Qualitative Analysis for Solutions of a Class of Nonlinear Ordinary Differential Equations

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Abstract

The boundary value problem (BVP) for a class of nonlinear ordinary differential equations is examined. It can be used to describe the eversion problem for a spherical shell composed of a class of transversely isotropic incompressible Mooney-Rivlin materials. The solutions of the nonlinear equation describing the relation among deformed thickness, initial thickness and material parameters of the spherical shell are obtained. The effects of structure parameter and material parameters on the thickness of the everted spherical shell are discussed by numerical examples.

Keywords : incompressible hyperelastic material, spherical shell, eversion, deformation, implicit solution

1. Introduction

Recently, the eversion problems of bodies composed of hyperelastic materials (such as rubber and rubber-like materials) widely arise from engineering design and aerospace fields, and so on. These problems can be described as a BVP for a class of

nonlinear ordinary differential equations. Many researchers studied the everted spherical shell theoretically and experimentally. In 1913, Armani [1] firstly considered the eversion problem of a spherical shell composed of a class of compressible hyperelastic materials. Ericksen [2] studied the existence and uniqueness of the everted solutions for a spherical shell composed of incompressible Mooney-Rivlin material. Antman [3], Szeri [4] gave further analysis for the existence of the everted solutions for compressible spherical shells. In 1993, Jeremiah [5] made a comparison between the qualitative features of the regular and everted shells composed of a special compressible material under internal pressure. It is found that the stress distribution in both cases is different and a larger pressure can be sustained by the everted shell. For some incompressible hyperelastic spherical shells, Chen and Haughton [6] proved that, if the material satisfied the Baker-Ericksen inequalities, no matter how the thickness was, there was a unique spherically symmetric everted solution, and they gave a sufficient condition of cavity formation for the everted spherical shell. Otherwise, they also found that thicker spherical shells could undergo a bifurcation on eversion.

In this paper, the eversion problem of a spherical shell composed of a class of transversely isotropic incompressible Mooney-Rivlin materials is considered. The corresponding mathematical model can be treated as a BVP. The implicit solution is obtained by using the incompressible condition and the semi-inverse method. The effects of structure parameter and material parameters on the thickness of the everted spherical shells are discussed by numerical examples.

2. Mathematical model and Solutions

Here we consider the solution of the eversion problem for an incompressible hyperelastic thin-walled spherical shell.

Since the deformation is spherical symmetry, the deformation configuration is given by

$$r = r(R), \quad 0 \leq A \leq R \leq B, \quad 0 < a \leq r \leq b, \quad \theta = \pi - \Theta, \quad \varphi = \Phi, \quad (1)$$

where R and r are the radii of undeformed and deformed shell, a and b are the inner and outer radii, respectively. Note that $b = r(A)$, $a = r(B)$.

The principal stretches are given by

$$\lambda_r = -r' = -\frac{dr}{dR}, \quad \lambda_\theta = \lambda_\varphi = \frac{r}{R} = \lambda, \quad (2)$$

where λ is a parameter describing the axial stretch rate. Since the material is

incompressible, we get $\lambda_r \lambda_\theta \lambda_\varphi = 1$, then $r^3 - a^3 = B^3 - R^3$. Moreover,

$$b^3 - a^3 = B^3 - A^3.$$

For different materials, there are different deformation modes. So for solving this problem, we need to give a specific strain-energy function. In this paper, suppose that the spherical shell is consist of radial transversely isotropic incompressible Mooney-Rivlin material, whose strain-energy function is ^[8]

$$W = \frac{\mu}{2} \left(\left(\frac{1}{2} + \beta \right) (I_1 - 3) + \left(\frac{1}{2} - \beta \right) (I_2 - 3) + \alpha (\lambda_r^2 - 1) \right), \quad (3)$$

where $\mu > 0$ is the shear modulus in infinitesimal deformation, β ($-1/2 \leq \beta \leq 1/2$) is a dimensionless material constant, α is a dimensionless parameter measuring the degree of anisotropy of the material, $I_1 = \lambda_r^2 + \lambda_\theta^2 + \lambda_\varphi^2$ and $I_2 = \lambda_r^2 \lambda_\theta^2 + \lambda_\theta^2 \lambda_\varphi^2 + \lambda_r^2 \lambda_\varphi^2$ are the principal invariants of the right Cauchy deformation tensor.

The corresponding principal components of the Cauchy stress tensor are given by

$$\sigma_{rr} = \mu (r')^2 \left(\frac{1}{2} + \beta + 2 \left(\frac{1}{2} - \beta \right) \lambda^2 + \alpha \right) - p, \quad \sigma_{\theta\theta} = \mu \lambda^2 \left(\frac{1}{2} + \beta + \left(\frac{1}{2} - \beta \right) ((r')^2 + \lambda^2) \right) - p, \quad (4)$$

where p is the hydrostatic pressure.

