Exact Solutions of Nonlinear Partial Differential Equations in Mathematical Physics Using the \((G'/G)\)-Expansion Method

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Abstract

In this paper, we explore new applications of the \((G'/G)\)-expansion method to the nonlinear general Burgers-Fisher, the \(K(n, n)\) and the \(K(n + 1, n + 1)\) equations where the balance numbers of which are not positive integers. Then, exact traveling wave solutions involving parameters are obtained. When these parameters are taken special values, the solitary waves are derived from the traveling waves.

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1 Introduction

In recent years, nonlinear wave equations have been played essential roles in many scientific and engineering areas such as fluid mechanics, nonlinear optics, quantum mechanics and so on. Thus, it has had a considerable attention to find explicit traveling wave solutions of those problems. Several methods have been presented to obtain new exact solutions for many nonlinear
equations such that the homogeneous balance method [1], the tanh-function method [2-6] the Jacobi elliptic function method [7], the F-expansion method [8], the auxiliary equation method [9-12], the \((G'/G)\)-expansion method [13-14], the variational iteration method [15], the homotopy perturbation method [16], the exp-function method [17,18], the sine-cosine method [19], the improved \((G'/G)\)-expansion method [20], Spectral collocation method \([21]\) and so on.

Wang et al [13] have introduced the \((G'/G)\)-expansion method to look for exact traveling wave solutions of nonlinear evolution equations where the balance numbers of which are positive integers. The motivation of the present article is to explore the possibilities of solving nonlinear equations where the balance numbers of which are not positive integers using the \((G'/G)\)-expansion method.

The objective of this article is to use the \((G'/G)\)-expansion method to solve directly the nonlinear general Burgers-Fisher, the \(K(n, n)\) and the \(K(n+1, n+1)\) equations \([21, 23-27]\) where the balance numbers of which are not positive integers, without using some special transformation. This idea has been done in Zhang’s article \([22]\) which has been applied to different nonlinear equations. These equations play a significant role in many fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics, optical fibers and so on.

2 Description of the \((G'/G)\)-expansion method

Suppose that a nonlinear equation is given by

\[
F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1}
\]

where \(u = u(x,t)\) is an unknown function, \(F\) is a polynomial in \(u = u(x,t)\) and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

The main steps of the \((G'/G)\)-expansion method \([13,14, 20, 22]\) are the following:

**Step1.** The traveling wave variable

\[
u(x, t) = u(\xi), \quad \xi = x - ct, \tag{2.2}
\]

where \(c\) is a constant, permits us reducing Eq. (2.1) into an ODE in the form

\[
P(u, u', u'', ...) = 0. \tag{2.3}
\]
Step2. Suppose that the solution of Eq. (2.3) can be expressed by a polynomial in \(\left(\frac{G'}{G}\right)\) as follows:

\[
\begin{align*}
    u(\xi) &= \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \\
    \quad \text{(2.4)}
\end{align*}
\]

where \(G = G(\xi)\) is the solution of the second order ODE in the form

\[
\begin{align*}
    G'' + \lambda G' + \mu G = 0, \\
    \quad \text{(2.5)}
\end{align*}
\]

where \(\alpha_i, \lambda, \mu\) are constants to be determined and \(\alpha_m \neq 0\), while \(m\) is called the balance number.

Step3. The positive integer \(m\) can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3).

Step4. Substituting (2.4) into (2.3) and using the ODE (2.5), collecting all terms with the same order of \(\left(\frac{G'}{G}\right)\) together, we get a polynomial in \(\left(\frac{G'}{G}\right)\). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations, which can be solved to get \(\alpha_i, c, \lambda\) and \(\mu\).

Step5. Since the general solution of Eq. (2.5) is well known to us, then substituting \(\alpha_i, c\) and the general solutions of (2.5) into (2.4), we have traveling wave solutions of Eq. (2.1).

In the subsequent sections, we will illustrate the validity and reliability of this method with some nonlinear evolution equations in mathematical physics where the balance number “\(m\)” in (2.4) is not positive integer.

3 Applications

In this section, we apply the \(\left(\frac{G'}{G}\right)\)-expansion method to find the traveling wave solutions for the nonlinear general Burgers-Fisher, the \(K(n, n)\) and the \(K(n+1, n+1)\) equations with power law nonlinearities where the balance numbers of which are not positive integers.

