v_2(t_0 + \tau)$ and opposite direction and independent of the body masses.

* In the case of no elastic collision, between bodies, $0 < k < 1$, if condition $m_1v_1(t_0) + m_2v_2(t_0) = 0$, or $\frac{m_1}{m_2} = \frac{v_2(t_0)}{v_1(t_0)}$ is satisfied, outgoing (post-collision) velocities of both bodies satisfied relation: $\frac{m_1}{m_2} = \frac{v_2(t_0 + \tau)}{v_1(t_0 + \tau)}$. In this case outgoing velocities are: $v_1(t_0 + \tau) = -kv_1(t_0)$ and $v_2(t_0 + \tau) = kv_2(t_0)$.

Kinetic energy of the bodies in translator motion in pre-collision kinetic state is in the form:

$$E_k(t_0) = \frac{1}{2}(m_1v_1^2(t_0) + m_2v_2^2(t_0))$$

and kinetic energy of these bodies after central collision (in post-collision kinetic state) is:

$$E_k(t_0 + \tau) = \frac{1}{2}(m_1v_1^2(t_0 + \tau) + m_2v_2^2(t_0 + \tau))$$

$$\Delta E_k = E_k(t_0 + \tau) - E_k(t_0) = -\frac{m_1m_2}{2(m_1 + m_2)}(1 - k^2)(v_1(t_0) - v_2(t_0))^2$$
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\[ \Delta E_{k,\text{plast}} = E_k(t_0 + \tau) - E_k(t_0) = -\frac{m_1 m_2}{2(m_1 + m_2)} (v_1(t_0) - v_2(t_0))^2 \]  

(10)

c* In the case of ideal elastic collision, \( k = 1 \), between bodies in translator motion no change of kinetic energy in comparison between pre-collision and post-collision kinetic states, and is equal to zero:

\[ \Delta E_{k,\text{elast}} = E_k(t_0 + \tau) - E_k(t_0) = 0. \]  

(11)

In this case of ideal elastic impact low of kinetic energy conservation is valid.

Listed expressions and relations (7)-(11) and conclusions a*-b*-c* of kinetic energy decreasing in comparison kinetic energy in pre-collision of kinetic state and post-collision kinetic state of the bodies in collision present Carnot’s theorem (Lazare Carnot 1753-1824., Principes fondamenteaux de l’équilibre et de movement - 1803) [25] of kinetic energy of bodies in translator kinetic states pre- and post collision (in arrival and outgoing kinetic states): “In the collision of two bodies in translator motion for arbitrary coefficient of the restitution, \( 0 < k < 1 \), lost of kinetic energy in decreasing during collision, is proportional to lost of the velocities.

\[ \Delta E_k = E_k(t_0) - E_k(t_0 + \tau) = 2E_{k,\text{ang}}(\tau) = \sum_{i=1}^{N} m_i [\vec{v}_i(\tau)]^2 \]  

(12)

II. Collision of two heavy rolling balls along horizontal trace

II.1. Kinematics of collision of two rolling balls along horizontal trace

II.1.1. Let’s start with analysis of the elements of kinematics of two rolling balls in the state pre-collision between them. We consider two heavy smooth balls with different masses, and different radii, \( r_1 \) and \( r_2 \), each in rolling kinetic state with momentary angular velocity, \( \vec{\omega}_{p,1} \) and \( \vec{\omega}_{p,2} \), along corresponding straight trace of rolling, which are linear. Momentary axes of each rolling lying in horizontal plane and is orthogonal to the rolling trace in each moment passing through point \( P \) (see Figure 2, and also Figure A1.b in Appendix) or for first and second rolling ball through point \( P_1 \) as in point \( P_2 \) (see Figure 3, a* and b*) or Figure 4. These points \( P_1 \) and \( P_2 \) are points of touch between rolling trace and corresponding rolling ball, and these point move along trace together with momentary axis of ball’s rolling.
Figure 2. Plan of component impact velocities of impact points of a ball for the different types of collisions two equal rolling balls

If momentary angular velocities, $\dot{\omega}_{p,1}$ and $\dot{\omega}_{p,2}$, of the rolling balls and corresponding axes of the rolling first and second heavy ball are known, and also, radiiuses of balls and mass densities of balls, then dynamics of each ball is fully determined. Then investigation of the heavy balls dynamics is simple’s task for obtaining all kinetic parameters of balls.

Let us consider possible component impact velocities in point $T$ at spherical surface as possible point of touch in kinetic state of collision between two rolling balls. If words are about collision of two equal dimensions (equal radiiuses) of the rolling balls all possible points $T_i$, $i = 1,2,3,...$ of central or skew collision between balls are at circle passing through mass centre of both ball, and balls’ common tangent plane through this point of balls collision is vertical. Taking into account that trajectories of mass centers of both rolling balls are horizontal and straight lines parallel to rolling trace, then both mass centers move translator with velocities $\vec{v}_{C,\text{translator}}$ (Figure 2) or $\vec{v}_{C,1,\text{translator}}$ and $\vec{v}_{C,2,\text{translator}}$ (Figure 3.a* and b*).

For the case equal balls in collision (Figure 3), each impact velocity $\vec{v}_{T,\text{impact}}$ of impact at pre-collision state have two components, one horizontal equal to $\vec{v}_{T,\text{translator}} = \vec{v}_{C,\text{translator}}$ and one vertical component $\vec{v}_{T,\text{rolling}}$ of self rotation with angular velocity $\dot{\omega}_c = \dot{\omega}_p$ around central axis parallel to instantaneous (momentary) axis of ball rolling along trace. This rolling component of impact velocity is dependent of the types of collision. If collision of balls is central with same line as a trace rolling both balls, then rolling component $\vec{v}_{T,0,\text{rolling}}$ of impact velocity of point $T_0$ is with maximal intensity and with intensity equal to product between ball radius $R$ and intensity of angular velocity $\dot{\omega}_c = \dot{\omega}_p$ of self rotation. In the case that rolling balls are in skew collision between balls, impact points are at
the point $T_{12}$ or $T_{13}$ (see Figure 2) and with angular velocities no parallel, and balls’ rolling traces are with intersection, or parallel, then point $T_{12}$ or $T_{13}$ of the collision of rolling balls are at the distance defined by $R \cos \alpha$ to the self rotation central axis of ball, where $\alpha$ is angle between trace rolling of corresponding ball and normal to the common tangent plane. Then intensity of the rolling component of arrival velocity is equal to the product between orthogonal distance $R \cos \alpha$ and intensity of angular velocity $\omega_c = \omega_p$ of self rotation. Outgoing components of the impact velocity of the impact point $T_0$ in central collision of the equal rolling balls are corresponding intensities and with opposite directions in vertical and in horizontal planes. After analysis of the post collision motion for this case we see that it is simple taking into consideration only that collision appear in rolling ball with corresponding angular velocities of rolling. This will be explained later and for detail about rolling ball kinematic and dynamic see Appendix.

II.1.2. In the case that rolling balls are with different dimensions (size) and masses and axial mass inertia moments for instantaneous axis of rolling, the kinematical plan of component velocities in collision are presented in Figures 3, 4 and 5.

In Figure 3, the kinematical plans of the impact velocities of possible points of collision of two rolling heavy balls different radiuses: $a^*$ and $c^*$ for first rolling smaller ball and $b^*$ and $d^*$ for second rolling bigger ball are presented.

In Figure 4, plans of the impact velocities of possible points at corresponding circles at same height of balls in central collision of two rolling heavy balls with different radiuses: left for first smaller ball and right for second bigger ball are presented.

From listed plans of the component velocities, we can see that at case of central collision of the rolling different dimension balls, the collision point is $T_0$ at both spherical surfaces. At smaller ball this point $T_0$ of ball central collision is upper to its mass center and at bigger ball this point $T_0$ is lower to the mass center of ball. Tangent plane of central collision, passing through point $T_0$, and is orthogonal to the radii of both balls from point $T_0$ to the corresponding ball center, $C_1$ and $C_2$.

In Figures 2, 3 and 4, the possible points of the impacts at balls with different size for different types of collision with corresponding kinematic plans of velocities are presented, but, in basic, this part is focused to central collisions of two rolling rigid smooth balls along staring trace (Figure 5) and circle trace (Figure 7).
**Figure 3.** Plans of the impact velocities of possible points of collision of two rolling heavy balls different radiiuses: a* and c* for first rolling smaller ball and b* and d* for second rolling biggest ball.
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Figure 4. Plans of the impact velocities of possible points at corresponding circles at same height of balls in central collision of two rolling heavy balls different radiuses: left for first smaller ball and right for second bigger ball.

In Figure 5, the kinematic plans of the component impact and outgoing velocities at point $T_0$ of central collision of two rolling heavy balls different radiuses are presented.

After similar analysis of the presented kinematic plans of arrival and outgoing component velocities, as in previous case III.1.2., all conclusions from listed three Figures 3, 4 and 5 directed us to a general central conclusion, that collision of the two rolling balls, is simpler to investigate in analogy with known classical theory and results of kinetic, kinematic and dynamic parameters of collision between two bodies in translator motion. See Appendix for detail about simple motion!
Figure 5. The kinematic pans of the component impact and outgoing velocities at point of central collision of two rolling heavy balls different radiuses

II.2. Dynamics of the central collision of two rolling balls along horizontal trace

Let’s start with application of mathematical analogy of the classical theory of dynamics of collision to the dybanics of the collision between two rolling balls, with mass $m_1$ and $m_2$, and axial mass inertia moments $J_{p1}$ and $J_{p2}$ for corresponding momentary axis of the rotation in rolling along trace with pre-impact (arrival) angular velocities $\tilde{\omega}_{P1,\text{impact}} = \tilde{\omega}_{p1}(t_0)$ and $\tilde{\omega}_{P2,\text{impact}} = \tilde{\omega}_{p2}(t_0)$. Mass ceneters $C_1$ and $C_2$ of the balls move transtatory with pre-impact (arrival) velocities $\tilde{v}_{C1,\text{impact}} = \tilde{v}_{c1}(t_0)$ and $\tilde{v}_{C2,\text{impact}} = \tilde{v}_{c2}(t_0)$. Angular velocities $\tilde{\omega}_{P1,\text{impact}} = \tilde{\omega}_{p1}(t_0)$ and $\tilde{\omega}_{P2,\text{impact}} = \tilde{\omega}_{p2}(t_0)$ we denote as arrival, or impact or pre-impact angular velocities at the moment $t_0$ (see Figures 3, 4 and 5). At this moment $t_0$ of the collision start between these rolling balls, contact of these two balls is at point $T_{12}$, in which both balls posses common tangent plane –plane of contact (touch). In the theory of collision, it is proposed that collision takes very shorth period time $(t_0, t_0 + \tau)$, and that $\tau$ tend to zeo. After this short period $\tau$ bodies-two rolling balls in collision separate and outgoing by post-impact outgoing angular velocities $\tilde{\omega}_{P1,\text{outgoing}} = \tilde{\omega}_{p1}(t_0 + \tau)$ and $\tilde{\omega}_{P2,\text{outgoing}} = \tilde{\omega}_{p2}(t_0 + \tau)$. 
Mass centers $C_1$ and $C_2$ of the balls move translatory with post-impact (outgoing) translator velocities $\vec{v}_{c1,\text{outgoing}} = \vec{v}_{c1}(t_0 + \tau)$ and $\vec{v}_{c2,\text{outgoing}} = \vec{v}_{c2}(t_0 + \tau)$. These translator velocities are possible to express, each by corresponding outgoing post-collision angular velocity and radius of the corresponding ball.

Elements of mathematical phenomenology [5] and phenomenological mappings [6] between rolling balls and translator bodies (balls), which are analogous dynamical impact systems (similar as electromechanical analogy between electrical oscillator with one degree of freedom and mechanical oscillator with one degree of freedom [9, 26, 27]). Translator motion of a body (bal) and rolling motion of a ball are motions and each with one degree of freedom. See Figure 1A and explanation in Appendix. The analogies between mass and axial mass inertia moment for the rolling momentary axis and also translator velocity and angular velocity around momentary axis of rolling follow from comparison of their mathematical description by differential equations of corresponding motion-translator and rolling kinetic states. This is visible and simple explanation!

