Axial Compression of a Transversely Isotropic Incompressible Rectangular Rubber Ring

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Abstract

The problem of finite deformation is examined for a rectangular rubber ring composed of a class of transversely isotropic incompressible neo-Hookean materials, where both ends of the ring are subjected to a uniform axial compression load. A system of implicit analytical solutions is derived by using the method of integral transformation and the incompressibility constraint. Then, the influences of axial load, structure parameters and anisotropy parameter of the material on finite deformation of the ring are discussed in detail by numerical examples. Specially, both the axial displacement and the radial displacement of the ring increase with the increasing value of the anisotropy parameter, as well as the axial compression ratio and the axial displacement.

Keywords: transversely isotropic; rectangular rubber ring; axial compress; finite deformation

1 Introduction

As is well known, hyperelastic materials such as rubber and rubber-like materials play an extremely important role in the production of human society and

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Economic advantages for their high elasticity, abrasion resistance, excellent insulation. The deformation problems of various structures of such materials have aroused extensive attention from the scholars [1]. Meanwhile, with the advancement of technology, transversely isotropic materials and anisotropic materials emerge in our vision frequently [2]. In all structures composed of the rubber material, rubber ring is used widely. Specially, the rectangular rubber ring has its own characteristics, such as resistance to twisting forces, finite deformation under high pressures, and so on. The finite deformation problems of rectangular rubber rings gradually become a hot research field based on the hyperelastic theory [3-5].

In this paper, we examine the finite deformation problem of a rectangular rubber ring composed of a class of transversely isotropic incompressible neo-Hookean materials under a uniform axial compression load at its both ends. The relationships of finite deformation among axial load, structure parameters and anisotropy parameter are obtained. The influences of these parameters on finite deformation of the ring are discussed qualitatively by numerical simulations.

2 Mathematical Model

Here we are concerned with the finite deformation problem of a rectangular rubber ring composed of a hyperelastic material under a uniform axial compression load. Let \((R, \Theta, Z)\) and \((r, \theta, z)\) be the cylindrical polar coordinates in the reference and current configurations, respectively. Under the assumption of axially symmetric deformation, according to the nonlinear field theory, the deformation configuration is given by

\[
\begin{align*}
B \leq r(R, Z) &
\leq A, \quad 0 = \Theta, \quad z = z(R, Z), -L \leq Z \leq L,
\end{align*}
\]

where \(B\) and \(A\) are the radii of the inner and the outer surfaces in the reference configuration, respectively, \(Z = 0\) is the central cross-section and \(L\) is a half of the initial axial height of the ring. The schematic description of the structure is shown in Fig. 1, in which the portions of deep color represent the rectangular rubber ring.

![Fig. 1 The schematic description of the structure](image)

This work considers that the rectangular rubber ring is composed of a class of transversely isotropic incompressible neo-Hookean materials and that the corresponding strain energy function is given by [2].
Axial compression

\[ W = \frac{\mu}{2} [(I_1 - 3) + a(I_3^2 - 2I_3 + 1)], \]  

(2)

where \( \mu \) is the shear modulus and \( a \) is the anisotropy parameter, \( I_1 = trC \), \( I_3 = C_{11} \), \( C = F^TF \) is the right Cauchy-Green strain tensor, \( F \) is the deformation gradient tensor, as follows:

\[ F = r_r e_r \otimes E_R + r_z e_r \otimes E_Z + (r/R)e_\theta \otimes E_\theta + z_r e_z \otimes E_R + z_z e_z \otimes E_Z, \]  

(3)

in which \( E_R, E_\theta, E_Z \) and \( e_r,e_\theta,e_z \) are the orthogonal bases associated with the cylindrical polar coordinates \((R,\Theta,Z)\) and \((r,\theta,z)\), respectively. Particularly, it should be noted that the subscript letter of \( r \) or \( z \) indicates the partial derivative with respect to it, such as \( r_z = \partial r/\partial Z \).