3.1 Example 1. The general Burgers-Fisher equation

Let us consider the following general Burger’s-Fisher equation [21, 23, 24]:

\[
\begin{align*}
    u_t + au^nu_x + bu_{xx} + du(1 - u^n) &= 0, \\
    \quad \text{(3.1)}
\end{align*}
\]
where \( a, b \) and \( d \) are nonzero constants. The traveling wave variable (2.2) permits us converting Eq. (3.1) into the following ODE:

\[
-cu' + au^n u' + bu'' + du - du^{n+1} = 0.
\]  

(3.2)

Consider the homogeneous balance between \( u'' \) and \( u^{n+1} \) in (3.2) we get \( m = \frac{2}{n} \). It should be noted that \( m \) is not a positive integer. However, with reference to [22] we may still choose the solution of Eq. (3.2) in the form

\[
 u(\xi) = A \left( \frac{G'}{G} \right)^{\frac{2}{n}}, n > 0,
\]  

(3.3)

where \( A \) is a constant to be determined and \( G \) satisfies Eq. (2.5). Substituting (3.3) into (3.2) we obtain the polynomial

\[
\left[ 2Anc\mu + 2Ab\lambda\mu(4-n) \right] \left( \frac{G'}{G} \right)^{\frac{n+2}{n}} + \left[ Adn^2 + 2Anc\lambda + 4Ab\lambda^2 + 8Ab\mu \right] \left( \frac{G'}{G} \right)^{\frac{2n+2}{n}} + \\
\left[ 2Anc + 2Ab\lambda(4+n) - 2aA^{1+n}\lambda \right] \left( \frac{G'}{G} \right)^{\frac{3n+2}{n}} + \left[ 2Ab(2+n) - A^{1+n}dn^2 - 2aA^{1+n}\lambda \right] \left( \frac{G'}{G} \right)^{\frac{2n+2}{n}} = 0.
\]  

(3.4)

On equating the coefficients of the polynomial (3.4) to be zero, we get a system of algebraic equations, which can be solved by Mathematica to obtain the following results:

\[
 A = \lambda^{-\frac{2}{n}}, \quad c = -\frac{dn(4+n)}{2\lambda(2+n)}, \quad \mu = 0, \quad a = 0, \quad b = \frac{dn^2}{\lambda^2(4+2n)}, \quad \lambda \neq 0.
\]  

(3.5)

From (3.3) and (3.5) we obtain the exact traveling wave solution

\[
 u(\xi) = \left[ \frac{c_2e^{-\lambda\xi}}{c_1 + c_2e^{-\lambda\xi}} \right]^{\frac{2}{n}},
\]  

(3.6)

where \( c_1 \) and \( c_2 \) are arbitrary constants. If we set \( c_1 = c_2 \) in (3.6) we obtain the solitary wave solution

\[
 u(\xi) = \left[ \frac{1}{2} - \frac{1}{2}\tanh \left( \frac{1}{2} \lambda\xi \right) \right]^{\frac{2}{n}},
\]  

(3.7)

where \( \xi = x + t \left[ \frac{dn(4+n)}{2\lambda(2+n)} \right] \).
3.2 Example 2. The $K(n,n)$ equation

In this section, we consider the following $K(n,n)$ equation [25]:

$$u_t + a(u^n)_x + b(u^n)_{xxx} = 0, \quad n > 1.$$  \hfill (3.8)

The traveling wave variable (2.2) gives the following ODE:

$$-cu'u^{3-n} + anu^2u' + bnu^2u'' + 3nb(n-1)uu'u'' + nb(n-1)(n-2)u^3 = 0. \hfill (3.9)$$

Consider the homogeneous balance between $u'u^{3-n}$ and $u^2u''$ in (3.9), we get $m = \frac{2}{1-n}$. We choose the solution of Eq. (3.9) in the form

$$u(\xi) = B \left( \frac{G'}{G} \right)^{\frac{2}{1-n}}, \quad n > 1,$$  \hfill (3.10)

where $B$ is a constant to be determined and $G$ satisfies Eq. (2.5). Substituting (3.10) into (3.9), we have the polynomial

$$\left( \frac{G'}{G} \right)^6 \left[ 4nbB^3 \mu^3 \left(-1 + 5n - 6n^2 \right) \right] + \left( \frac{G'}{G} \right)^{\frac{12-6n}{1-n}} \left[-4nbB^3(1+n) \right]$$

$$+ 2B^{4-n}c(1 - 2n + n^2)] + \left( \frac{G'}{G} \right)^{\frac{11-5n}{1-n}} \left[-6B^3bn\lambda(1+2n+n^2) \right]$$

