Advances in Theoretical and Applied Mechanics, Vol. 7, 2014, no. 2, 67 - 78
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/atam.2014.469

Analysis of the Small Oscillations of a Pendulum Partially Filled by a Heavy Capillary Liquid

H. Essaouini*, J. Elbahaoui and L. Elbakkali

Université Abdelmalek Essaâdi, Faculté des Sciences, M2SM, ER28/FS/05, 93000, Tétouan, Morocco

P. Capodanno

Université de Franche-Comté, 2B, Rue des jardins, F-25000 Besançon, France

Abstract

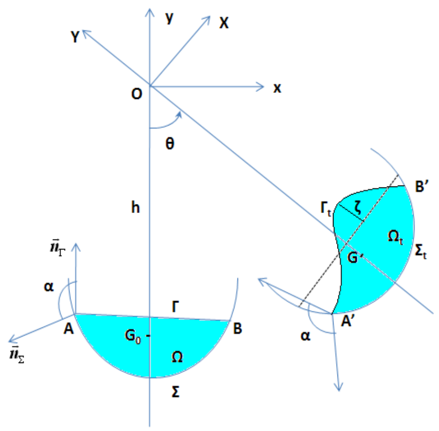
The authors study the small oscillations of a compound pendulum partially filled by a heavy liquid, in presence of surface tensions.
Using the variational formulation of the problem, they prove that, under a very simple geometrical condition, it is a classical vibration problem.
On the other hand, introducing the operatorial equations of motion, they prove that the eigenvalues equation can be obtained by equaling to zero an absolutely convergent infinite determinant.

Keywords: capillary liquid, small oscillations, spectral problem.

1. Introduction

The problem of the small oscillations of a heavy system formed by a container partially filled by an inviscid liquid has been the subject of a number of works (Moiseyev and Rumiantsev 1968).
The same problem has been studied more recently in the case of a liquid submitted to surface tensions and oscillating in a fixed container (Morand et Ohayon 1992, Kopachevsky and Krein, Vol 1, 2001).

In this work, the authors consider the problem of a moving container partially filled by a capillary liquid, restricting themselves to the simple case of the compound pendulum.
Using the variational formulation of the problem and suitable Hilbert spaces, they prove that, under a simple geometrical condition, the problem is a classical vibration problem.
On the other hand, introducing the operatorial equations of motion, they obtain the eigenvaluesequation by equaling to zero an absolutely convergent infinite determinant.



Figure

2. Position of the Problem

The container is a compound pendulum oscillating about a fixed point O.
We use the axes Ox, Oy, Oy vertical upwards (unit vectors $\vec{x}$, $\vec{y}$).
In the equilibrium position, the container is symmetrical with respect to Oy. It contains an incompressible inviscid liquid (density ρ) that occupies a domain Ω bounded by the wetted part Σ of the wall of the container and the horizontal free line Γ or AB $(AB = \sigma)$, the equation of which is $y = -h$ $(h > 0)$. The mass of the pendulum is denoted by m, its center of mass by G_0 $\left(\overrightarrow{OG_0} = -d\vec{y};\ d > 0\right)$; g is the acceleration of the gravity.
At the instant t, the liquid occupies the domain Ω_t, bounded by the wetted part Σ_t of the wall of the container and the free line Γ_t or $A'B'$. If G is the position of G_0 at the instant t, we set $\widehat{\overrightarrow{OG_0}, \overrightarrow{OG}} = \theta(t)$, θ and its derivatives being small. Introducing the axes OX, OY, $\widehat{\overrightarrow{Ox}, \overrightarrow{OX}} = \theta$, we denote by $Y = -h + \zeta(X, t)$ the equation of Γ_t, ζ and its derivatives being small.

We suppose that the pressure above the free line is equal to zero and we take into account the constant surface tension τ.
We are going to study the possible small oscillations of the system pendulum-liquid about its equilibrium position, in linear theory.

3. Equations of motion

3.1. Let us consider the liquid

If P is the pressure, we introduce the dynamic pressure p by

$$P = -\rho g\left(y+h\right) + p$$

Then, if $\vec{u}\left(x,y,t\right)$ is the small displacement of a particle from its equilibrium position, we have

(1) $$\rho\ddot{\vec{u}} = -\overrightarrow{\mathrm{grad}}\,p \quad \text{(Euler's equation);}$$

$$\mathrm{div}\,\dot{\vec{u}} = 0 \quad \text{(incompressibility)}$$

Integrating from the equilibrium position to the instant t, we have

(2) $$\mathrm{div}\,\vec{u} = 0$$

The kinematic condition can be written

(3) $$u_{n|_\Sigma} \overset{\text{def}}{=} \vec{u}\cdot\vec{n}_\Sigma = \theta\left(xn_{\Sigma y} - yn_{\Sigma x}\right) \;;$$

(4) $$u_{n|_\Gamma} \overset{\text{def}}{=} \vec{u}\cdot\vec{n}_\Gamma = \zeta + \theta x \;,$$

$\vec{n}_\Sigma$ and $\vec{n}_\Gamma$ being the unit vectors normal to Σ and Γ.