Taking into account that translator motion of two bodies in central collision is simpler motion of two bodies, defined by corresponding inertia properties expressed by mass, $m_1$ and $m_2$, of each body, and also by corresponding translator pre-impact velocities, $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ at the moment before collision and by post-impact-outgoing translator velocities $\vec{v}_1(t_0 + \tau)$ and $\vec{v}_2(t_0 + \tau)$ it is possible to establish a analogy with collision between two rolling balls. Explanation is in following form.

Also, rolling balls along horizontal strength trace is simple rotation motion defined only by inertia properties in the axial ball mass inertia moments $J_{p1}$ and $J_{p2}$ for corresponding momentary axis of rotation in rolling along trace with pre-impact (arrival) angular velocities $\tilde{\omega}_{p1,\text{impact}} = \tilde{\omega}_{p1}(t_0)$ and $\tilde{\omega}_{p2,\text{impact}} = \tilde{\omega}_{p2}(t_0)$ and corresponding outgoing post-impact-outgoing angular velocities $\tilde{\omega}_{p1,\text{outgoing}} = \tilde{\omega}_{p1}(t_0 + \tau)$ and $\tilde{\omega}_{p2,\text{outgoing}} = \tilde{\omega}_{p2}(t_0 + \tau)$.

Using Petrović’s theory of elements of mathematical phenomenology and phenomenological mappings [5-14] in parts of qualitative and mathematical analogies, we can indicate a qualitative and mathematical analogy between system of the translator dynamics and central collision (impact) dynamics of two bodies in translator motion pre-impact and post impact dynamics phenomena and system of the rolling two ball dynamics and central collision (impact) dynamics of two rolling balls in rolling motion pre-impact and post impact dynamics phenomena.

On the basis of these indicated qualitative and mathematical analogies, it is possible list analogous kinetic parameters of these systems.

The axial ball mass inertia moments $J_{p1}$ and $J_{p2}$ for corresponding momentary axis of rotation in rolling along trace are analogous to the corresponding bodies with masses $m_1$ and $m_2$ of two bodies in collision in translatory motion.
Pre-impact (arrival) angular velocities \( \tilde{\omega}_{p1,\text{impact}} = \tilde{\omega}_{p1}(t_0) \) and 
\( \tilde{\omega}_{p2,\text{impact}} = \tilde{\omega}_{p2}(t_0) \) of the rolling balls around corresponding momentary axis are analogus to corresponding translator pre-impact velocities, \( \tilde{v}_{1}(t_0) \) and \( \tilde{v}_{2}(t_0) \) of two bodies at the moment before collision.

Post-impact outgoing post-impact-outgoing angular velocities 
\( \tilde{\omega}_{p1,\text{outgoing}} = \tilde{\omega}_{p1}(t_0 + \tau) \) and 
\( \tilde{\omega}_{p2,\text{outgoing}} = \tilde{\omega}_{p2}(t_0 + \tau) \) of the rolling balls are analogous to the corresponding post-impact-outgoing translator velocities \( \tilde{v}_{1}(t_0 + \tau) \) and \( \tilde{v}_{2}(t_0 + \tau) \) of two bodies in translator motion to collision.

On the basis of Petrović’s theory [5-7] and qualitative and mathematical analogies considered in previous part, is possible on the basis of theorem of conservation of linear momentum (impulse) (1) of impact dynamics of two bodies in translator motion pre-collision and post-collision, formulate analogous theorem of conservation of angular momentum (moment of impulse for corresponding momentary axis) of impact dynamics of two rolling balls pre-collision and post-collision motion in the following relation:

\[
J_{p1}\tilde{\omega}_{p1}(t_0) + J_{p2}\tilde{\omega}_{p2}(t_0) = J_{p1}\tilde{\omega}_{p1}(t_0 + \tau) + J_{p2}\tilde{\omega}_{p2}(t_0 + \tau)
\] (13)

and analogous with (2), the coefficient of the restitution of rolling balls collision is in the form:

\[
k = \frac{\omega_{p1}(t_0 + \tau)}{\omega_{p1}(t_0)} = \frac{\omega_{p2}(t_0 + \tau) - \omega_{p2}(t_0 + \tau)}{\omega_{p1}(t_0) - \omega_{p2}(t_0)}
\] (14)

as ratio between difference of angular velocities of rolling balls post-collision and pre-collision kinetic states.

Equation (13) is starting and important kinetic parameter of the system of two colliding rolling balls as angular momentum (sum of the moment of impulse of each of the colliding rolling balls for corresponding momentary axis of rolling) of the colliding and rolling balls before – pre-collision and after – post-collision kinetic state of the system in analogy with and on the same level as equation (1) of linear momentum (impulse) for two colliding bodies (balls) in translator motion before-pre collision and after post-collision of bodies in translator motion.

The restitution coefficient \( k \) expressed by (2) is determined by Newton’s classical theory of impact dynamics of rigid bodies, as the ratio of the normal velocity components after and before the impact for the case of central collision between two bodies (balls) in translator motion after and before collision. In present paper coefficient of the restitution \( k \) by expression (14) is introduced by angular velocities after and before collision of the two rolling balls. It is normally and in mathematical and qualitative analogy on the basis of theory of Elements of mathematical phenomenology and Phenomenological mappings [5-8] founded by Mihailo Petrović (one of three doctoral students of Julius Henri Poincare) using analogous kinetic elements of translator motion of two balls and of the rotation motion of two rolling balls, each of them with one degrees of freedom!
Also, in analogy with the expressions (4)-(5) of post-collision –outgoing body translator velocities is possible to write expressions of post-collision –outgoing rolling balls angular velocities in the following forms:

\[
\omega_{p_1}(t_0 + \tau) = \frac{(J_{p_1} - kJ_{p_2})\omega_{p_1}(t_0) + (1 + k)J_{p_2}\omega_{p_2}(t_0)}{J_{p_1} + J_{p_2}}
\]

\[
\omega_{p_1}(t_0 + \tau) = \omega_{p_1}(t_0) - \frac{1 + k}{1 + \frac{J_{p_1}}{J_{p_2}}} (\omega_{p_1}(t_0) - \omega_{p_2}(t_0))
\]

\[
\omega_{p_2}(t_0 + \tau) = \frac{(J_{p_2} - kJ_{p_1})\omega_{p_2}(t_0) + (1 + k)J_{p_1}\omega_{p_1}(t_0)}{J_{p_1} + J_{p_2}}
\]

\[
\omega_{p_2}(t_0 + \tau) = \omega_{p_2}(t_0) + \frac{1 + k}{1 + \frac{J_{p_2}}{J_{p_1}}} (\omega_{p_1}(t_0) - \omega_{p_2}(t_0))
\]

Previous obtained expressions (16) of post-collision –outgoing rolling balls angular velocities are new and original results obtained on the basis of Petrović’s theory of elements of mathematical phenomenology (see Reference [5]). Also expression (13) for the theorem of conservation of angular momentum (moment of impulse for corresponding momentary axis) of impact dynamics of two rolling balls pre-collision and post-collision motion is new introduced relation in impact dynamics as well as expression (14) for the coefficient of restitution in collision of two rolling balls with different size and in central collision. All these results are analytical and present basis for applications in other kind of collisions. 

In analogy of expression (6) of the impulses (linear momentum) of collision two bodies in pre-collision and post-collision translator motions, it is possible to compose analogous moment of impulse (kinetic moment, angular momentum) of collision two rolling balls in pre-collision and post-collision dynamics, in the following form:

\[
L_{M,\text{impact}} = J_{p_1}(\omega_{p_1}(t_0 + \tau) - \omega_{p_1}(t_0)) = -\frac{J_{p_1}J_{p_2}}{J_{p_1} + J_{p_2}} (1 + k)(\omega_{p_1}(t_0) - \omega_{p_2}(t_0))
\]

As it is known from classical literature, in analogy with coefficient of the restitution of two body collision in translator motion [25], for coefficient of collision of two rolling balls, also depend of kind of collisions: 1* for pure no elastic (plastic) collision coefficient of collision is equal to zero- \( k = 0 \); 2* for pure ideal elastic collision coefficient of collision is equal to unique, \( k = 1 \); and 3* for arbitrary case between ideal plasrtic and ideal elastic collision coefficient of collision is in interval between zero and unique, \( 0 < k < 1 \).

From comparison between outgoing (post-collision) angular velocities in no elastic collision of two rolling balls in pre-collision state, we can point out the following conclusions:
* in the case of pure plastic collision of two rolling balls, \( k = 0 \), outgoing (post-collision) angular velocities are equal one to other;

* in the case of ideal elastic collision of two rolling balls , \( k = 1 \), outgoing (post-collision) angular velocities of the rolling balls, ball with larger pre-collision impact angular velocity is smaller, and outgoing (post-collision) angular velocity of the rolling ball with smaller pre-collision impact angular velocity is larger; in this case, ideal elastic impact , \( k = 1 \), if both pre-collision impact angular velocities of the rolling balls are equal, then, both outgoing (post-collision) angular velocities of the both balls are equal and independent of the balls axial inertia moments.

* In the case of no elastic collision between rolling balls, \( 0 < k < 1 \), if condition
\[
J_{p1}\omega_{p1}(t_0) + J_{p2}\omega_{p2}(t_0) = 0,
\]
and kinetic energy of the rolling balls in pre-collision kinetic state is in the form:
\[
E_k(t_0) = \frac{1}{2}(J_{p1}\omega_{p1}^2(t_0) + J_{p2}\omega_{p2}^2(t_0))
\]
and kinetic energy of these rolling balls after collision (in post-collision kinetic state) is:
\[
E_k(t_0 + \tau) = \frac{1}{2}(J_{p1}\omega_{p1}^2(t_0 + \tau) + J_{p2}\omega_{p2}^2(t_0 + \tau))
\]

a* In the case of arbitrary coefficient of restitution, \( 0 < k < 1 \), of collision, rate of decreasing kinetic energy in comparison between pre-collision and post-collision kinetic state of the rolling balls is equal:
\[
\Delta E_k = E_k(t_0 + \tau) - E_k(t_0) = \frac{J_{p1}J_{p2}}{2(J_{p1} + J_{p2})}\left(1 - k^2\right)(\omega_{p1}(t_0) - \omega_{p2}(t_0))^2
\]

b* For ideal plastic collision, \( k = 0 \), rate of the kinetic energy decreasing in comparison between pre-collision and post-collision kinetic state of the rolling balls is equal:
\[
\Delta E_{k,\text{plast}} = E_k(t_0 + \tau) - E_k(t_0) = \frac{J_{p1}J_{p2}}{2(J_{p1} + J_{p2})}(\omega_{p1}(t_0) - \omega_{p2}(t_0))^2
\]

c* In the case of ideal elastic collision, \( k = 1 \), between rolling balls no change of kinetic energy in comparison between pre-collision and post-collision kinetic state of rolling balls and is equal to zero:
\[
\Delta E_{k,\text{elast}} = E_k(t_0 + \tau) - E_k(t_0) = 0.
\]
In this case of ideal elastic impact low of kinetic energy conservation is valid.