$$+ 2B^{4-n}c\lambda(1 - 2n + n^2)] + \left( \frac{G'}{G} \right)^{\frac{10-4n}{1-n}} \left[-2aB^3n(1 - 2n + n^2) \right]$$

$$- 2B^3bn\lambda^2(1 + 4n + 7n^2) - 4B^3bn\mu(1 + n + 4n^2) + 2B^{4-n}c\mu$$

$$+ \left( \frac{G'}{G} \right)^{\frac{9-3n}{1-n}} \left[-2aB^3n\lambda(1 - 2n + n^2) - 8B^3bn^3\lambda^3 \right]$$

$$+ \left( \frac{G'}{G} \right)^{\frac{8-2n}{1-n}} [2B^3n\lambda(1 - 2n + n^2) - nb(n-1)(n-2)u^3]$$

where $B$ is a constant to be determined and $G$ satisfies Eq. (2.5). Substituting (3.10) into (3.9), we have the polynomial

$$\left( \frac{G'}{G} \right)^6 \left[ 4nbB^3 \mu^3 \left(-1 + 5n - 6n^2 \right) \right] + \left( \frac{G'}{G} \right)^{\frac{12-6n}{1-n}} \left[-4nbB^3(1+n) \right]$$

$$+ 2B^{4-n}c(1 - 2n + n^2)] + \left( \frac{G'}{G} \right)^{\frac{11-5n}{1-n}} \left[-6B^3bn\lambda(1+2n+n^2) \right]$$

$$+ 2B^{4-n}c\lambda(1 - 2n + n^2)] + \left( \frac{G'}{G} \right)^{\frac{10-4n}{1-n}} \left[-2aB^3n(1 - 2n + n^2) \right]$$

$$- 2B^3bn\lambda^2(1 + 4n + 7n^2) - 4B^3bn\mu(1 + n + 4n^2) + 2B^{4-n}c\mu$$

$$+ \left( \frac{G'}{G} \right)^{\frac{9-3n}{1-n}} \left[-2aB^3n\lambda(1 - 2n + n^2) - 8B^3bn^3\lambda^3 \right]$$

where $B$ is a constant to be determined and $G$ satisfies Eq. (2.5). Substituting (3.10) into (3.9), we have the polynomial

$$\left( \frac{G'}{G} \right)^6 \left[ 4nbB^3 \mu^3 \left(-1 + 5n - 6n^2 \right) \right] + \left( \frac{G'}{G} \right)^{\frac{12-6n}{1-n}} \left[-4nbB^3(1+n) \right]$$

$$+ 2B^{4-n}c(1 - 2n + n^2)] + \left( \frac{G'}{G} \right)^{\frac{11-5n}{1-n}} \left[-6B^3bn\lambda(1+2n+n^2) \right]$$

$$+ 2B^{4-n}c\lambda(1 - 2n + n^2)] + \left( \frac{G'}{G} \right)^{\frac{10-4n}{1-n}} \left[-2aB^3n(1 - 2n + n^2) \right]$$

$$- 2B^3bn\lambda^2(1 + 4n + 7n^2) - 4B^3bn\mu(1 + n + 4n^2) + 2B^{4-n}c\mu$$

$$+ \left( \frac{G'}{G} \right)^{\frac{9-3n}{1-n}} \left[-2aB^3n\lambda(1 - 2n + n^2) - 8B^3bn^3\lambda^3 \right]$$

where $B$ is a constant to be determined and $G$ satisfies Eq. (2.5).
\[-4B^3bn\lambda\mu(1-2n+13n^2)] + \left(\frac{G'}{G}\right)^{\frac{8-2n}{1-n}} [-4B^3an\mu(1-2n+n^2)] \]

\[-2B^3bn\lambda^2\mu(1-8n+19n^2) - 4B^3bn\mu^2(1-5n+10n^2)] + \]

\[\left(\frac{G'}{G}\right)^{\frac{7-n}{1-n}} [-6B^3bn\lambda\mu^2(1-6n+9n^2)] = 0. \quad (3.11)\]

On equating the coefficients of the polynomial (3.11) to be zero, we get a system of algebraic equations, which can be solved, to obtain the following results:

\[a = 0, \quad \mu = 0, \quad \lambda = 0, \quad B = \left(\frac{2bn(1+n)}{(n-1)^2c}\right)^{\frac{1}{1-n}}. \quad (3.12)\]

Consequently, we obtain the exact traveling wave solution

\[u(\xi) = \left[\frac{2bn(1+n)c^2}{(n-1)^2c(c_1\xi+c_2)^2}\right]^{\frac{1}{1-n}}. \quad (3.13)\]

If we set \(c_1 = c_2\) in (3.13), we obtain the solitary wave solution

\[u(\xi) = \left[\frac{2bn(1+n)}{(n-1)^2c(\xi+1)^2}\right]^{\frac{1}{1-n}}, \quad (3.14)\]

where \(\xi = x - ct\).