3.2. Let us write the equations deduced from the laws of the capillarity

a) Denoting by α the angle $\vec{n}_\Gamma, \vec{n}_\Sigma$ at the points A and B, the conditions of contact angle are (Morand et Ohayon 1992)

(5) $$\zeta'\left(\pm\frac{\sigma}{2},t\right) = \pm\frac{1}{R\sin\alpha}\zeta\left(\pm\frac{\sigma}{2},t\right), \quad \left(\zeta' = \frac{\partial\zeta}{\partial x}\right),$$

where R is the radius of curvature of Σ at the points A and B.

b) The Laplace law on the free line Γ in the equilibrium position is verified, because the pressure on the line $y=-h$ is equal to zero.

We obtain easily the Laplace law on Γ_t (Morand et Ohayon 1992)

(6) $$p_{|\Gamma} = \rho g(\zeta + \theta x) - \tau\zeta'' ,$$

ζ'' being classically the curvature of Γ_t .

3.3 . The theorem of the moment of momentum for the system pendulum-liquid gives.

$$I\ddot{\theta} + \int_\Omega \rho(x\ddot{u}_y - y\ddot{u}_x)d\Omega = -mgd\theta - \rho g\int_{\Omega_t} xd\Omega_t ,$$

I being the moment of intertia of the pendulum about O.
We can write

$$\rho g\int_{\Omega_t} xd\Omega_t = \rho g\int_{\Omega_t}(X - \theta Y)d\Omega_t = \rho g\theta\int_\Omega yd\Omega - \rho g\int_{\Omega_t} Xd\Omega_t$$

But we have

$$\rho\int_\Omega yd\Omega = m_\ell y_{G_\ell} ,$$

where m_ℓ is the mass of the liquid and G_ℓ its center of mass in the equilibrium position, and

$$\int_{\Omega_t} Xd\Omega_t = \int_\Omega Xd\Omega + \int_{\Omega_t - \Omega} XdXdY = \int_\Gamma\left(\int_{-h}^{-h+\zeta} dY\right)XdX = \int_\Gamma x\zeta d\Gamma .$$

On the other hand, we have, using the Euler's equation (1) and the Green formula

$$\int_\Omega \rho(x\ddot{u}_y - y\ddot{u}_x)d\Omega = -\int_\Omega\left[\frac{\partial(xp)}{\partial y} - \frac{\partial(yp)}{\partial x}\right]d\Omega$$
$$= -\int_\Sigma p_{|\Sigma}(xn_{\Sigma y} - yn_{\Sigma x})d\Sigma - \int_\Gamma p_{|\Gamma} xd\Gamma$$

Finally, using the condition (6), we obtain the equation

(7) $$I\ddot{\theta} = -K^2\theta + \int_\Sigma p_{|\Sigma}(xn_{\Sigma y} - yn_{\Sigma x})d\Sigma - \tau\int_\Gamma x\zeta'' d\Gamma$$

where

$$J = \rho\int_\Gamma x^2 d\Gamma \ ; \quad K^2 = g\left(md - m_\ell y_{G_\ell} - J\right),$$

K^2 being positive if the pendulum is preponderant.

4. Variational formulation of the problem

4.1) Let use introduce $\tilde{\theta}$ arbitrary in R, a smooth function $\vec{\tilde{u}}$ defined in Ω and verifying

$$\mathrm{div}\vec{\tilde{u}} = 0 \ , \text{ in } \Omega \ , \ \tilde{u}_{n|_\Sigma} = \tilde{\theta}(xn_{\Sigma y} - yn_{\Sigma x}),$$

and $\tilde{\zeta}$ by $\tilde{u}_{n|_\Gamma} = \tilde{\zeta} + \tilde{\theta}x$.