Listed analytical expressions and relations (18)-(22) and conclusions a*- b* -c* of the kinetic energy decreasing in comparison, kinetic energy of two rolling balls in pre-collision of kinetic state and post-collision kinetic state of the balls in collision present generalized Carnot’s theorem (Lazare Carnot 1753-1824, Principes fondamenteaux de l’équilibre et de movement - 1803) of the kinetic energy of two rolling balls in kinetic states pre- and post collision (in arrival and outgoing kinetic states): “In the collision of two rolling balls in rolling motion for arbitrary coefficient of the restitution, 0 < k < 1, lost of kinetic energy in decreasing during collision, is proportional to lost of angular velocities.

\[ \Delta E_k = E_k(t_0) - E_k(t_0 + \tau) = 2E_{k,acc}(\tau) = \sum_{i=1}^{N} J_{P_i} [\bar{\omega}_{P_i}(\tau)]^2, \] (23)

**Examples:** Masses of the balls are \( m_1 = \rho_1 \frac{4}{8} r_1^3 \pi \) and \( m_2 = \rho_2 \frac{4}{8} r_2^3 \pi \), and axial mass inertia moments for momentary axis of rolling balls are:

\[ J_{P_1} = J_{c_1} + m_1 r_1^2 = \frac{2}{5} m_1 r_1^2 + m_1 r_1^2 = \frac{7}{5} m_1 r_1^2 = \frac{7}{5} \rho_1 \frac{4}{8} r_1^3 \pi = 7 \rho_1 r_1^5 \pi \]
\[ J_{P_2} = J_{c_2} + m_2 r_2^2 = \frac{7}{10} \rho_2 r_2^5 \pi \]

For \( \lambda_1 = \frac{R}{r_1} \) and \( \lambda_2 = \frac{r_2}{r_1} \) ratio of the axial mass inertia moments is:

\[ \frac{J_{P_1}}{J_{P_2}} = \left( \frac{r_2}{r_1} \right)^5 \left( \frac{\rho_1}{\rho_2} \right)^5 = \lambda_2^5 \]

For different ratio between axial mass inertia moments, balls’ outgoing angular velocities around instantaneous axis at post collision kinetic state between balls are:

**a** for \( \lambda_2 = \left( \frac{r_2}{r_1} \right)^5 = \frac{1}{32} = \frac{1}{33} \)
\[ \omega_{P_1}(t_0 + \tau) = \omega_{P_1}(t_0) - \frac{32}{33} (1 + k)(\omega_{P_1}(t_0) - \omega_{P_2}(t_0)) \]
\[ \omega_{P_2}(t_0 + \tau) = \omega_{P_2}(t_0) + \frac{1}{33} (1 + k)(\omega_{P_1}(t_0) - \omega_{P_2}(t_0)) \]

**b** for \( \lambda_2 = \frac{r_2}{r_1} = \frac{1}{81} \cdot 3 = \frac{1}{243} \)
\[ \omega_{P_1}(t_0 + \tau) = \omega_{P_1}(t_0) - \frac{243}{244} (1 + k)(\omega_{P_1}(t_0) - \omega_{P_2}(t_0)) \]
\[ \omega_{p2}(t_0 + \tau) = \omega_{p2}(t_0) + \frac{1}{244} (1 + k) \left( \omega_{p1}(t_0) - \omega_{p2}(t_0) \right) \]

\[ c^* \lambda_2 = \frac{r_2}{r_1} = \frac{2^5}{3} = \frac{32}{81.3} = \frac{32}{244} : \frac{1}{1 + \frac{J_{p2}}{J_{p1}}} = \frac{1}{1 + \frac{243}{32}} = \frac{32}{275} \]

\[ \omega_{p1}(t_0 + \tau) = \omega_{p1}(t_0) - \frac{243}{275} (1 + k) \left( \omega_{p1}(t_0) - \omega_{p2}(t_0) \right) \]

\[ \omega_{p2}(t_0 + \tau) = \omega_{p2}(t_0) + \frac{32}{275} (1 + k) \left( \omega_{p1}(t_0) - \omega_{p2}(t_0) \right) \]

III. Vibro-impact dynamics of multiple collisions of two different rolling heavy balls along circle trace in vertical plane

In the References [28-29] phase trajectory portrait of the vibro-impact forced dynamics of two heavy mass particles motions along rough circle is investigated, and also vibro-impact of a heavy mass particle moving along a rough circle with two impact limiters was considered and studied.

In References [30-35] series of mass particle motion along smooth or rough curvilinear line are studied and result presented. In this part vibro-impact dynamics of multiple collisions of two rolling heavy balls along circle trace in vertical plane is studied and results are presented.

In Figure 6, a model of two heavy homogeneous rolling balls, with radiuses \( r_1 \) and \( r_2 \), along a circle, with radius \( R \), in vertical plane is presented. Let’s start with theory of dynamics of collision between these two rolling balls, with mass \( m_1 \) and \( m_2 \), and axial mass inertia moments \( J_{p1} \) and \( J_{p2} \) for corresponding momentary axis of rotation in rolling along curvilinear trace in the form of circle line in vertical plane, with pre-impact (arrival) angular velocities \( \omega_{p1,\text{impact}} = \omega_{p1}(t_0) \) and \( \omega_{p2,\text{impact}} = \omega_{p2}(t_0) \). Mass centers \( C_1 \) and \( C_2 \) of the balls move transatatory along two circles, with radius \( R - r_1 \) and \( R - r_2 \), respectively, and with pre-impact (arrival) velocities \( \tilde{v}_{C1,\text{impact}} = \tilde{v}_{C1}(t_0) \) and \( \tilde{v}_{C2,\text{impact}} = \tilde{v}_{C2}(t_0) \).

Angular velocities \( \tilde{\omega}_{p1,\text{impact}} = \tilde{\omega}_{p1}(t_0) \) and \( \tilde{\omega}_{p2,\text{impact}} = \tilde{\omega}_{p2}(t_0) \) we denote as arrival, or impact or pre-collision angular velocities at the moment \( t_0 \) (see Figure 7). At this moment \( t_0 \) of the collision start between these rolling balls, contact of these two balls is at point \( T_{12} \), in which both balls posses common tangent plane-plane of contact (touch). In theory of the collision, it is proposed that collision takes very shorth period time \( (t_0, t_0 + \tau) \), and that \( \tau \) tend to zero. After this short period \( \tau \) bodies-two rolling balls in collision separate and outgoing by post-collision-outgoing angular velocities \( \tilde{\omega}_{p1,\text{outgoing}} = \tilde{\omega}_{p1}(t_0 + \tau) \) and \( \tilde{\omega}_{p2,\text{outgoing}} = \tilde{\omega}_{p2}(t_0 + \tau) \).
Mass centers $C_1$ and $C_2$ of the balls move transitory with post-collision (auto-going) translator velocities $\vec{v}_{C1,\text{outgoing}} = \vec{v}_{C1}(t_0 + \tau)$ and $\vec{v}_{C2,\text{outgoing}} = \vec{v}_{C2}(t_0 + \tau)$. These translator velocities is possible to express, each by corresponding angular velocity and radius of the corresponding ball.

Taking into account that translator motion along ideal curvilinear line of two bodies in central collision (as collision of two mass particle moving along curvilinear line) is simpler motion of two mass particle, defined by corresponding inertia properties expressed by mass, $m_1$ and $m_2$, of each body and also by corresponding translator pre-impact velocity, $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ at the moment before collision and by post-impact-outgoing translator velocities $\vec{v}_1(t_0 + \tau)$ and $\vec{v}_2(t_0 + \tau)$ is possible to compare with collision of two rolling balls along curvilinear line. Explanation is similar as in the case that pre- and post-collision traces are straight lines, presented in previous part.

Also, rolling balls along curvilinear circle line-trace is simple rotation motion defined only by inertia properties in the axial mass inertia moments $J_{P1}$ and $J_{P2}$ for corresponding momentary axis of rotation in rolling along curvilinear circle trace with pre-impact (arrival) angular velocities $\vec{\omega}_{P1,\text{impact}} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,\text{impact}} = \vec{\omega}_{P2}(t_0)$ and corresponding outgoing post-impact-outgoing angular velocities $\vec{\omega}_{P1,\text{outgoing}} = \vec{\omega}_{P1}(t_0 + \tau)$ and $\vec{\omega}_{P2,\text{outgoing}} = \vec{\omega}_{P2}(t_0 + \tau)$. But for rolling motion between two collisions must to take that balls are in rolling dynamics under the conservative force caused by gravitation field, in which is balls and circle. But this is only necessary to take into account during balls motion between two collisions, and for to oblation pre-collision angular velocities as angular velocities at end of one previous interval of rolling each of balls in gravitational field.

### III.1. Kinetic parameters of a rolling heavy ball motion along circle in vertical plane

Let us consider rolling dynamics of one heavy smooth ball (first) along curvilinear circle line trace in vertical plane and in gravitational field (rolling pendulum in References [25-27]). For that reason, the kinetic and potential energies are expressed by central angle $\phi_1$ with respect to circle centre $C_0$ (see Figure 6, and Reference [26]):

$$E_{k,1} = \frac{1}{2} m_1 (R - r_1) \dot{\phi}_1^2$$  \hspace{1cm} (24)

$$E_{p,1} = m_1 g h_{c1} = m_1 g (R - r_1)(1 - \cos \phi_1)$$  \hspace{1cm} (25)

where translator velocity of ball mass center $\vec{v}_{C1}$ and angular velocity of central axis $\vec{\omega}_{C1}$ and angular velocity of momentary axis of rolling $\vec{\omega}_{P1}$ is in the following relations:
\[ v_{cl} = (R - r_1) \dot{\phi}_1 = r_1 \omega_{p1} = r_1 \omega_{cl} \]
\[ \omega_{cl} = \omega_{p1} = \left( \frac{R}{r_1} - 1 \right) \dot{\phi}_1 = \frac{(R - r_1)}{r_1} \dot{\phi}_1 \]  
(26)

**Figure 6.** Mechanical system of collision of two rolling heavy balls along circle trace in vertical plane

and the first ball axial mass inertia moment for instantaneous axis of rolling is:

\[ J_{p1} = m_1 \left( \frac{J_{cl}}{m_1} + r_1^2 \right), \]
(27)

and the coefficient of rolling of first rolling ball along circle line in vertical plane is:

\[ \kappa_1 = \frac{J_{cl}}{m_1 r_1^2} + 1 = \frac{I_{cl}^2}{r_1^2} + 1 \]  
(28)

For that reason, it is necessary to obtain corresponding ordinary differential equations of rolling each of balls along curvilinear line in vertical plane in gravitational field.
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The ordinary nonlinear differential equation of first ball rolling curvilinear line in vertical plane in gravitational field is:

\[ \ddot{\phi}_1 + \frac{g}{\kappa_1 (R - r)} \sin \phi_1 = 0 \]  \hspace{1cm} (29)

Integral of the energy of first ball in rolling dynamics along circle trace in gravitational field is:

\[ E_1 = E_{k,1} + E_{p,1} = \frac{1}{2} \kappa_1 m_1 (R - r_1)^2 \dot{\phi}_1^2 + m_1 g (R - r_1) (1 - \cos \phi_1) = C_1 = \text{const} \]  \hspace{1cm} (30)

and present expression of total mechanical energy of the rolling ball at arbitrary moment and arbitrary position on the circle trace. The total mechanical energy of the first rolling ball along circle trace at initial moment is:

\[ E_{1,0} = E_{k,1,0} + E_{p,1,0} = \frac{1}{2} \kappa_1 m_1 (R - r_1)^2 \dot{\phi}_{1,0}^2 + m_1 g (R - r_1) (1 - \cos \phi_{1,0}) = C_1 = \text{const} \]  \hspace{1cm} (31)

where \( \phi_{1,0} = \phi_1(0) \) and \( \dot{\phi}_{1,0} = \dot{\phi}_1(0) \) are initial values of the generalized angular coordinate and generalized angular velocity.

First integral of the ordinary nonlinear differential equation (29) of the rolling dynamics of first ball rolling curvilinear circle line is possible to obtain from integral of energy (30)-(31) in the following form:

\[ \dot{\phi}_1^2 + \frac{2g}{\kappa_1 (R - r)} (1 - \cos \phi_1) = \dot{\phi}_{1,0}^2 + \frac{2g}{\kappa_1 (R - r)} (1 - \cos \phi_{1,0}) \]  \hspace{1cm} (32)

or in the form:

\[ \dot{\phi}_1^2 = \dot{\phi}_{1,0}^2 + \frac{2g}{\kappa_1 (R - r)} (\cos \phi_1 - \cos \phi_{1,0}) \]  \hspace{1cm} (33)

This previous non-linear equation (33) present equation of the phase trajectory in phase plane \((\phi_1, \dot{\phi}_1)\) and is visible that are curves of constant total mechanical energy of the rolling ball between two collision, and that total mechanical energy in this interval is constant, but depend of initial conditions, \(\phi_{1,0} = \phi_1(0)\) and \(\dot{\phi}_{1,0} = \dot{\phi}_1(0)\), in each of the interval between two successive collisions. After each collision of the balls, set of angular velocities of rolling balls are outgoing angular velocities as post-impact angular velocities of rolling balls as the initial velocities for dynamics in next post-collision interval of corresponding period.