The Cauchy stress tensor \( \sigma \) and the Piola-Kirchhoff stress tensor \( S \) associated with the material (2) are respectively given by

\[ \sigma = \begin{pmatrix} N \end{pmatrix} B, \quad S = T F F^T, \]  

(4)

in which \( I \) is the unit tensor, \( p = p(R,Z) \) is the hydrostatic pressure related to the incompressibility constraint, \( B = F F^T \) is the left Cauchy-Green strain tensor, \( N_{i,j} = F_{i,j} F_{j,i}, i,j = 1,2,3 \), \( W_q = \partial W/\partial q, q = 1,S \).

In the absence of body force, the equilibrium equation \( \text{div} S = 0 \) can be reduced to

\[ \begin{align*}
&\left(S_{rr}\right)_r + \left(S_{zz}\right)_z + R^{-1} (S_{rr} - S_{zz}) = 0, \\
&\left(S_{rZ}\right)_z + \left(S_{Zr}\right)_z + R^{-1} S_{rZ} = 0.
\end{align*} \]  

(5)

Firstly, the inner and the outer surfaces of the rubber ring are traction free in the undeformed configuration, the following equations are valid, i.e.,

\[ S_{rr}(B) = 0, \quad S_{rr}(A) = 0, \quad S_{rZ}(B) = 0, \quad S_{rr}(A) = 0. \]  

(6)

Moreover, assume that both ends of the ring are fixed on rigid bodies, and so the connected areas are not variable during the course of deformation, this requires that

\[ r(R,-L) = r(R,L) = R. \]  

(7)

Since the inner surface of the rubber ring is assumed to be supported by a rigid cylinder which sufficiently contacts and lubricates the ring, the inner radius of the ring is invariant, namely,

\[ r(B,Z) = B, \quad -L \leq Z \leq L. \]  

(8)

Finally, both ends of the rubber ring are subjected to a uniform axial load \( q \), it leads to

\[ \int_0^{2\pi} \int_B^L [\sigma_{zz}]_{z=\pm L} RdRd\Theta = q \pi (A^2 - B^2). \]  

(9)