Using the equation (1) and the Green formula, we have

$$\int_\Omega \rho\ddot{\vec{u}} \cdot \bar{\tilde{\vec{u}}} d\Omega = -\int_\Omega \overrightarrow{\text{grad}} p \cdot \bar{\tilde{\vec{u}}} d\Omega = -\int_\Sigma p_{|_\Sigma} \bar{\tilde{u}}_{n|_\Sigma} d\Sigma - \int_\Gamma p_{|_\Gamma} \bar{\tilde{u}}_{n|_\Gamma} d\Gamma \quad .$$

Using the condition (6), we obtain

$$I\ddot{\theta}\bar{\tilde{\theta}} + \int_\Omega \rho\ddot{\vec{u}} \cdot \bar{\tilde{\vec{u}}} d\Omega + K^2\theta\bar{\tilde{\theta}} + \rho g \int_\Gamma u_{n|_\Gamma} \bar{\tilde{u}}_{n|_\Gamma} d\Gamma - \tau\int_\Gamma u''_{n|_\Gamma} \bar{\tilde{u}}_{n|_\Gamma} d\Gamma + \bar{\tilde{\theta}}\tau\int_\Gamma x\zeta'' d\Gamma = 0$$

Integrating by parts the last two integrals and taking into account the condition (5), we obtain the variational equation

$$(8) \qquad \begin{cases} I\ddot{\theta}\bar{\tilde{\theta}} + \int_\Omega \rho\ddot{\vec{u}} \cdot \bar{\tilde{\vec{u}}} d\Omega + \left[K^2 + \tau\sigma\left(1 - \dfrac{\sigma}{2R\sin\alpha}\right)\right]\theta\bar{\tilde{\theta}} + \rho g \int_\Gamma u_{n|_\Gamma} \bar{\tilde{u}}_{n|_\Gamma} d\Gamma \\ +\tau\left[\int_\Gamma u'_{n|_\Gamma} \bar{\tilde{u}}'_{n|_\Gamma} d\Gamma - \dfrac{1}{R\sin\alpha}\left\{u_{n|_\Gamma}\left(\dfrac{\sigma}{2}\right)\bar{\tilde{u}}_{n|_\Gamma}\left(\dfrac{\sigma}{2}\right) + u_{n|_\Gamma}\left(-\dfrac{\sigma}{2}\right)\bar{\tilde{u}}_{n|_\Gamma}\left(-\dfrac{\sigma}{2}\right)\right\}\right] \\ -\tau\left(1 - \dfrac{\sigma}{2R\sin\alpha}\right)\left[\theta\left\{\bar{\tilde{u}}_{n|_\Gamma}\left(\dfrac{\sigma}{2}\right) - \bar{\tilde{u}}_{n|_\Gamma}\left(-\dfrac{\sigma}{2}\right)\right\} + \bar{\tilde{\theta}}\left\{u_{n|_\Gamma}\left(\dfrac{\sigma}{2}\right) - u_{n|_\Gamma}\left(-\dfrac{\sigma}{2}\right)\right\}\right] = 0 \end{cases}$$

so that the problem is self adjoint.

We are going to study the equation (8).

4.2) Since $\int_\Gamma u_{n|_\Gamma} d\Gamma = 0$, we look for $u_{n|_\Gamma}$ in the space $\tilde{H}^1(\Gamma) = \left\{v \in H^1(\Gamma);\ \int_\Gamma v d\Gamma = 0\right\}$.

We are going to prove that:

$$\lambda = \inf_{v\in\tilde{H}^1(\Gamma)} \frac{\int_{-\sigma/2}^{\sigma/2} v'^2 dx}{v^2\left(\dfrac{\sigma}{2}\right) + v^2\left(-\dfrac{\sigma}{2}\right)} = \frac{2}{\sigma}$$

It is sufficient to seek this $\inf$ in the space $\tilde{C}^2(\bar{\Gamma}) = \left\{v \in C^2(\bar{\Gamma}),\ \int_\Gamma v d\Gamma = 0\right\}$, since $\tilde{C}^2(\bar{\Gamma})$ is dense in $\tilde{H}^1(\Gamma)$.

Introducing the multipliers λ and μ , we write

$$\delta\left[\int_{-\sigma/2}^{\sigma/2} v'^2 dx - \lambda\left\{v^2\left(\frac{\sigma}{2}\right) + v^2\left(-\frac{\sigma}{2}\right)\right\}\right] - 2\mu\delta\int_{-\sigma/2}^{\sigma/2} v dx = 0\,, \quad \forall \delta v \in C^2(\bar{\Gamma})$$

or

$$\int_{-\sigma/2}^{\sigma/2} v'\delta v' dx - \lambda\left[v\left(\frac{\sigma}{2}\right)\delta v\left(\frac{\sigma}{2}\right) + v\left(-\frac{\sigma}{2}\right)\delta v\left(-\frac{\sigma}{2}\right)\right] - \mu\int_{-\sigma/2}^{\sigma/2} \delta v dx = 0$$

or, after integration by parts

$$\int_{-\sigma/2}^{\sigma/2}(-v''-\mu)\delta v dx+\left[v'\left(\frac{\sigma}{2}\right)-\lambda v\left(\frac{\sigma}{2}\right)\right]\delta v\left(\frac{\sigma}{2}\right)-\left[v'\left(-\frac{\sigma}{2}\right)+\lambda v\left(-\frac{\sigma}{2}\right)\right]\delta v\left(-\frac{\sigma}{2}\right)=0\,.$$