For that reason expression of the momentary angular velocity of rolling ball is necessary to express by generalized coordinate \( \phi_1 \) in the following form:

\[ \omega_{\phi_1} = \left( \frac{R}{r_1} - 1 \right) \dot{\phi}_1 = \left( \frac{R}{r_1} - 1 \right) \sqrt{\dot{\phi}_{1,0}^2 + \frac{2g}{\kappa_1 (R - r)} (\cos \phi_1 - \cos \phi_{1,0})} \]  \hspace{1cm} (34)

Angular velocity of the first rolling ball is function of the initial central angular velocity \( \dot{\phi}_1(0) = \dot{\phi}_{0,1} \) in relation of circle centre \( C_0 \) and coordinate \( \phi_{1,\text{impac},1} \) of position at circle where first collision appears:
and for next impact angular velocity of each of the rolling balls depends of the outgoing angular velocity in previous collision of the balls and coordinate \( \varphi_{\text{impact},1} \) of position where next collision appear:

\[
\omega_{\text{p1,impact},2} = \omega_{\text{p1,outgoing},1} = \left( \frac{R}{r_1} - 1 \right) \omega_{\text{p1,impact},1} = \left( \frac{R}{r_1} - 1 \right) \tilde{\varphi}_1 \left( \varphi_{\text{impact},1} \right) = \left( \frac{R}{r_1} - 1 \right) \sqrt{\frac{\varphi_{\text{impact},1}^2 + \frac{2g}{\kappa_1(R-r_1)}(\cos \varphi_{\text{impact},1} - \cos \varphi_{\text{impact},0})}{\kappa_1(R-r_1)}}
\]

(35)

III.2. Using previous expressions for first rolling ball along same circle trace in vertical plane, for second rolling ball, kinetic and potential energies are expressed by central angle \( \varphi_2 \), and are in the form:

\[
E_{k,2} = \frac{1}{2} \kappa_2 m_2 (R-r_2) \tilde{\varphi}_2^2
\]

(37)

\[
E_{p,2} = m_2 g h_{c1} = m_2 g(r_i - r_2)(1 - \cos \varphi_2)
\]

(37*)

where velocity of ball mass center \( \vec{v}_{c2} \) and angular velocity of central axis \( \vec{\omega}_{c2} \) and angular velocity of momentary axis of rolling \( \vec{\omega}_{p2} \) are in the following relations:

\[
\vec{v}_{c2} = (R-r_2) \tilde{\varphi}_2 = r_i \vec{\omega}_{p2} = r_i \omega_{c2}
\]

\[
\omega_{c2} = \omega_{p2} = \left( \frac{R}{r_2} - 1 \right) \tilde{\varphi}_2 = \left( \frac{R}{r_2} - 1 \right) \tilde{\varphi}_2
\]

(38)

and axial second ball mass inertia moment for instantaneous axis of rolling is:

\[
J_{p2} = m_2 \left( \frac{J_{c2}}{m_2} + r_2^2 \right)
\]

(39)

and the coefficient of the rolling of second rolling ball along circle trace in vertical plane is:

\[
\kappa_2 = \frac{J_{c2}}{m_2 r_2^2} + 1 = \frac{I_{c2}}{r_2^2} + 1
\]

(40)

Ordinary non-linear differential equation of second ball rolling is:

\[
\ddot{\varphi}_2 + \frac{g}{\kappa_2(R-r_2)} \sin \varphi_2 = 0
\]

(41)

Integral of energy of second ball rolling along circle in gravitational field is:
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\[ E_2 = E_{k,2} + E_{p,2} = \frac{1}{2} \kappa_2 m_2 (R - r_2)^2 \dot{\phi}_2^2 + m_2 g (R - r_2) (1 - \cos \phi_2) = C_2 = \text{const} \quad (42) \]

and present expression of total mechanical energy of the second rolling ball at arbitrary moment and arbitrary position on the circle trace. The total mechanical energy of the second rolling ball along same circle trace at initial moment is:

\[ E_{2,0} = E_{k,2,0} + E_{p,2,0} = \frac{1}{2} \kappa_2 m_2 (R - r_2)^2 \dot{\phi}_{2,0}^2 + m_2 g (R - r_2) (1 - \cos \phi_{2,0}) = C_2 = \text{const} \quad (43) \]

where \( \phi_{2,0} = \phi_2(0) \) and \( \dot{\phi}_{2,0} = \dot{\phi}_2(0) \) are initial values of the generalized angular coordinate and generalized angular velocity.

First integral of the ordinary nonlinear differential equation (41) of the rolling dynamics of second ball along curvilinear circle line is possible to obtain from integral of energy (42)-(43) in the following form:

\[ \dot{\phi}_2^2 + \frac{2g}{\kappa_2 (R - r_2)} (1 - \cos \phi_2) = \dot{\phi}_{2,0}^2 + \frac{2g}{\kappa_2 (R - r_2)} (1 - \cos \phi_{2,0}) \quad (44) \]

or in the form:

\[ \dot{\phi}_2^2 = \dot{\phi}_{2,0}^2 + \frac{2g}{\kappa_2 (R - r_2)} (\cos \phi_2 - \cos \phi_{2,0}) \quad (44^*) \]

This previous nonlinear equation (44) present equation of the phase trajectory in phase plane \((\phi_2, \dot{\phi}_2)\) and is visible that are curves of constant energy of the rolling second ball between two collisions and that total mechanical energy in this interval is constant, but depend of initial conditions \( \phi_{2,0} = \phi_2(0) \) and \( \dot{\phi}_{2,0} = \dot{\phi}_2(0) \) in each of the interval between two successive collisions. After each collision of the balls, angular velocities of rolling balls are outgoing angular velocities as post-impact angular velocities of rolling balls as a set of initial velocities for dynamics in next post-collision interval of corresponding period.

For that reason expression of the momentary angular velocity of the rolling ball is necessary to express by generalized angle coordinate \( \phi_1 \) in the following form:

\[ \omega_{p,2} = \left( \frac{R}{r_2} - 1 \right) \dot{\phi}_2 = \left( \frac{R}{r_2} - 1 \right) \sqrt{\dot{\phi}_{2,0}^2 + \frac{2g}{\kappa_2 (R - r_2)} (\cos \phi_2 - \cos \phi_{2,0})} \quad (45) \]

Angular velocity of the second ball rolling along instantaneous axis is function of the initial central angular velocity \( \dot{\phi}_2(0) = \dot{\phi}_{2,0} \) with respect to circle center \( C_0 \) and coordinate \( \phi_{2,\text{impact}} \) of position where second collision appear:

\[ \omega_{p,2,\text{impact},1} = \left( \frac{R}{r_2} - 1 \right) \dot{\phi}_2(\phi_{2,\text{impact}}) = \left( \frac{R}{r_2} - 1 \right) \sqrt{\dot{\phi}_{2,0}^2 + \frac{2g}{\kappa_2 (R - r_2)} (\cos \phi_{2,\text{impact}} - \cos \phi_{2,0})} \quad (46) \]
and for next impact angular velocity of the second ball rolling depends of the outgoing angular velocity in previous collision of the second balls and coordinate \( \phi_{2, \text{impact}, 2} \) of position where next (second) collision appear:

\[
\omega_{\text{P2,impact}, 2} = \omega_{\text{P2,outgoing}, 1} = \left( \frac{R}{r_2} - 1 \right) \phi_2 \left( \phi_{2, \text{impact}, 2} \right)
\]

\[
\omega_{\text{P2,impact}, 2} = \omega_{\text{P2,outgoing}, 1} = \left( \frac{R}{r_2} - 1 \right) \phi_2 \left( \phi_{2, \text{impact}, 2} \right) + \frac{2g}{\kappa_2 (R - r_2)} \left( \cos \phi_{2, \text{impact}, 2} - \cos \phi_{2, \text{outgoing}, 1} \right)
\]

Central angle coordinates of positions of the balls in state of the collisions are in the following relation: \( \phi_{2, \text{impact}, k} = \phi_{1, \text{impact}, k} + \beta \), where angle \( \beta \) depends on geometrical parameters of circle line radius \( R \), and of the both balls radiuses: \( r_1 \) and \( r_2 \) and is defined by expression in the form:

\[
\beta = \arccos \left( \frac{(R - r_1)^2 + (R - r_2)^2 - (r_1 + r_2)^2}{2(R - r_1)(R - r_2)} \right) = \arccos \left( \frac{\lambda_1 (\lambda_2 - 1) - 1}{\lambda_1 - 1} \right)
\]

where \( \lambda_1 = \frac{R}{r_1} \) and \( \lambda_2 = \frac{r_2}{r_1} \).

III.3. Non-linear vibro-impact dynamics and phase trajectories with successive central collisions of two rolling heavy smooth balls along circle trace in vertical plane

Let us to consider vibro-impact dynamics of two rolling heavy balls along circle in vertical plane. Using obtained, in previous part of the paper, the ordinary differential equations and the equations of the phase trajectory of two separate rolling balls along circle line, we can consider as dynamics of two rolling balls in vibro-impact dynamics along circle trace in vertical plane, taking into account that these equations are valid for the balls dynamics between two successive collisions of balls.

III.3.1. Solution of governing nonlinear differential equations with respect to time duration of the rolling balls at circle line

For each interval of the non-linear dynamics of balls, between two collisions, for initial conditions must to take into account position of the corresponding impact and post-collision outgoing angular velocity of corresponding ball. Measure of the time, we take from zero at each next interval between two successive collisions. Also, it is necessary to obtain time for each next collision in relation to initial moment of motion, or from starting interval of motion post-previous-collision.
For that reason, let introduce following denotations:

\[
\omega_1 = \frac{2g}{\kappa_1 (R-r_1)}, \quad \omega_2 = \frac{2g}{\kappa_2 (R-r_2)},
\]

\[
k_1^2 = \frac{2\omega_1^2}{\phi_{1,0}^2 + 2\omega_1^2 \sin^2 \phi_{1,0}} = \frac{4g}{\kappa_1 (R-r_1)} \frac{1}{2}
\]

\[
k_2^2 = \frac{2\omega_2^2}{\phi_{2,0}^2 + 2\omega_2^2 \sin^2 \phi_{2,0}} = \frac{4g}{\kappa_2 (R-r_2)} \frac{1}{2}
\]

then equation (44) of the phase trajectory of each of the rolling heavy balls along circle line in vertical plane is possible to write in following forms:

\[
\dot{\phi}_1 = \sqrt{\phi_{1,0}^2 + \omega_1^2 \left( \cos \phi_1 - \cos \phi_{1,0} \right)} \quad \text{and} \quad \dot{\phi}_2 = \sqrt{\phi_{2,0}^2 + \omega_2^2 \left( \cos \phi_2 - \cos \phi_{2,0} \right)}
\]

Previous first differential equation from (52) is possible to solve with respect to time \( t \), and for that reason must to introduce trigonometric relation: \( \cos \phi_1 = 1 - 2 \sin^2 \frac{\phi_1}{2} \) and after transformation, time \( t \) of duration of rolling a ball along circle trace between two ball positions \( \phi_{1,0} = \phi_1(0) \) and \( \phi_1 = \phi_1(t) \), on the circle line, is expresses by an integral in the form:

\[
t = \int_{\phi_{1,0}}^{\phi_1} \frac{d\phi_1}{\sqrt{\phi_{1,0}^2 + \omega_1^2 \left( \cos \phi_1 - \cos \phi_{1,0} \right)}} = \int_{\phi_{1,0}}^{\phi_1} \frac{d\phi_1}{\sqrt{\phi_{1,0}^2 + 2\omega_1^2 \sin^2 \phi_{1,0}} - \frac{1}{2} - 2\omega_1^2 \sin^2 \phi_1}
\]

or in the form:

\[
t = \frac{2}{\omega_{1,0}} \int_{\phi_0}^{\phi_1} \frac{d\phi_1}{\sqrt{1 - k_1^2 \sin^2 \phi_1}}
\]
Next transformation of the previous expression is by introducing relations: \( u = \sin \theta = \sin \frac{\phi_1}{2} \) and \( u_{1,0} = \sin \left( \frac{1}{2} \phi_{1,0} \right) \), that previous integral for obtaining time \( t \) of duration of rolling a ball along circle between two ball positions \( \phi_{1,0} = \phi_1(0) \) and \( \phi_1 = \phi_1(t) \), on the circle line, turn the following form:

\[
t = \frac{2}{\omega_{0,1}} \frac{\sin \frac{\phi_1}{2}}{\sin \frac{\phi_{1,0}}{2}} \int \frac{du}{\sqrt{(1-u^2)(1-k_1^2u^2)}}
\]  

(54)

Obtained integral in expression (54) for time \( t \) of duration of rolling a ball along circle between two ball positions \( \phi_{1,0} = \phi_1(0) \) and \( \phi_1 = \phi_1(t) \) on the circle line, is normal elliptic integral, known as Legendre elliptic integral first kind (see Reference by Rašković [15] and Mitrinović, Djoković [36]).