### 3.3 Example 3. The \(K(n + 1, n + 1)\) equation

Let us consider the following \(K(n + 1, n + 1)\) equation[26, 27]:

\[u_t + a\left(u^{n+1}\right)_x + b\left(u^n\right)_{xx} = 0. \quad (3.15)\]

Consequently, we get the following ODE:

\[-cu'u^{2-n} + a(n+1)u^2u' + bnu^2u'' + bn(3n-2)uu'u'' + \]
\[(n - 1)^2 bnu'^3 = 0. \quad (3.16)\]

Consider the homogeneous balance between \( u' u^{2-n} \) and \( u^2 u''' \) in (3.16) we get \( m = -\frac{2}{n} \). We choose the solution of Eq. (3.16) in the form

\[u(\xi) = E \left( \frac{G'}{G} \right)^{-\frac{2}{n}}, \quad n > 0, \quad (3.17)\]

where \( E \) is a constant to be determined and \( G \) satisfies Eq. (2.5). From (3.17) and (3.16) we have the following polynomial:

\[
\left( \frac{G'}{G} \right)^{-\frac{3n-6}{n}} [12E^3 b\mu^3 (1 + 2n)] + \left( \frac{G'}{G} \right)^{-\frac{2n-6}{n}} [E^3 b\lambda \mu^2 (32 + 54n)] + \\
\left( \frac{G'}{G} \right)^{-\frac{n-6}{n}} [2aE^3 \mu (1 + n) + E^3 b\lambda^2 \mu (28 + 38n) + E^3 b\mu^2 (28 + 40n)] + \\
\left( \frac{G'}{G} \right)^{\frac{n-6}{n}} [2aE^3 (1 + n) + 2bE^3 \lambda^2 (10 + 7n) + 2bE^3 \mu (10 + 8n) - 2E^{3-n}c\mu] + \\
\left( \frac{G'}{G} \right)^{\frac{2n-6}{n}} [2E^5 b\lambda (8 + 3n) - 2E^{3-n}c\lambda] + \left( \frac{G'}{G} \right)^{\frac{3n-6}{n}} [4E^3 b - 2E^{3-n}c] + \\
\left( \frac{G'}{G} \right)^{\frac{n-6}{n}} [2aE^3 \lambda (1 + n) + 8bE^3 \lambda^3 (1 + n) + bE^3 \lambda \mu (48 + 52n)] = 0. \quad (3.18)\]

On equating the coefficients of the polynomial (3.18) to be zero, we get a system of algebraic equations which can be solved to obtain the following results:

\[a = 0, \quad \mu = 0, \quad \lambda = 0, \quad E = \left( \frac{2b}{c} \right)^{-\frac{1}{n}}. \quad (3.19)\]

Consequently, we obtain the exact traveling wave solution
\[ u(\xi) = \left[ \frac{2bc^2}{c(c_1\xi + c_2)^2} \right]^{-\frac{1}{n}}. \]  

(3.20)

If we set \( c_1 = c_2 \) in (3.20), we obtain the solitary wave solution

\[ u(\xi) = \left[ \frac{2b}{c(\xi + 1)^2} \right]^{-\frac{1}{n}}, \]  

(3.21)

where \( \xi = x - ct. \)

4 Conclusions

The applications of the \((G'/G)\)-expansion method are still limited to nonlinear evolution equations where the balance numbers of which are positive integers, (see for example [13,14,20]). Recently, Zhang [22] has presented a new application of this method to some nonlinear evolution equations where the balance numbers of which are not positive integers and has obtained new types of exact traveling wave solutions. In this article, we have followed the work of Zhang [22] to present a wider applicability for handling several types of the nonlinear general Burgers- Fisher, the \(K(n, n)\) and the \(K(n + 1, n + 1)\) equations where the balance numbers of which are not positive integers. We have obtained exact traveling wave solutions involving parameters. The solitary waves follow from the traveling waves, when these parameters are taken special values.

References


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