So, we obtain for calculating λ the Steklov eigenvalues problem:

$$v''+\mu=0\,;\quad \int_{-\sigma/2}^{\sigma/2}vdx=0\,;\quad v'\left(\frac{\sigma}{2}\right)-\lambda v\left(\frac{\sigma}{2}\right)=0\,;\quad v'\left(-\frac{\sigma}{2}\right)+\lambda v\left(-\frac{\sigma}{2}\right)=0$$

We find easily two values for λ : $\frac{2}{\sigma}$ and $\frac{6}{\sigma}$, so that $\lambda=\frac{2}{\sigma}$.

4.3) Now, we consider the hermitiansesquilinear form, continuous in $\tilde{H}^1(\Gamma)\times\tilde{H}^1(\Gamma)$:

$$b\left(u_{n|_\Gamma},\tilde{u}_{n|_\Gamma}\right)=\int_\Gamma u'_{n|_\Gamma}\overline{\tilde{u}}'_{n|_\Gamma}d\Gamma-\frac{1}{R\sin\alpha}\left[u_{n|_\Gamma}\left(\frac{\sigma}{2}\right)\overline{\tilde{u}}_{n|_\Gamma}\left(\frac{\sigma}{2}\right)+u_{n|_\Gamma}\left(-\frac{\sigma}{2}\right)\overline{\tilde{u}}_{n|_\Gamma}\left(-\frac{\sigma}{2}\right)\right]$$

Using the precedent result, we have

$$b\left(u_{n|_\Gamma},u_{n|_\Gamma}\right)\geq\frac{2}{\sigma}\left(1-\frac{1}{R\sin\alpha}\right)\left[\left|u_{n|_\Gamma}\left(\frac{\sigma}{2}\right)\right|^2+\left|u_{n|_\Gamma}\left(-\frac{\sigma}{2}\right)\right|^2\right]\,.$$

In the following, we will suppose

(9) $$1-\frac{\sigma}{2R\sin\alpha}>0\,,$$

so that

$$K^2+\tau\sigma\left(1-\frac{\sigma}{2R\sin\alpha}\right)>0 \text{ and } b\left(u_{n|_\Gamma},u_{n|_\Gamma}\right)\geq 0$$

It is easy to see that de condition (9) express that the center of curvature of Σ in A must be of the right of the axis Oy.

4.4) Let use prove that

$$a_0(v,v)=\rho g\int_\Gamma|v|^2d\Gamma+\tau b(v,v)$$

defines on $\tilde{H}^1(\Gamma)$ the square of a norm that is equivalent to the classical norm $\|\cdot\|_1$ of $H^1(\Gamma)$, i.e that there exist $c>0$ such that

$$\frac{a_0(v,v)}{\|v\|_1^2}\geq c\,.$$

Indeed, if c does not exist, there is a sequence $\{v_n\}\in\tilde{H}^1(\Gamma)$ so that $\|v_n\|_1=1$ and $a_0(v_n,v_n)\to 0$ when $n\to\infty$. From the sequence $\{v_n\}$ bounded in $\tilde{H}^1(\Gamma)$, we can deduce a sequence, denoted still by $\{v_n\}$, that is strongly convergent in the space $\tilde{L}^2(\Gamma)=\left\{v\in L^2(\Gamma),\ \int_\Gamma vd\Gamma=0\right\}$ to $\tilde{v}\in\tilde{H}^1(\Gamma)\subset\tilde{L}^2(\Gamma)$.

From $a_0(v_n-v_m,v_n-v_m)\to 0$ when $n,\ m\to\infty$, we deduce easily

$b(v_n - v_m, v_n - v_m) \to 0$ and then $\|v'_n - v'_m\|_{\tilde{L}^2(\Gamma)} \to 0$.

Therefore, the sequence $\{v_n\}$ is strongly convergent in $\tilde{H}^1(\Gamma)$, so that $\|\tilde{v}\|_1 = 1$, in contradiction with $\tilde{v} = 0$.