Using development of terms of functions in previous integral (54) in series:

\[
(1-k_1^2u^2)^{\frac{1}{2}} = 1 + \frac{1}{2} k_1^2u^2 + \frac{1.3}{2.4} k_1^4u^4 + \frac{1.3.5}{2.4.6} k_1^6u^6 + \ldots = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n k_1^{2n}u^{2n}
\]  

(55)
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\[
(1-u^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}u^2 + \frac{1}{2!} \frac{3}{2 \cdot 4} u^4 + \frac{1}{3!} \frac{3 \cdot 5}{2 \cdot 4 \cdot 6} u^6 + \ldots = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n u^{2n}
\]

(56)

where:

\[
(-1)^n \left( -\frac{1}{2} \right)^n = \frac{(2n-1)!}{(2n)!}
\]

(57)

is not difficult to obtain approximate values of the integral (54) in the following form:

\[
t \approx \frac{2}{\omega_{0,1}} \left[ u + \frac{1}{2} k^2 u^3 + \frac{1}{3!} \frac{3}{2 \cdot 4 \cdot 6} k^4 u^5 \right]^{\sin \frac{\theta_1}{2}}_{\omega_{0,1} \sin \frac{\theta_1}{2}} + \frac{2}{\omega_{0,1}} \left[ \frac{1}{2} u^3 + \frac{1}{4 \cdot 5} k^2 u^4 + \frac{1}{3 \cdot 5} \frac{3}{2 \cdot 4 \cdot 7} k^4 u^6 \right]^{\sin \frac{\theta_1}{2}}_{\omega_{0,1} \sin \frac{\theta_1}{2}} +
\]

\[
+ \frac{2}{\omega_{0,1}} \left[ \frac{1}{3 \cdot 5} u^5 + \frac{1}{2 \cdot 4 \cdot 7} k^2 u^6 + \frac{1}{9 \cdot 2 \cdot 4} \left( \frac{1}{2 \cdot 4} k^4 u^7 \right) \right]^{\sin \frac{\theta_1}{2}}_{\omega_{0,1} \sin \frac{\theta_1}{2}}
\]

(58)

Previous obtained expression is approximate value and present the time \( t \) duration of the first ball rolling along circle line between two ball positions \( \varphi_{0,1} = \varphi_1(0) \) and \( \varphi_1 = \varphi_1(t) \) on the circle line, from initial position \( \varphi_{0,1} \) of ball to arbitrary position \( \varphi_1 \) on the curvilinear circle trace, where \( \varphi_{0,1} \) is coordinate angle at initial position of the first ball initial at moment.

In the analogy with previous obtained approximate value (58) of the time \( t \) duration of the first ball rolling from initial position to arbitrary position \( \varphi_1 \) of the on the curvilinear circle line, for expression of approximate value of the time \( t \) duration of the second ball rolling from initial position \( \varphi_{0,2} = \varphi_2(0) \) to arbitrary position \( \varphi_2(t) \) on the curvilinear circle line, we obtain:

\[
t \approx \frac{2}{\omega_{0,1}} \left[ u + \frac{1}{2 \cdot 3} k^2 u^3 + \frac{1}{3 \cdot 5} \frac{3}{2 \cdot 4 \cdot 6} k^4 u^5 \right]^{\sin \frac{\theta_2}{2}}_{\omega_{0,1} \sin \frac{\theta_2}{2}} + \frac{2}{\omega_{0,1}} \left[ \frac{1}{2 \cdot 3} u^3 + \frac{1}{4 \cdot 5} k^2 u^4 + \frac{1}{3 \cdot 5} \frac{3}{2 \cdot 4 \cdot 7} k^4 u^6 \right]^{\sin \frac{\theta_2}{2}}_{\omega_{0,1} \sin \frac{\theta_2}{2}} +
\]

\[
+ \frac{2}{\omega_{0,1}} \left[ \frac{1}{3 \cdot 5} u^5 + \frac{1}{2 \cdot 4 \cdot 7} k^2 u^6 + \frac{1}{9 \cdot 2 \cdot 4} \left( \frac{1}{2 \cdot 4} k^4 u^7 \right) \right]^{\sin \frac{\theta_2}{2}}_{\omega_{0,1} \sin \frac{\theta_2}{2}}
\]

(59)

III.3.2. System non-linear dynamics in interval from initial position to first collision of balls

For obtaining the coordinates of ball’s positions at configuration of first collision between rolling heavy balls at circle line in vertical plane, it is necessary to obtain time \( t_{\text{impact},1} \) of first collision at which both balls are in the configuration of first collision. Propose that mass center \( C_{1,\text{impact},1} \) of first ball is in position defined by angle coordinate \( \varphi_{1,\text{impact},1} = \varphi_{\text{impact},1} \), then coordinate of mass center...
$C_{2,\text{impact},1}$ of second ball is defined by angle coordinate:  
$\phi_2(t_{\text{impact},1}) = \phi_1(t_{\text{impact},1}) + \beta$, where angle $\beta$ is defined by expression (48) and (49). Using approximate expressions (58) and (59) for time $t_{\text{impact},1}$, the duration of balls motion from corresponding initial positions, $\phi_{1,0}$ and $\phi_{2,0}$, to the position, $\phi_{1,\text{impact},1}$ and $\phi_{2,\text{impact},1}$, the first collision between rolling balls we can write the following:

$$t_{\text{impact},1} = t_{1,\text{impact},1} \approx \frac{2}{\omega_{0,1}} \left( u + \frac{1}{2 \cdot 3} k_1^2 u^3 + \frac{1}{2 \cdot 4 \cdot 6} k_1^4 u^5 \right) \frac{\sin \phi_{1,\text{impact},1}}{\sin \phi_{0,2}} +$$

$$+ \frac{2}{\omega_{0,1}} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{4 \cdot 5} k_2^2 u^5 + \frac{1}{2 \cdot 4 \cdot 7} k_2^4 u^7 \right) \frac{\sin \phi_{1,\text{impact},1}}{\sin \phi_{0,2}} +$$

$$+ \frac{2}{\omega_{0,1}} \left( \frac{1}{2 \cdot 4 \cdot 5} u^5 + \frac{1}{2 \cdot 4 \cdot 7} k_2^2 u^7 + \left( \frac{1}{2 \cdot 4} \right)^2 k_2^4 u^9 \right) \frac{\sin \phi_{1,\text{impact},1} + \beta}{\sin \phi_{0,2}}$$

$$+ \frac{2}{\omega_{0,2}} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{4 \cdot 5} k_2^2 u^5 + \frac{1}{2 \cdot 4 \cdot 7} k_2^4 u^7 \right) \frac{\sin \phi_{2,\text{impact},1} + \beta}{\sin \phi_{0,2}} +$$

$$+ \frac{2}{\omega_{0,2}} \left( \frac{1}{2 \cdot 3} k_2^2 u^3 + \frac{1}{2 \cdot 4 \cdot 7} k_2^4 u^7 \right) \frac{\sin \phi_{2,\text{impact},1} + \beta}{\sin \phi_{0,2}}$$

Taking into account, that both balls starting from initial positions, $\phi_{1,0}$ and $\phi_{2,0}$, must to arrive at configuration of the first position of first collision, defined by coordinates $\phi_{1,\text{impact},1}$ and $\phi_{2,\text{impact},1}$, let know that expressions (60) and (61) are equal, first to other, and as result is a nonlinear transcendent equation with respect to unknown angle coordinate $\phi_1(t_{\text{impact},1}) = \phi_{1,\text{impact},1}$ of the mass center position of first ball at position of first collision between balls. This task, to find first real root of this transcendent equation, is not possible to solve analytically and it is necessary to use some of numerical methods as well as some commercial software tool. In this paper we deal with ideas and analytical approach to the defined task of vibro-impact dynamics. We propose that we have first real root of this transcendent equation obtained numerically.
Then, we suppose that we have angle coordinate \( \varphi_1(t_{\text{impact},1}) = \varphi_{1,\text{impact},1} \) of mass center \( C_{1,\text{impact},1} \) of first ball at position of the first collision between balls, and also angle coordinate \( \varphi_2(t_{\text{impact},1}) = \varphi_{1,\text{impact},1} + \beta \) of mass center \( C_{2,\text{impact},1} \) of the second ball at position of the first collision between balls, then is possible to compose pre-first collision impact angular velocities \( \omega_{P1,\text{impact},1} \) and \( \omega_{P2,\text{impact},1} \) of the heavy rolling balls using expressions (34) and (45), in the following forms:

\[
\omega_{P1,\text{impact},1} = (\lambda_1 - 1) \varphi_1(t_{\text{impact},1}) = (\lambda_1 - 1) \sqrt{\phi_{1,0}^2 + \omega_1^2 (\cos \varphi_{1,\text{impact},1} - \cos \varphi_{1,0})} 
\]

(62)

\[
\omega_{P2,\text{impact},1} = (\lambda_2 - 1) \varphi_2(t_{\text{impact},1}) = (\lambda_2 - 1) \sqrt{\phi_{2,0}^2 + \omega_2^2 (\cos \varphi_{1,\text{impact},1} + \beta) - \cos \varphi_{2,0}}
\]

(63)

For obtaining post-first-collision outgoing angular velocities \( \omega_{P1}(t_{u,1} + \tau) = \omega_{P1,\text{outgoing},1} \) and \( \omega_{P2}(t_0 + \tau) = \omega_{P2,\text{outgoing},1} \) of the rolling balls along circle line, at same position of the first collision balls, determined by generalized coordinates \( \varphi_{1,\text{impact},1} \) and \( \varphi_{2,\text{impact},1} \), we use expressions (15)-(16) and we obtain the following expressions:

\[
\omega_{P1}(t_{u,1} + \tau) = \omega_{P1,\text{outgoing},1} = \omega_{P1,\text{impact},1}(t_{u,1}) - \frac{1 + k}{1 + \frac{J_{p1}}{J_{p2}}} \left( \omega_{P1,\text{impact},1}(t_{u,1}) - \omega_{P2,\text{impact},1}(t_{u,1}) \right)
\]

(64)

\[
\omega_{P2}(t_0 + \tau) = \omega_{P2,\text{outgoing},1} = \omega_{P2,\text{impact},1}(t_{u,1}) + \frac{1 + k}{1 + \frac{J_{p2}}{J_{p1}}} \left( \omega_{P1,\text{impact},1}(t_{u,1}) - \omega_{P2,\text{impact},1}(t_{u,1}) \right)
\]

(65)

or in developed forms:

\[
\omega_{P1}(t_0 + \tau) = \omega_{P1,\text{outgoing},1} = (\lambda_1 - 1) \sqrt{\phi_{1,0}^2 + \omega_1^2 (\cos \varphi_{1,\text{impact},1} - \cos \varphi_{1,0})} -
- \frac{1 + k}{1 + \frac{J_{p1}}{J_{p2}}} \left( (\lambda_1 - 1) \sqrt{\phi_{1,0}^2 + \omega_1^2 (\cos \varphi_{1,\text{impact},1} - \cos \varphi_{1,0})} -
- (\lambda_2 - 1) \sqrt{\phi_{2,0}^2 + \omega_2^2 (\cos \varphi_{1,\text{impact},1} + \beta) - \cos \varphi_{2,0}} \right)
\]

(66)

\[
\omega_{P2}(t_0 + \tau) = \omega_{P2,\text{outgoing},1} = (\lambda_2 - 1) \sqrt{\phi_{2,0}^2 + \omega_2^2 (\cos \varphi_{1,\text{impact},1} + \beta) - \cos \varphi_{2,0}} +
+ \frac{1 + k}{1 + \frac{J_{p2}}{J_{p1}}} \left( (\lambda_2 - 1) \sqrt{\phi_{2,0}^2 + \omega_2^2 (\cos \varphi_{1,\text{impact},1} + \beta) - \cos \varphi_{2,0}} -
- (\lambda_2 - 1) \sqrt{\phi_{2,0}^2 + \omega_2^2 (\cos \varphi_{1,\text{impact},1} + \beta) - \cos \varphi_{2,0}} \right)
\]

(67)
For obtaining kinetic parameters of the rolling balls in the form of pre-collision and post-collision angular velocities and angle coordinate of the ball position at series of the collisions between balls we must to use the approach similar as presented in this part III.3.2.