3 Solutions

Since both ends of the rectangular rubber ring considered in this paper is fixed on rigid bodies and the initial axial height of the ring is very short, each
cross-section of the ring is always assumed to be a plane and be perpendicular to
the $Z$-axis, mathematically,
\[ z(R,Z) = z(Z). \]  
(10)
Using Eq. (10) and the incompressibility constraint $I_3 = \det C = 1$, we have
\[ r(R,Z) = \lambda^{1/2}(R^2 + C)^{1/2}, \]  
(11)
where both $\lambda = \lambda(Z) = z_Z(Z)$ and $C = C(Z)$ are functions with respect to $Z$.
It is easy to derive from the boundary condition (8) that
\[ C(Z) = B^2(\lambda - 1), \]  
(12)
Substituting the component of $S$ into Eq.(5), using Eq.(11) and the
incompressibility constraint $1 3 \frac{\partial}{\partial z} = 0$, we get
\[ \left( \mu \lambda_z + rR^{-1}(r_z p_R - r_R p_Z) \right) = 0, \]
(13)
In what follows, it is convenient to introduce the following dimensionless
notations
\[ m = B/L, \quad n = A/B, \quad H = Z/L, \quad h = z/l, \quad Q = q(n^2 - 1)/2\mu, \quad D^*_3 = D^*_3/B^2. \]  
(23)
Using the method of integral transformation to Eq.(13) and using the boundary
conditions, the system of implicit analytic solutions are as follows:
\[ \frac{m}{\lambda(H)} \left( \frac{I(\lambda(w),m,n,a)}{K(\lambda(w),m,n,a)} \right)^{\frac{1}{3}} d\lambda(w) = 1 - H, \quad 0 \leq H \leq 1; \]  
(13)
\[ -\frac{m}{\lambda(H)} \left( \frac{I(\lambda(w),m,n,a)}{K(\lambda(w),m,n,a)} \right)^{\frac{1}{3}} d\lambda(w) = -1 + H, \quad -1 \leq H \leq 0. \]  
(14)
where $I(\lambda(H),m,n,a) = (n^2 - 1)^2 - 2\lambda(n^2 - 1) + 2\lambda^2 \ln \frac{n^2 + \lambda - 1}{\lambda}$,
\[ K(\lambda(H),m,n,a) = 8\lambda^3(n^2 - 1) + 16\lambda^3(n^2 - 1) + 8\lambda^3 \ln \frac{n^2 + \lambda - 1}{\lambda(n^2 - 1)^2} - 8\lambda^3 \ln \frac{n^2 + \lambda - 1}{\lambda} \]
\[ - 16a\lambda^3 \ln \frac{n^2 + \lambda - 1}{\lambda} + 32a\lambda^3 \ln \frac{n^2 + \lambda - 1}{\lambda} + 192a\lambda^3 \ln \frac{n^2 + \lambda - 1}{(n^2 - 1)^2} + 128a\lambda^3 \ln \frac{n^2 + \lambda - 1}{(n^2 - 1)^2} \]
\[ + \frac{96a\lambda^3}{(n^2 - 1)^2} \ln \frac{n^2 + \lambda - 1}{\lambda} + 16a\lambda^3 \ln \frac{n^2 + \lambda - 1}{\lambda} + 8a\lambda(n^2 - 1) - 16a\lambda^2(n^2 - 1) - 104a\lambda^2 \]
\[ - \frac{24a\lambda^3}{(n^2 - 1)^2(n^2 + \lambda - 1)^2} - \frac{48a\lambda^3}{(n^2 - 1)^2(n^2 + \lambda - 1)^2} - \frac{72a\lambda^3}{(n^2 - 1)^2(n^2 + \lambda - 1)^2} + \frac{24a\lambda^3}{(n^2 - 1)^2(n^2 + \lambda - 1)^2} \]
\[ - \frac{144a\lambda^3}{n^2 - 1} - 8a - \frac{96a^4\lambda^3}{(n^2 - 1)^3} - \frac{32an^2\lambda^3}{n^2 - 1} + 64a^4\lambda^2 \ln \frac{n^2 + \lambda - 1}{\lambda} + \frac{32an^2\lambda^3}{(n^2 - 1)^3} - 32an^2\lambda^2 \]
\[ - \frac{16an^4\lambda^2}{n^2 - 1} + \frac{32an^4\lambda^3}{(n^2 - 1)^3} - 32Q\lambda^3 - 32D^*_3\lambda^3. \]  
(15)
4 Numerical simulations and result analyses

Next, we discuss the influences of the dimensionless axial load $Q$, the length $m$, the thickness $n$ and the anisotropy parameter $a$ of the material on finite deformation of the rectangular rubber ring in detail, as well as the axial compression ratio and the axial displacement. Numerical simulations are as follows.

As shown in Fig. 2, it is found that the more $Q$ increases, the more the lateral surface extends along the radial direction and both ends of the ring shrink along the axial direction, which is similar to experiment results, as well as with the increasing value of $a$, which shows the influence of the anisotropy parameter. Moreover, with the decrease value of $m$, the expansion of the profile along radial direction is more and more obvious, the shrinkage degree of both ends along the axial direction also increases.

In Fig. 3, curves of $\lambda(H) \sim H$ and $H - h \sim H$ are also plotted. It is found that the axial compression ratio is the smallest at the central cross-section of the ring;
however, it changes very fast near the two ends and achieves the maximum at both ends, which agrees with the change of the whole ring. Furthermore, $\lambda$ increases with the increasing value of $a$, which shows the influence of the anisotropy parameter on the axial compression ratio. Moreover, the axial displacement is zero at the central cross-section; the axial displacement achieves the maximum at the endpoints. Similarly, with the increasing value of $a$, the axial displacement also increases.

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