4.5) In the variational equation (8), we consider the hermitian sesquilinear form

$$a_1\left(\theta,\vec{u};\tilde{\theta},\vec{\tilde{u}}\right) \overset{\text{def}}{=} \left[K^2 + \tau\sigma\left(1 - \frac{\sigma}{2R\sin\alpha}\right)\right]\theta\bar{\tilde{\theta}} + a_0\left(u_{n|_\Gamma}, \tilde{u}_{n|_\Gamma}\right)$$

$$-\tau\left(1 - \frac{\sigma}{2R\sin\alpha}\right)\left[\theta\left\{\bar{\tilde{u}}_{n|_\Gamma}\left(\frac{\sigma}{2}\right) - \bar{\tilde{u}}_{n|_\Gamma}\left(-\frac{\sigma}{2}\right)\right\} + \bar{\tilde{\theta}}\left\{u_{n|_\Gamma}\left(\frac{\sigma}{2}\right) - u_{n|_\Gamma}\left(-\frac{\sigma}{2}\right)\right\}\right]$$

By means of the precedent result and little long calculation, it is possible to prove that, under the condition (9), $\left[a_1\left(\theta,\vec{u};\tilde{\theta},\vec{\tilde{u}}\right)\right]^{1/2}$ defines on $\mathbb{C}\times\tilde{H}^1(\Gamma)$ a norm that is equivalent to the classical norm of this space.

4.6) Since, by the Lagrange's theorem, $\vec{u}$ is a gradient, we introduce the space

$$V = \left\{\begin{array}{l} U = (\theta,\vec{u})^t;\ \theta\in\mathbb{C};\ \vec{u} = \overrightarrow{\text{grad}}\varphi;\ \varphi\in\tilde{H}^1(\Omega) \overset{\text{def}}{=} \left\{\varphi\in H^1(\Omega),\ \int_\Gamma \varphi_{|_\Gamma} d\Gamma = 0\right\}; \\ \text{div}\vec{u} = \Delta\varphi = 0 \ \text{ in } \ \Omega;\ u_{n|_\Sigma} = \frac{\partial\varphi}{\partial n}\Big|_\Sigma = \theta\left(xn_{\Sigma y} - yn_{\Sigma x}\right);\ u_{n|_\Gamma} = \frac{\partial\varphi}{\partial n}\Big|_\Gamma \in \tilde{H}^1(\Gamma) \end{array}\right\}$$

equipped with the hilbertian norm defined by

$$\|U\|_V^2 = |\theta|^2 + \int_\Omega |\vec{u}|^2\, d\Omega + \left\|u_{n|_\Gamma}\right\|_1^2 ,$$

and the space χ , completion of V for the norm defined by

$$\|U\|_V^2 = I\,|\theta|^2 + \int_\Omega \rho|\vec{u}|^2\, d\Omega .$$

Writing $a\left(U,\tilde{U}\right)$ instead of $a_1\left(\theta,\vec{u};\tilde{\theta},\vec{\tilde{u}}\right)$, we obtain the precise variational formulation of the problem:

To find $U(\cdot)\in V$ such that

$$\left(\ddot{U},\tilde{U}\right)_\chi + a\left(U,\tilde{U}\right) = 0 \ \forall\tilde{U}\in V \tag{10}$$

4.7) Using a method and calculations that can be found in the book (Sanchez Hubert, Sanchez Palencia 1989, pp 66-68)- so that we omit the proof- we see that the problem (10) is a classical vibration problem.

Therefore, if the pendulum is preponderant and under the simple condition $\sigma < 2R\sin\alpha$, there exists a countable infinity of positive real eigenvalues

$$0 < \omega_1^2 \le \omega_2^2 \le \ldots \le \omega_n^2 \le \ldots \ \omega_n^2 \to +\infty$$

The corresponding eigenelements $U_n = (\theta_n, \vec{u}_n)^t$ form an orthonormal basis in χ and an orthogonal basis in V equipped with the scalar product $a(U, \tilde{U})$.