In the case that we deal with numerical data a discussion is possible about directions of outgoing angular velocities of corresponding rolling ball, depending of relation between intensities and directions of the arrival pre-collision angular velocities and position of the collision when balls. It is possible different cases, that after the considered collision balls departures are in opposite directions or in same direction depending of listed kinetic parameters. But this is task with numerical analysis.

### III.3.3. System non-linear dynamics in interval from position of first collision to second collision of the balls

Next period of motion of the rolling balls, between first and second collisions of balls, is starting with measures of time interval with zero, and initial conditions are equal to outgoing kinetic parameters at post-first-collision state of rolling balls:

* for first rolling ball initial coordinate are \( \varphi_{\text{impact,1}} = \varphi_{\text{outgoing,1}} \) and initial angular velocity of the first ball mass center with respect to circle trace centre is

\[
\dot{\varphi}_1(t_{\text{impact,1}}) = \dot{\varphi}_{\text{outgoing,1}} = \frac{\omega_{\text{outgoing,1}}}{(\lambda_1 - 1)}
\]

and equation of the phase trajectory branch of the first ball dynamics in interval between first and second collision is:

\[
\dot{\varphi}_1 = \sqrt{\dot{\varphi}_{\text{outgoing,1}}^2 + \omega_1^2 \left( \cos \varphi_1 - \cos \varphi_{\text{impact,1}} \right)}
\]  \( (68) \)

and

* for second rolling ball initial coordinate is

\( \varphi_{2,\text{impact,1}} = \varphi_{2,\text{impact,2}} = \varphi(t_{\text{impact,1}}) + \beta = \varphi_{1,\text{impact,1}} + \beta \) and initial angular velocity is

\[
\dot{\varphi}_2(t_{\text{impact,1}}) = \dot{\varphi}_{2,\text{outgoing,1}} = \frac{\omega_{\text{outgoing,1}}}{(\lambda_2 - 1)}
\]  of the second ball mass center with respect to circle trace centre, and equation of phase trajectory branch of the second ball dynamics in interval between first and second collision is:

\[
\dot{\varphi}_2 = \sqrt{\dot{\varphi}_{2,\text{outgoing,1}}^2 + \omega_2^2 \left[ \cos \varphi_2 - \cos (\varphi_{1,u,1} + \beta) \right]}
\]  \( (69) \)

For obtaining time \( t_{\text{impact,2}} = t_{1,\text{impact,2}} = t_{2,\text{impact,2}} \) of the second collision between rolling balls, time duration between first and second collision between balls, it is necessary to use previous approach, and of this basis write the following integrals:
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\[ t_{\text{impact},2} = \int_{\phi_{\text{impact},1}}^{\phi_{\text{impact},2}} \frac{d\phi_1}{\sqrt{\phi_{\text{outgoing},1}^2 + \phi_1^2 (\cos \phi_1 - \cos \phi_{\text{impact},1})}} \] (71)

\[ t_{\text{impact},2} = \int_{\phi_{\text{impact},1} + \beta}^{\phi_{\text{impact},2} + \beta} \frac{d\phi_2}{\sqrt{\phi_{\text{outgoing},1}^2 + \phi_2^2 (\cos \phi_2 - \cos \phi_{\text{impact},1} + \beta)}} \] (72)

On the basis of previous explanation an analogy with approximate expressions (60) and (61), for time \( t_{\text{impact},2} = t_{\text{impact},1} = t_{\text{impact},2} \) of duration of the intervals between first and second collisions of the rolling balls dynamics, it is possible to write:

* \( t_{\text{impact},2} \) for fist ball

\begin{align*}
\frac{t_{\text{impact},2}}{t_{\text{impact},1}} & \approx \frac{2}{\omega_{0,1}} \left( u + \frac{1}{2 \cdot 3} k_2^4 u^4 + \frac{1}{2 \cdot 4} k_4^2 u^6 \right) \frac{\sin \theta_{\text{impact},2}}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2}}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{3}{2 \cdot 3} u^5 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2}}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2}}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2}}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
\end{align*}

(73)

* \( t_{\text{impact},2} \) for send ball

\begin{align*}
\frac{t_{\text{impact},2}}{t_{\text{impact},1}} & \approx \frac{2}{\omega_{0,2}} \left( u + \frac{1}{2 \cdot 3} k_2^4 u^4 + \frac{1}{2 \cdot 4} k_4^2 u^6 \right) \frac{\sin \theta_{\text{impact},2} + \beta}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2} + \beta}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{1}{2 \cdot 3} u^5 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2} + \beta}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2} + \beta}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
+ \frac{2}{\omega_0} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4} k_2^4 u^7 + \frac{1}{2 \cdot 3} u^5 \right) \frac{\sin \theta_{\text{impact},2} + \beta}{2} \sin \frac{\theta_{\text{impact},1}}{2} + \\
\end{align*}

(74)

Taking into account that both balls, starting from position of the fist collision, now initial position of both rolling balls along circle trace, in interval between first and second collisions, must to arrive at configuration of position of second collision for equal time \( t_{\text{impact},2} = t_{\text{impact},1} = t_{\text{impact},2} \), let know that expressions (73) and (74) are equal first to other, and as result is a non-linear transcendental equation with respect to unknown angle coordinate \( \phi(t_{\text{impact},2}) = \phi_{\text{impact},2} \) of mass center \( C_{\text{impact},2} \) of first ball at position of second colli-
sion between balls. This task is to find first real root of this transcendent equation, and this task is not solvable analytically and it is necessary to use some of numerical methods as well as some commercial software tool. In this paper we deal with ideas and analytical approach to the defined task of vibro-impact dynamics. We propose that we have first real root of this transcendent equation obtained numerically.

Suppose that we have angle coordinate \( \varphi_{t_{\text{impact},2}} \) of the mass center \( C_{t_{\text{impact},2}} \) of first ball at position of second collision between balls, and also angle coordinate \( \varphi_{t_{\text{impact},1}} \) of the second ball at position of second collision between balls, then is possible to compose pre-second-collision impact angular velocities \( \omega_{t_{\text{impact},2}} \) and \( \omega_{t_{\text{impact},2}} \) of the heavy rolling of balls around corresponding instantaneous axis, using expression (34) and (45), in the following forms:

\[
\omega_{t_{\text{impact},2}} = (\lambda_1 - 1) \dot{\varphi}_{t_{\text{impact},2}} \quad \text{(75)}
\]

\[
\omega_{t_{\text{impact},2}} = (\lambda_2 - 1) \dot{\varphi}_{t_{\text{impact},2}} \quad \text{(76)}
\]

For obtaining post-second-collision outgoing angular velocities \( \omega_{t_{\text{impact},2} + \tau} = \omega_{t_{\text{outgoing},2}} \) and \( \omega_{t_{\text{impact},2} + \tau} = \omega_{t_{\text{outgoing},2}} \) of the rolling balls along circle line we use expressions (15)-(16) and we obtain the following expressions:

\[
\omega_{t_{\text{impact},2} + \tau} = \omega_{t_{\text{outgoing},2}} = \omega_{t_{\text{impact},2}}(t_{u,2}) - \frac{1+k}{1+J_{P_{t_{\text{impact},2}}}} \left( \omega_{t_{\text{impact},2}}(t_{u,2}) - \omega_{t_{\text{impact},2}}(t_{u,2}) \right) \quad \text{(77)}
\]

\[
\omega_{t_{\text{impact},2} + \tau} = \omega_{t_{\text{outgoing},2}} = \omega_{t_{\text{impact},2}}(t_{u,2}) + \frac{1+k}{1+J_{P_{t_{\text{impact},2}}}} \left( \omega_{t_{\text{impact},2}}(t_{u,2}) - \omega_{t_{\text{impact},2}}(t_{u,2}) \right) \quad \text{(78)}
\]

III.3.4. System non-linear dynamics in interval from position of \( n \)-th to \( n+1 \)-th collision of balls.

On the previous consideration of present series of successive collisions between balls, it is possible to make a generalization of the expressions for kinetic parameters between two succeed collisions of the balls.
Next period of the motion of two rolling balls, after $n$-th collision, $n \geq 2$, between balls, with measures of time interval $t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)}$ starting with zero, and initial conditions equal to outgoing kinetic parameters at position of $n$-th collision state of the rolling balls:

* for first rolling ball initial coordinate is $\phi_{1,\text{impact},n} = \phi_{1,\text{outgoing},n}$, $n \geq 2$ and initial angular velocity is

$$\dot{\phi}_1(t_{\text{impact},n}) = \dot{\phi}_{1,\text{outgoing},n} = \frac{\omega_{1,\text{outgoing},n}}{(\lambda_1 - 1)}$$

and the equation of phase trajectory branch of the first ball dynamics between $n$-th and $n+1$-th collisions of the rolling balls is:

$$\phi_1 = \sqrt{\phi_{1,\text{outgoing},n}^2 + \omega_1^2 (\cos \phi_1 - \cos \phi_{1,\text{impact},n})}, \text{ between } n\text{-th and } n+1\text{-th collisions of rolling balls:}$$

$$n \geq 2 \quad (79)$$

and

* for second rolling ball, the initial coordinate is $\phi_{2,\text{impact},n} = \phi_{1,\text{impact},n} + \beta$ and initial angular velocity between $n$-th and $n+1$-th collisions of the rolling balls is

$$\dot{\phi}_2(t_{\text{impact},n}) = \dot{\phi}_{2,\text{outgoing},n} = \frac{\omega_{2,\text{outgoing},n}}{(\lambda_2 - 1)}$$

and equation of the phase trajectory branch of the second ball dynamics after $n$-th collision and in interval between $n$-th and $n+1$-th collisions of the second rolling balls is:

$$\phi_2 = \sqrt{\phi_{2,\text{outgoing},n}^2 + \omega_2^2 (\cos \phi_2 - \cos (\phi_{1,\text{impact},n} + \beta))}, \text{ between } n\text{-th and } n+1\text{-th collisions of rolling balls:}$$

$$n \geq 2 \quad (80)$$

For obtaining time $t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)}$ of the $(n+1)$-th collisions between rolling balls is necessary to use previous approach, and of this basis, write the following integrals:

$$t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)} = \int_{\phi_{1,\text{outgoing},n}}^{\phi_{1,\text{outgoing},n}} d\phi_1 \frac{d\phi_1}{\sqrt{\phi_{1,\text{outgoing},n}^2 + \omega_1^2 (\cos \phi_1 - \cos \phi_{1,\text{impact},n})}} \quad (81)$$

$$t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)} = \int_{\phi_{1,\text{outgoing},n} + \beta}^{\phi_{1,\text{outgoing},n} + \beta} d\phi_2 \frac{d\phi_2}{\sqrt{\phi_{2,\text{outgoing},n}^2 + \omega_2^2 (\cos \phi_2 - \cos (\phi_{1,\text{impact},n} + \beta))}} \quad (82)$$

On the basis of previous explanation in part III.3.2, an analogy with approximate expressions (60) and (61), for interval $t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)} = t_{\text{impact},(n+1)}$ between $n$-th and $n+1$-th collisions of the rolling balls is possible to write:
\[ t_{\text{impact}, (n+1)} = t_{1, \text{impact}, (n+1)} \approx \frac{2}{\omega_{b, 1}} \left( u + \frac{1}{2 \cdot 3} k^2 u^3 + \frac{1}{2 \cdot 4 \cdot 6} k^4 u^5 \right) \frac{\sin \phi_{1, \text{impact}(n+1)}}{\sin \phi_{1, \text{impact}(n)}} + \]

\[ + \frac{2}{\omega_{b, 1}} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4 \cdot 7} k^2 u^5 + \frac{1}{2 \cdot 4 \cdot 7} k^4 u^7 \right) \frac{\sin \phi_{1, \text{impact}(n+1)}}{\sin \phi_{1, \text{impact}(n)}} + \]  

\[ (83) \]

\[ t_{\text{impact}, (n+1)} = t_{2, \text{impact}, (n+1)} \approx \frac{2}{\omega_{b, 2}} \left( u + \frac{1}{2 \cdot 3} k^2 u^3 + \frac{1}{2 \cdot 4 \cdot 6} k^4 u^5 \right) \frac{\sin \phi_{2, \text{impact}(n+1)}}{\sin \phi_{2, \text{impact}(n)}} + \]

\[ + \frac{2}{\omega_{b, 2}} \left( \frac{1}{2 \cdot 3} u^3 + \frac{1}{2 \cdot 4 \cdot 7} k^2 u^5 + \frac{1}{2 \cdot 4 \cdot 7} k^4 u^7 \right) \frac{\sin \phi_{2, \text{impact}(n+1)}}{\sin \phi_{2, \text{impact}(n)}} + \]  

\[ (84) \]

Taking into account that both balls, starting from position of \( n \)-th, \( n \geq 2 \) collision, now initial position of the rolling balls along circle, in interval between \( n \)-th and \((n + 1)\)-th, \( n \geq 2 \) collisions, must to arrive at configuration of the position of \((n + 1)\)-th \( n \geq 2 \) collision, let know that expressions (83) and (84) are equal, first to other, and as result is a non-linear transcendent equation with respect to unknown angle coordinate \( \phi \left( t_{\text{impact}, (n+1)} \right) = \phi_{1, \text{impact}(n+1)} \) of the mass center \( C_{1, \text{impact}(n+1)} \) of first ball at position of \( n+1 \)-th, \( n \geq 2 \) collision between balls. This task is to find first real root of this transcendent equation, and is not analytically solvable and it is necessary to use some of numerical methods as well as some commercial software tool, as we explain in previous parts.