In the absence of body force, the equilibrium differential equations can be reduced to

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = 0. \quad (5)$$

Integrating Eq. (5) with respect to r , we have

$$\sigma_{rr}(r) = \sigma_{rr}(a) + 2 \int_a^r \left(\frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} \right) dr. \quad (6)$$

Assume that the inner and outer surfaces of the everted spherical shell are traction-free, the boundary conditions are as follows

$$\sigma_{rr}(a) = \sigma_{rr}(b) = 0. \quad (7)$$

Then form Eq. (6), we get

$$\int_a^b \left(\frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} \right) dr = 0. \quad (8)$$

Substituting Eq. (4) into Eq. (8), we have

$$\int_a^b \left(\left(\lambda^2 - (r')^2 \right) \left(\frac{1}{2} + \beta + \left(\frac{1}{2} - \beta \right) \lambda^2 \right) - \alpha (r')^2 \right) \frac{dr}{r} = 0. \quad (9)$$

For convenience, we introduce the following notations

$$\delta = \frac{A}{B}, \quad m = \frac{a}{B}. \quad (10)$$

Then we obtain

$$M = \frac{b}{A} = \left(\frac{1 - \delta^3 + m^3}{\delta^3} \right)^{1/3}, \quad \eta = \frac{a}{b} = \left(\frac{m^3}{1 - \delta^3 + m^3} \right)^{1/3},$$

$$\lambda = \frac{r}{R} = \left(\frac{B^3 + a^3}{r^3} - 1 \right)^{-1/3}, \quad \frac{dr}{r} = \frac{r^3}{\lambda^3} \frac{d\lambda}{\lambda} = \frac{B^3 + a^3}{1 + \lambda^3} \frac{d\lambda}{\lambda}. \quad (11)$$

Using the above notations, we rewrite Eq. (9) as

$$\left\{ \left(\frac{1}{2} - \beta \right) \lambda - \left(\frac{1}{2} + \alpha + \beta \right) \frac{1}{\lambda} + \frac{1}{4} \left(\frac{1}{2} + \alpha + \beta \right) \frac{1}{\lambda^4} + \frac{1}{2} \left(\frac{1}{2} - \beta \right) \frac{1}{\lambda^2} + \frac{\alpha}{3} \ln(\lambda + 1) \right. \\ \left. - \frac{\alpha}{3} \left(\frac{1}{2} \ln \left(\left(\lambda - \frac{1}{2} \right)^2 + \frac{3}{4} \right) + \sqrt{3} \arctan \left(2 \frac{\sqrt{3}}{3} \left(\lambda - \frac{1}{2} \right) \right) \right) \right\} \Bigg|_m^M = 0. \quad (12)$$

Obviously, Eq. (12) describing the finite deformation of the everted spherical shell is a nonlinear equation with respect to δ and m . From Eqs. (11), (12), we can get the relations among the structure parameter δ , the material parameters α , β and the thickness η of the everted spherical shell.

3. Numerical simulations

Figs. 1-2 show the effects of the initial thickness δ , the material parameters α and β on the finite deformation of the everted spherical shell composed of the incompressible material (3).

Fig. 1 Relation curves of $\delta \sim m, \eta$ for different values of α ($\beta = 0.2$)

Fig. 2 Relation curves of $\delta \sim m, \eta$ for different values of β ($\alpha = 2$)

From Fig. 1(a), it is seen that, for the given values of β , the inner radius of the everted spherical shell increases with the increasing initial thickness δ , and the increasing relations are linear approximately. Meanwhile, the inner radius of the everted spherical shell increases with the increasing the material parameter α , that is to say, the influence of the material parameter α on the inner radius and thickness of the everted thickness is significant. From Fig. 1(b), it is seen that, for the given values of β , the thickness of the everted spherical shell increases with the increasing initial

thickness δ . However, the influence of the material parameter α on the thickness of the everted thickness is not obvious. From Fig. 2(a), it is seen that, for the given values of α , the inner radius of the everted spherical shell increases with the increasing initial thickness δ . Moreover, the effect of the material parameter β on the inner radius of the everted spherical shell is not obvious. From Fig. 2(b), it is seen that, for the thickness of the everted spherical shell, there are some similar properties to the inner radius in Fig. 2(a).

4. Conclusions

The nonlinear equations describing the finite deformation for an everted spherical shell composed of radial transversely isotropy Mooney-Rivlin material is considered. The implicit solution is obtained by using the incompressible condition and the semi-inverse method. The conclusions show that

- (1) The thinner the initial spherical shell is, the bigger the deformed inner radius is.
- (2) The thinner the initial spherical shell is, the thinner the everted spherical shell is.
- (3) The influence of the parameter α measuring the degree of anisotropy of the material on the inner radius and thickness of the everted thickness is significant.
- (4) The influence of the material parameter β on the inner radius and thickness of the everted thickness is not obvious.

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