5. Operatorial equations of motion

5.1) We introduce the displacement potential $\varphi(x, y, t)$ defined by

$$\vec{u} = \overrightarrow{\text{grad}}\varphi \ .$$

Using the equations (2), (3), (4), we see that φ verifies

$$\Delta\varphi = 0 \ \text{ in } \ \Omega \ ; \qquad \frac{\partial \varphi}{\partial n}\Big|_{\Sigma} = \theta\left(x n_{\Sigma y} - y n_{\Sigma x}\right) \ ; \qquad \frac{\partial \varphi}{\partial n}\Big|_{\Gamma} = \zeta + \theta x$$

We are going to seek φ in the form

$$\varphi(x, y, t) = \varphi^*(x, y, t) + \theta(t)\Phi(x, y) \ ,$$

φ^* and Φ being solutions of the Neumann's problems

$$\Delta\varphi^* = 0 \ \text{ in } \ \Omega \ ; \quad \frac{\partial \varphi^*}{\partial n}\Big|_{\Sigma} = 0 \ ; \quad \frac{\partial \varphi^*}{\partial n}\Big|_{\Gamma} = \zeta \ ; \quad \int_\Gamma \varphi^*_{|\Gamma} d\Gamma = 0 \ ;$$

$$\Delta\Phi = 0 \ \text{ in } \ \Omega ; \quad \frac{\partial \Phi}{\partial n}\Big|_{\Gamma+\Sigma} = x n_y - y n_x \ ; \quad \int_\Gamma \Phi_{|\Gamma} d\Gamma = 0 \ .$$

$\Phi(x, y)$ is a Joukowski's potential, depending on the form of Ω .

On the other hand, if $\zeta \in \tilde{L}^2(\Gamma)$, it is well-known that the first problem has a generalized solution

$$\varphi^* \in \tilde{H}^1(\Omega) = \left\{ \varphi^* \in H^1(\Omega), \ \int_\Gamma \varphi^*_{|\Gamma} d\Gamma = 0 \right\}$$

and only one and that

$$\varphi^*_{|\Gamma} = \mathrm{K}\zeta \ ,$$

K beingan operator bounded from $\tilde{L}^2(\Gamma)$ into $\tilde{L}^2(\Gamma)$, self-adjoint, positive definite and compact.

From the Euler's equation (1), we deduce

$$p = -\rho\ddot{\varphi} + c(t)$$

So that

$$p_{|\Gamma} = -\rho\left(\mathrm{K}\ddot{\zeta} + \ddot{\theta}\ \Phi_{|\Gamma}\right) + c(t) \ .$$

Using the condition (6), we obtain the equation

$$-\rho\left(\mathrm{K}\ddot{\zeta} + \ddot{\theta}\ \Phi_{|\Gamma}\right) + c(t) = \rho g\left(\zeta + \theta x\right) - \tau\zeta''$$

Integratingon Γ, we calculate $c(t)$ and we obtain finally a first equation between $\theta(t)$ and $\zeta(x, t)$

(11) $$\rho\left(\mathrm{K}\,\ddot{\zeta}+\Phi_{|_\Gamma}\ddot{\theta}\right)+\rho g\left(\zeta+\theta x_{|_\Gamma}\right)-\tau\zeta''+\frac{\tau}{\sigma R\sin\alpha}\left[\zeta\left(\frac{\sigma}{2}\right)+\zeta\left(-\frac{\sigma}{2}\right)\right]=0$$

5.2) Now, we transform the left-hand side of the equation of the moment of momentum.
We have

$$\int_\Omega \rho\left(x\ddot{u}_y-y\ddot{u}_x\right)d\Omega=\int_\Omega \rho\left[\frac{\partial(x\ddot{\varphi})}{\partial y}-\frac{\partial(y\ddot{\varphi})}{\partial x}\right]d\Omega$$
$$=\rho\int_{\partial\Omega}\ddot{\varphi}\left(xn_y-yn_x\right)d\left(\partial\Omega\right)\quad ,$$

i.e

$$\int_\Omega \rho\left(x\ddot{u}_y-y\ddot{u}_x\right)d\Omega=\rho\int_{\partial\Omega}\ddot{\varphi}\frac{\partial\Phi}{\partial n}d\left(\partial\Omega\right)$$

$\ddot{\varphi}$ and Φ being harmonic, we can write

$$\int_{\partial\Omega}\ddot{\varphi}\frac{\partial\Phi}{\partial n}d\left(\partial\Omega\right)=\int_{\partial\Omega}\Phi\frac{\partial\ddot{\varphi}}{\partial n}d\left(\partial\Omega\right)=\int_{\partial\Omega}\Phi\left(\frac{\partial\ddot{\varphi}^*}{\partial n}+\ddot{\theta}\frac{\partial\Phi}{\partial n}\right)d\left(\partial\Omega\right)$$

and finally

$$\int_{\partial\Omega}\ddot{\varphi}\frac{\partial\Phi}{\partial n}d\left(\partial\Omega\right)=\int_\Gamma \Phi_{|_\Gamma}\ddot{\zeta}\,d\Gamma+\ddot{\theta}\int_\Omega \overrightarrow{\mathrm{grad}}^2\Phi d\Omega$$

Setting

$$I+\rho\int_\Omega \overrightarrow{\mathrm{grad}}^2\Phi d\Omega=I_0\quad ;\qquad md-m_\ell y_{G_\ell}=m_0a_0\ \left(m_0=m+m_\ell\ ;\ a_0>0\right),$$

we obtain another equation for $\theta(t)$ and $\zeta(x,t)$.