Suppose that we have necessary angle coordinate \( \phi \left( t_{\text{impact}, (n+1)} \right) = \phi_{1, \text{impact}(n+1)} \) of mass center \( C_{1, \text{impact}(n+1)} \) of the first ball at position of \( n+1 \)-th, \( n \geq 2 \) collision between balls, and also angle coordinate \( \phi_2 \left( t_{\text{impact}, (n+1)} \right) = \phi_1 \left( t_{\text{impact}, (n+1)} \right) + \beta \) of the mass center \( C_{2, \text{impact}, (n+1)} \) of second ball at position of \( n+1 \) collision between balls, then it is possible to compose pre-(\( n+1 \))-th-collision impact angular velocities \( \omega_{P1, \text{impact}, (n+1)} \) and \( \omega_{P2, \text{impact}, (n+1)} \) of the rolling of heavy balls, using expressions (34) and (45), in the following forms:
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\[ \omega_{p1,\text{impact},(n+1)} = (\lambda_1 - 1) \dot{\phi}_{1,\text{impact},(n+1)} \]

\[ \omega_{p1,\text{impact},(n+1)} = (\lambda_1 - 1) \sqrt{\dot{\phi}_{1,\text{impact},n}^2 + \omega_1^2 (\cos \phi_{1,\text{impact},(n+1)} - \cos \phi_{1,\text{impact},n})} \] (85)

\[ \omega_{p2,\text{impact},(n+1)} = (\lambda_2 - 1) \dot{\phi}_{2,\text{impact},(n+1)} \]

\[ \omega_{p2,\text{impact},(n+1)} = (\lambda_2 - 1) \sqrt{\dot{\phi}_{2,\text{impact},n}^2 + \omega_2^2 (\cos \phi_{1,\text{impact},(n+1)} + \beta - \cos \phi_{1,\text{impact},n} + \beta)} \] (86)

For obtaining post-(n+1)-th, n \geq 2-central collision outgoing angular velocities \( \omega_{p1}(t_{\text{impact},(n+1)} + \tau) = \omega_{p1,\text{outgoing},(n+1)} \) and \( \omega_{p2}(t_{\text{impact},(n+1)} + \tau) = \omega_{p2,\text{outgoing},(n+1)} \) around corresponding instantaneous axis of each of the rolling balls along circle line, we use expressions (15)-(16) and we obtain the following expressions:

\[ \omega_{p1}(t_{u,(n+1)} + \tau) = \omega_{p1,\text{outgoing},(n+1)} \]

\[ \omega_{p1}(t_{u,(n+1)} + \tau) = \omega_{p1,\text{impact},(n+1)}(t_{u,(n+1)}) - \frac{1 + k}{1 + \frac{\omega_{p1,\text{impact},(n+1)}(t_{u,(n+1)}) - \omega_{p2,\text{impact},(n+1)}(t_{u,(n+1)})}{\int_{p2}^{1}} \] (87)

\[ \omega_{p2}(t_{u,(n+1)} + \tau) = \omega_{p2,\text{outgoing},(n+1)} \]

\[ \omega_{p2}(t_{u,(n+1)} + \tau) = \omega_{p2,\text{impact},2}(t_{u,(n+1)}) + \frac{1 + k}{1 + \frac{1}{\int_{p1}^{2}}} \left[ \omega_{p1,\text{impact},(n+1)}(t_{u,(n+1)}) - \omega_{p2,\text{impact},(n+1)}(t_{u,(n+1)}) \right] \] (88)

Next period motion of the rolling balls, after (n+1)-th collision, n \geq 2, between balls, with measures of time \( t_{\text{impact},(n+2)} \) interval starting with zero, and initial conditions equal to outgoing kinetic parameters at post-(n+1)-th -collision state of the rolling balls:

* for first rolling ball, the initial coordinate is \( \phi_{1,\text{impact},(n+1)} \), n \geq 2 and initial angular velocity around circle line center \( C_0 \), is

\[ \dot{\phi}_{1}(t_{\text{impact},(n+1)}) = \dot{\phi}_{1,\text{outgoing},(n+1)} = \frac{\omega_{p1,\text{outgoing},(n+1)}}{\lambda_1 - 1} \]

and equation of phase trajectory branch of the first ball dynamics between \( n+1 \)-th and \( n+2 \)-th, \( n \geq 2 \) is:

\[ \dot{\phi}_1 = \sqrt{\dot{\phi}_{1,\text{outgoing},(n+1)}^2 + \omega_1^2 (\cos \phi_1 - \cos \phi_{1,\text{impact},(n+1)})}, \quad n+1 \text{-th and } n+2 \text{-th,} \]

\[ n \geq 2 \] (89)

and

* for second rolling ball, the initial coordinate is \( \phi_{2,\text{impact},(n+1)} = \phi_{1,\text{impact},(n+1)} + \beta \) and initial angular velocity, between \( n+1 \)-th and \( n+2 \)-th, \( n \geq 2 \) central collisions, is

\[ \dot{\phi}_2(t_{\text{impact},(n+1)}) = \dot{\phi}_{2,\text{outgoing},(n+1)} = \frac{\omega_{p2,\text{outgoing},(n+1)}}{\lambda_2 - 1} \],
and equation of phase trajectory branch of second ball dynamics after \((n+1)\)-th collision, \(n \geq 2\), and between \(n+1\)-th and \(n+2\)-th, \(n \geq 2\) collisions of the balls is:

\[
\phi_2 = \sqrt{\phi_{2, \text{outgoing} (n+1)}^2 + \omega^2} \cos \phi_2 - \cos(\phi_{1, \text{impact} (n+1)} + \beta), \quad \text{between } n+1\text{-th and } n+2\text{-th, } n \geq 2. \tag{90}
\]

### III.3.5 Sketch of the phase trajectory branches of the rolling ball dynamics between successive central collisions of two rolling heavy balls along circle trace in vertical plane

In Figure 8, phase trajectory portraits of vibro-impact dynamics of two rolling balls along curvilinear circle line in vertical plane with successive two first collisions are presented: (upper) for second rolling ball and (lower) for first rolling ball non-linear dynamics. In Figure 9, plans of the configurations of the rolling balls in vibro-impact dynamics of rolling heavy balls along curvilinear circle line in vertical plane with successive first two collisions are presented.

Let us to explain how to obtain phase trajectories of vibro-impact nonlinear dynamics of two rolling balls along circle starting from initial conditions defined by corresponding initial position and initial angular velocity of the rolling balls: \((\phi_{1,0}, \phi_{1,0})\) and \((\phi_{2,0}, \phi_{2,0})\). Balls in this configuration is presented in Figure 9 (upper and left). In Figure 8, phase portraits for different initial conditions are presented: for first ball (lower) portrait in phase plane \((\phi_{1}, \phi_{1})\) and for the second ball (upper) portrait in phase plane \((\phi_{2}, \phi_{2})\), for the cases that single ball rolling along circle lines. We can see that on the phase portraits are visible three types of phase trajectories. Closed phase trajectory corresponds to oscillatory motions with constant total mechanical energy of the nonlinear oscillation dynamics. Open trajectories correspond to progressive ball rolling along circle line in one direction. Trajectories with cross sections passing through unstable saddle type singular points are separatrices and holoclinic trajectories. Saddle points at the phase portrait correspond to ball upper position on the circle line and present no stable equilibrium position. And stable centre type singular point correspond to lower position of the ball at the circle line, and present stable equilibrium position of the ball at circle line.

Using these phase portraits for single ball rolling along circle line, we start to construct phase trajectory branches for the vibro-impact dynamics of each of the rolling balls at circle line, starting by corresponding initial position and initial angular velocity. In then phase portrait starting kinetic states of the balls are presented by phase representative points: \(N_{1.0}(\phi_{1,0}, \phi_{1,0})\) and \(N_{2.0}(\phi_{2,0}, \phi_{2,0})\), taking
Figure 8. Phase trajectory portraits of vibro-impact dynamics of rolling balls along curvilinear circle line in vertical plane with two first successive collisions: (upper) for second rolling ball and (lower) for first rolling ball non-linear dynamics.
into account that is \( \dot{\phi}_{1,0} = \frac{\alpha_{1,0}}{\lambda_1 - 1} \) and \( \dot{\phi}_{2,0} = \frac{\alpha_{2,0}}{\lambda_2 - 1} \). Interval of the balls non-linear dynamics is along corresponding phase trajectory of the single ball non-linear dynamics between points from \( N_{1,0}(\phi_{1,0}, \phi_{1,0}) \) to \( N_{1,\text{impact},1}(\phi_{1,\text{impact},1}, \dot{\phi}_{1,\text{impact},1}) \), for first rolling ball and from \( N_{2,0}(\phi_{2,0}, \phi_{2,0}) \) to \( N_{2,\text{impact},1}(\phi_{2,\text{impact},1}, \dot{\phi}_{2,\text{impact},1}) \) for second rolling ball.

Taking into account that is \( \dot{\phi}_{1,\text{impact},1} = \frac{\alpha_{1,\text{impact},1}}{\lambda_1 - 1} \) and \( \dot{\phi}_{2,\text{impact},1} = \frac{\alpha_{2,\text{impact},1}}{\lambda_2 - 1} \), equation of corresponding branch of phase trajectory for first and second rolling ball along circle line is defined, respectively, by (33) and (44). Phase representative points \( N_{1,\text{impact},1}(\phi_{1,\text{impact},1}, \dot{\phi}_{1,\text{impact},1}) \) and \( N_{2,\text{impact},1}(\phi_{2,\text{impact},1}, \dot{\phi}_{2,\text{impact},1}) \), in phase portraits in Figure 8, correspond to pre-first-collision state, and phase representative points \( N_{1,\text{outgoing},1}(\phi_{1,\text{outgoing},1}, \dot{\phi}_{1,\text{outgoing},1}) \) and \( N_{2,\text{outgoing},1}(\phi_{2,\text{outgoing},1}, \dot{\phi}_{2,\text{outgoing},1}) \) correspond to post-first-collision kinetic state of the rolling balls along circle line. From these representative points at phase portrait a jumps in velocity for each of the ball dynamics appear and this jump is jump from one to other phase trajectory depending of outgoing angular velocity for each of the rolling balls defined by expressions (64) and (65) or (66) and (67).

Jump on the one branch of the one phase trajectory to other branch of other trajectory appears between representative points: from \( N_{1,\text{impact},1}(\phi_{1,\text{impact},1}, \dot{\phi}_{1,\text{impact},1}) \) to \( N_{1,\text{outgoing},1}(\phi_{1,\text{outgoing},1}, \dot{\phi}_{1,\text{outgoing},1}) \) for first ball and from \( N_{2,\text{impact},1}(\phi_{2,\text{impact},1}, \dot{\phi}_{2,\text{impact},1}) \) to \( N_{2,\text{outgoing},1}(\phi_{2,\text{outgoing},1}, \dot{\phi}_{2,\text{outgoing},1}) \) to second ball. This is produced by the change of the angular velocities of rolling balls pre- and post-collision kinetic state at same position and caused by the change of angular velocity directions of both rolling balls after collision between them. Next corresponding branch of the corresponding phase trajectory of first and second ball are defined by expressions (68) and (69) respectively. These new branches are defined and bounded by the pairs of the following representative points: for first ball rolling from representative point \( N_{1,\text{outgoing},1}(\phi_{1,\text{outgoing},1}, \dot{\phi}_{1,\text{outgoing},1}) \) to \( N_{1,\text{impact},2}(\phi_{1,\text{impact},2}, \dot{\phi}_{1,\text{impact},2}) \), and for second ball rolling from representative point \( N_{2,\text{outgoing},1}(\phi_{2,\text{outgoing},1}, \dot{\phi}_{2,\text{outgoing},1}) \) to \( N_{2,\text{impact},2}(\phi_{2,\text{impact},2}, \dot{\phi}_{2,\text{impact},2}) \), respectively.

Next jumps appear from the representative points \( N_{1,\text{impact},2}(\phi_{1,\text{impact},2}, \dot{\phi}_{1,\text{impact},2}) \) and \( N_{2,\text{impact},2}(\phi_{2,\text{impact},2}, \dot{\phi}_{2,\text{impact},2}) \) to the representative points \( N_{1,\text{outgoing},2}(\phi_{1,\text{outgoing},2}, \dot{\phi}_{1,\text{outgoing},2}) \) and \( N_{2,\text{outgoing},2}(\phi_{2,\text{outgoing},2}, \dot{\phi}_{2,\text{outgoing},2}) \), respectively.

Next branches of trajectories are defines by expressions (79) and (80), bounded by the representative points: for first rolling ball from \( N_{1,\text{outgoing},n}(\phi_{1,\text{outgoing},n}, \dot{\phi}_{1,\text{outgoing},n}) \) to \( N_{1,\text{impact},(n+1)}(\phi_{1,\text{impact},(n+1)}, \dot{\phi}_{1,\text{impact},(n+1)}) \), and for second
from $N_{2,\text{outgoing},n}\left(\phi_{2,\text{impact},n},\phi_{2,\text{outgoing},n}\right)$ to $N_{2,\text{impact},(n+1)}\left(\phi_{2,\text{impact},(n+1)},\phi_{2,\text{outgoing},(n+1)}\right)$, respectively.

**Figure 9.** Plan of configurations of rolling balls in vibro-impact dynamics of rolling heavy balls along curvilinear circle line in vertical plane with successive first two collisions.

Next jumps appear from the points $N_{1,\text{impact},(n+1)}\left(\phi_{1,\text{impact},(n+1)},\phi_{1,\text{outgoing},(n+1)}\right)$ and $N_{2,\text{impact},(n+1)}\left(\phi_{2,\text{impact},(n+1)},\phi_{2,\text{outgoing},(n+1)}\right)$ to the points $N_{1,\text{outgoing},(n+1)}\left(\phi_{1,\text{impact},(n+1)},\phi_{1,\text{outgoing},(n+1)}\right)$ and $N_{2,\text{outgoing},(n+1)}\left(\phi_{2,\text{impact},(n+1)},\phi_{2,\text{outgoing},(n+1)}\right)$, respectively, for $n = 2,3,4,5,6,\ldots$. 
Table 1. Mathematical and qualitative analogies between kinetic parameters of two system in central collision dynamics: collision of two bodies in translator motion and collision of two rolling balls

<table>
<thead>
<tr>
<th>Configuration of the systems in collision state and plane of bodies' collisions</th>
<th>Collision of two bodies in translator motion</th>
<th>Collision of two rolling balls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analogous theorems of conservation of linear momentum (impulse) or angular momentum</td>
<td>Theorem of conservation of linear momentum (impulse) in collision of two bodies in translator motion: ( m_1 \mathbf{v}_1(t_0 + \tau) + m_2 \mathbf{v}_2(t_0 + \tau) = m_1 \mathbf{v}_1(t_0) + m_2 \mathbf{v}_2(t_0) )</td>
<td>Theorem of conservation of angular momentum (kinetic moment) in collision of two rolling balls: ( \mathbf{I}_1 \mathbf{\omega}_1(t_0 + \tau) + \mathbf{I}_2 \mathbf{\omega}_2(t_0 + \tau) = \mathbf{I}_1 \mathbf{\omega}_1(t_0) + \mathbf{I}_2 \mathbf{\omega}_2(t_0) )</td>
</tr>
<tr>
<td>Coefficient of restitution of two body collision</td>
<td>Coefficient of the restitution in collision of two bodies in translator motion: ( k = \frac{v_1(t_0 + \tau) - v_2(t_0 + \tau)}{v_1(t_0) - v_2(t_0)} )</td>
<td>Coefficient of the restitution in collision of two rolling balls: ( k = \frac{\mathbf{\omega}_1(t_0 + \tau) - \mathbf{\omega}_2(t_0 + \tau)}{\mathbf{\omega}_1(t_0) - \mathbf{\omega}_2(t_0)} )</td>
</tr>
<tr>
<td>Outgoing velocities of third bodies at post-collision moment</td>
<td>Outgoing velocities of the bodies in translator motion at post-collision moment: ( v_1(t_0 + \tau) = v_1(t_0) + k(v_1(t_0) - v_2(t_0)) )</td>
<td>Outgoing angular velocities of the rolling balls at post-collision moment: ( \mathbf{\omega}_1(t_0 + \tau) = \mathbf{\omega}_1(t_0) + k(\mathbf{\omega}_1(t_0) - \mathbf{\omega}_2(t_0)) )</td>
</tr>
<tr>
<td>Impulse (linear momentum) of collision</td>
<td>Impulse (linear momentum) of collision of impact forces: ( \mathbf{I}_{\text{imp}} = m_1(v_1(t_0 + \tau) - v_1(t)) - m_2(v_2(t_0 + \tau) - v_2(t)) )</td>
<td>Moment of impulse (linear momentum) of collision of impact couple (moment of impact forces): ( \mathbf{I}_{\text{imp}} = \mathbf{J}_1(\mathbf{\omega}_1(t_0 + \tau) - \mathbf{\omega}_1(t)) - \mathbf{J}_2(\mathbf{\omega}_2(t_0 + \tau) - \mathbf{\omega}_2(t)) )</td>
</tr>
<tr>
<td>Kinetic energy change from pre-collision to post-collision kinetic state</td>
<td>Kinetic energy change from pre-collision to post-collision kinetic state: ( \Delta K_{\text{kin},\text{pre}} = K_1(t) - K_1(t_0) = \frac{m_1 v_1(t)^2}{2} - \frac{m_1 v_1(t_0)^2}{2} )</td>
<td>Kinetic energy change from pre-collision to post-collision kinetic state: ( \Delta K_{\text{kin},\text{pre}} = K_1(t) - K_1(t_0) = \frac{\mathbf{\omega}_1(t)^2}{2} - \frac{\mathbf{\omega}_1(t_0)^2}{2} )</td>
</tr>
</tbody>
</table>
III.3.6. Energy analysis of the vibro-impact non-linear dynamics with successive central collisions of two rolling heavy balls along circle trace in vertical plane

From the phase portraits of the rolling heavy balls along circle line in vertical plane it is possible to conclude that both ball nonlinear dynamics between impacts are conservative with constant total mechanical energies for each of the rolling balls. Jumps of the representative point in corresponding ball phase portrait in pre- and post collision caused of the change of total mechanical energy of each ball, from upper to lower total mechanical energy for one and opposite for other ball. If impacts are ideal elastic in sum total mechanical energies of both balls are constant, if is no ideal elastic collision this sum of total mechanical energy decreases and after numerous collisions tends to zero. But in this also case of the no ideal elastic collisions appear jumps from one to other phase trajectory branch.

For the case of ideal elastic collisions between rolling balls in the vibro-impact dynamics of whole system is with constant mechanical energy, and appear the change of mechanical energy between balls in each of the collisions, but this vibro-impact dynamics continued in infinite period and with infinite numbers of the collisions. For the case of no ideal elastic collisions between rolling balls in the vibro-impact dynamics of whole system is with no constant mechanical energy, energy dissipation appear in each collision and appear the change of mechanical energy between balls in each collision, then this vibro-impact dynamics continued in finite period and with finite numbers of collisions up to rest of the system after finite numbers of the collisions.

Taking into account that non-linear dynamics of the single heavy ball rolling along circle in vertical plane is conservative motion, and that for each ball is presented energy integrals in the forms: (42)-(43) for first rolling ball and (51)-(52) for second rolling ball along circle line and that phase trajectories in phase portraits present also curves of the constant system total mechanical energy for each of single ball motion it is possible to made some conclusions concerning vibro-impact dynamics of the two rolling balls. In each collision rolling ball with large angular velocity of the ball after collision is smaller and its total mechanical energy obtain a jump from upper level to lower level, and rolling ball with smaller angular velocity after collision obtain larger angular velocity and its total mechanical energy obtain a jump from lower to upper level. There jumps of total mechanical energy of each ball appear after each collision of the balls.

IV. Concluding Remarks

In concluding remarks it is necessary to point out importance of Petrović’s theory of Elements of mathematical phenomenology and Phenomenological Mapping for obtaining original results of kinetic parameters of two rolling balls in the central collision when balls roll along straight trace as well as along curvilinear trace in vertical plane, on the basis of analogy with kinetic parameters of the central collision between two bodies in translator motion. In Table 1 on the
basis of mathematical and qualitative analogies between kinetic parameters of two system central collision dynamics are presented corresponding analogous kinetic parameters of the central collision of two bodies in translator motion and central collision of two rolling balls.

Also, kinetic parameters of the collision between two rolling balls presented in this paper are used to present vibro-impact dynamics of two rolling heavy balls along curvilinear circle line in vertical plane. For this vibro-impact dynamics a sketch of phase trajectory is presented.

At the end, it is useful to conclude that obtained kinetic parameters of the central collision of two rolling balls is possible to use in study of the skew collision of two rolling balls that role along two straight line trace with intersection as well as parallel at distance smaller them sum of the balls radiuses.

Aim of this paper is not to present a review about generalization of the all results in area of the collision of two rolling balls different properties of balls and collisions. The paper is focused to central collision of two rolling rigid and heavy smooth balls and using elements of mathematical phenomenology and phenomenological mapping to obtain corresponding new expressions for the post-collision and outgoing angular velocity of each ball and applied these results for investigation of the vibro-impact dynamics of two rolling balls along circle trace. This task is fully analytical solved and obtained analytical results are original and new! Also, these results can be fundamental for next development and investigation of the special class of collision of the rigid and/or deformable balls and also in application in different area of engineering systems with coupled rotations (in rolling bearings, rolling vibro-impact dampers - mechanisms for dynamic absorption of torsional vibrations, or other ).

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Appendix:

Mathematical analogy between translator motion of two bodies and rolling motion of two balls

Mathematical analogy between translator motion of two bodies and rolling motion of two balls in collision is simple explain by use mathematical analogy and elements of mathematical phenomenology.

a* For translator motion of the bodies, corresponding velocity, \( v_{c1} \) or \( v_{c2} \), is coupled for body mass center, \( C_1 \) or \( C_1 \). Differential equation of the body translator motion is \( m_k \ddot{v}_{c_k} = F_k', \ k = 1,2 \), based on the theorem of change of the linear momentum in relation with applied force.
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**Figure A 1.** Models of two simple motions of the bodies with necessary kinetic and inertia properties: a* the translator motion of two bodies, each with one degrees of freedom; and b* the rolling motions of two, different size balls, each with one degree of freedom.

b* For rolling motion of the balls, corresponding angular velocity, $\vec{\omega}_p$ or $\vec{\omega}_s$ is coupled for corresponding momentary rolling axis through corresponding pole $P_1$ or $P_2$. Axial ball mass inertia moment $J_{p1}$ or $J_{p2}$ for corresponding momentary axis of rolling is body inertia properties (coefficients). Differential equation of rolling ball motion is $J_{pk}\vec{\omega}_{pk} = M_{pk}$, $k = 1,2$, based on the theorem of change angular momentum for pole or axis of rolling in relation with corresponding couple for same pole or axis!