(12) $$\rho\int_\Gamma \Phi_{|_\Gamma}\ddot{\zeta}\,d\Gamma+I_0\ddot{\theta}+m_0a_0g\theta+\rho g\int_\Gamma x\zeta\,d\Gamma=0$$

5.3) We transform the equation (11) by introducing the unbounded operator A_0 of $\tilde{L}^2(\Gamma)$ defined by

$$\mathrm{A}_0\,\zeta=-\tau\zeta''+\rho g\zeta+\frac{\tau}{\sigma R\sin\alpha}\left[\zeta\left(\frac{\sigma}{2}\right)+\zeta\left(-\frac{\sigma}{2}\right)\right]$$
$$D\left(\mathrm{A}_0\right)=\left\{\zeta\in H^2(\Gamma);\ \int_\Gamma \zeta\,d\Gamma=0\ ;\ \zeta'\left(\pm\frac{\sigma}{2}\right)=\pm\frac{1}{R\sin\alpha}\zeta\left(\pm\frac{\sigma}{2}\right)\right\}$$

It is easy to verify that

$$\left(\mathrm{A}_0\,\zeta,\tilde{\zeta}\right)_{\tilde{L}^2(\Gamma)}=a_0\left(\zeta,\tilde{\zeta}\right)\ \forall\zeta,\quad \tilde{\zeta}\in D\left(\mathrm{A}_0\right),$$

so that theFriedrich's extension of A_0, denoted still by A_0 is the unbounded operator associated to the form $a_0\left(\zeta,\tilde{\zeta}\right)$, continuous and coercive in

$\tilde{H}^1(\Gamma)\times\tilde{H}^1(\Gamma)$ and the pair $\left(\tilde{H}^1(\Gamma),\tilde{L}^2(\Gamma)\right)$.

Then the equation (11) takes the form

$$\rho\left(\mathrm{K}\,\ddot{\zeta}+\Phi_{|\Gamma}\,\ddot{\theta}\right)+\rho g x_{|\Gamma}\,\theta+\mathrm{A}_0\,\zeta=0 \tag{13}$$

In order to obtain the equations with bounded operators, we set

$$\mathrm{A}_0^{1/2}\,\zeta=\eta\in\tilde{L}^2(\Gamma).$$

Applying $\mathrm{A}_0^{-1/2}$ to the equation (13), we obtain

$$\rho\left(\mathrm{A}_0^{-1/2}\,\mathrm{K}\,\mathrm{A}_0^{-1/2}\,\ddot{\eta}+\mathrm{A}_0^{-1/2}\,\Phi_{|\Gamma}\,\ddot{\theta}\right)+\rho g\,\mathrm{A}_0^{-1/2}\,x_{|\Gamma}\,\theta+\eta=0 \tag{14}$$

and the equation (12) can be written

$$\rho\left(\mathrm{A}_0^{-1/2}\,\Phi_{|\Gamma},\ddot{\eta}\right)_{\tilde{L}^2(\Gamma)}+I_0\ddot{\theta}+m_0a_0g\theta+\rho g\left(\mathrm{A}_0^{-1/2}\,x_{|\Gamma},\eta\right)_{\tilde{L}^2(\Gamma)}=0 \tag{15}$$

6. The eigenvalues equation

We set

$$\mathrm{N}=\mathrm{A}_0^{-1/2}\,\mathrm{K}\,\mathrm{A}_0^{-1/2}\ .$$

N is obviously compact from $\tilde{L}^2(\Gamma)$ into $\tilde{L}^2(\Gamma)$ and, since

$$\left(\mathrm{N}\eta,\tilde{\eta}\right)_{\tilde{L}^2(\Gamma)}=\left(\mathrm{K}\,\zeta,\tilde{\zeta}\right)_{\tilde{L}^2(\Gamma)}\ ,$$

it is self-adjoint and positive definite.

Therefore, N has a countable infinity of positive eigenvalues

$$\mu_1^2\geq\mu_2^2\geq\ldots\geq\mu_n^2\geq\ldots\ \mu_n^2\to 0 \text{ when } n\to\infty$$

If $\varphi_n(\mathrm{x})$, $n=1,2,\ldots$ are the orthonormalizedeigenfunctions, we have

$$\mathrm{N}\varphi_n=\mu_n^2\varphi_n\ ,\qquad n=1,2,\ldots$$

Since $\eta(x,t)$, $\mathrm{N}\eta(x,t)$, $\mathrm{A}_0^{-1/2}\,x_{|\Gamma}$, $\mathrm{A}_0^{-1/2}\,\Phi_{|\Gamma}$ belong to $\tilde{L}^2(\Gamma)$, we can write

$$\eta(x,t)\sim\sum_n\eta_n(t)\varphi_n(x);\qquad \mathrm{N}\eta(x,t)\sim\sum_n\mu_n^2\eta_n(t)\varphi_n(x);$$

$$\mathrm{A}_0^{-1/2}\,x_{|\Gamma}\sim\sum_n\alpha_n\varphi_n(x);\qquad \mathrm{A}_0^{-1/2}\,\Phi_{|\Gamma}\sim\sum_n\beta_n\varphi_n(x)$$

(the sign $\sim$ indicates the convergence in $\tilde{L}^2(\Gamma)$)

The equation (14) gives the denumerable infinity of equations

$$\rho\left(\mu_n^2\ddot{\eta}_n+\beta_n\ddot{\theta}\right)+\rho g\alpha_n\theta+\eta_n=0\ ,\qquad n=1,2,\ldots \tag{16}$$

and the equation (15) takes the form

(17) $$\rho\sum\beta_n\ddot{\eta}_n + I_0\ddot{\theta} + m_0 ag\theta + \rho g\sum\alpha_n\eta_n = 0$$

Seeking the solutions of (16), (17) in the form $\theta(t) = \theta_0 e^{i\omega t}$, $\eta_n(t) = \eta_{0n}e^{i\omega t}$, θ_0 and the η_{0n} being constants and ω being real, we obtain the system of a countable infinity of linear equations for θ_0 and the η_{0n}.

(18) $$\begin{cases} \theta_0\left(1-\dfrac{I_0\omega^2}{m_0 a_0 g}\right)+\dfrac{\rho}{m_0 ag}\sum\eta_{0n}\left(g\alpha_n-\omega^2\beta_n\right)=0 \\ \theta_0\rho\left(g\alpha_n-\beta_n\omega^2\right)+\eta_{0n}\left(1-\rho\mu_n^2\omega^2\right)=0; \qquad n=1,2,... \end{cases}$$

The infinite determinant of this system is

$$\begin{vmatrix} 1-\dfrac{I_0\omega^2}{m_0a_0g} & \dfrac{\rho\left(g\alpha_1-\omega^2\beta_1\right)}{m_0a_0g} & \dfrac{\rho\left(g\alpha_2-\omega^2\beta_2\right)}{m_0a_0g} \cdots & \dfrac{\rho\left(g\alpha_n-\omega^2\beta_n\right)}{m_0a_0g} & \cdots \\ \rho\left(g\alpha_1-\beta_1\omega^2\right) & 1-\rho\mu_1^2\omega^2 & 0 & 0 & \cdots \\ \rho\left(g\alpha_2-\beta_2\omega^2\right) & 0 & 1-\rho\mu_2^2\omega^2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho\left(g\alpha_n-\beta_n\omega^2\right) & 0 & 0 & 1-\rho\mu_n^2\omega^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

It is a Von Koch's determinant (Riesz, 1913) and it is absolutely convergent because the series
$\sum\mu_n^2$, $\sum\alpha_n^2$, $\sum\beta_n^2$, $\sum\alpha_n\beta_n$ are convergent.
Equaling to zero, we obtain the equation for the eigenvalues ω_n^2 , $n = 1, 2, \ldots$.
Truncating this determinant, it is possible to calculate the first eigenvalues.

Acknowledgements. The authors are grateful to the referee and the editorial board for some useful comments that improved the presentation of the paper.

References

[1] N.D. Kopachevskii and S.G. Krein, *Operator approach to linear problems of hydrodynamics*, Vol. **1**, Birkhauser, Basel, 2001.

[2] N.N. Moiseyev and V.V. Rumyantsev,*Dynamic Stability of Bodies Containing Fluid*, Springer, Berlin, 1968.

[3] H. J-P. Morand, R.Ohayon, *Interactions fluides-structures*. Masson, Paris, 1992.

[4] F. Riesz , *Les systèmes d'équations linéaires à une infinité d'inconnues*. Gauthier Villars, Paris, 1913.

[5] J. Sanchez Hubert, E. Sanchez Palencia, *Vibration and coupling of continuous systems- Asymptotic methods*. Springer Verlag, Berlin, 1989.

Received: July 2, 2014