Stability at Principal Resonance of Multi-
Parametrically andExternally
Excited Mechanical System

Y. A. Amer* and M. Sayed**

*Department of Mathematics and Statistics, Faculty of Science, Taif University, El-Taif, El-Haweiah, P.O. Box 888, Zip Code 21974, Kingdom of Saudi Arabia

*Department of Math., Faculty of Science, Zagazig University, Zagazig, Egypt

**Department of Eng. Math., Faculty of Electronic Engineering
Menouf University 32952, Egypt
moh_6_11@yahoo.com

Abstract

A general parametrically and externally excited mechanical system is considered. The response of one-degree-of freedom, non-linear system under multi-parametric and external excitation forces simulating the vibration of the cantilever beam is studied. The solution of this system up to and including second order approximation is determined applying the multiple time scale perturbation. The stability at principal parametric resonance has been studied applying both frequency response function and phase-plane methods. The effects of the different parameters on the vibrating system are investigated. Both available and applicable resonance cases are reported.

Keywords: Stability, Response equation, Multi excitation force, Phase plane

1. INTRODUCTION

Systems of weakly non-linear oscillations have been investigated and studied for quite a long period. Various methods such as averaging, harmonic balance and multiple scales have been applied to obtain an approximate or semi closed form solutions for such systems.
El-Nagger and Alhanadwah [1], studied the principal parametric resonance for one-degree-of-freedom system with quadratic and cubic nonlinearities under parametric excitation only. Eissa [2], investigated the non-linear mechanical oscillators subjected to parametric and excitation forces. Eissa and El-Bassiouny [3] obtained the analytical and numerical solutions of non-linear rolling response of a ship in regular beam seas. Eissa [4] and Eissa and El-Gananwi [4, 5] studied some mechanical systems under parametric and external forces Al-Qaisia and Hamdan [6], studied the steady state periodic response of single-mode and two-modes having the same period as the excitation of strongly non-linear oscillators using harmonic balance and multiple time scales methods. Queini and Nayfeh [7] studied the problem of suppressing the vibration of a structure of a cantilever beam subjected to a principal parametric excitation.

Mahmoud et. al. [8], studied the periodic solutions of parametrically excited complex non-linear dynamical systems using the generalized averaging method. Kamel and Amer [9] investigated a system of non-linear differential equations representing the different vibrating modes of a cantilever beam having both non-linear quadratic damping and cubic stiffness subject to multi self-excitation. Eissa and Amer [10] studied the vibration control of a cantilever beam under external and parametric excitation using cubic velocity feedback.

Ioannis et al. [11] studied the problem of nonlinear resonance of a hinged-hinged slender marine structure due to parametrically imposed motion. They focuses on a specific case when the excitation frequency is equal to double of the structure's first lateral natural frequency. Kamel [12] investigated the stability of the periodic solution of the vibration of nonlinear coupled Van Der Pol oscillations under external and parametric excitation.

In this paper, we consider a general model describing parametrically and externally excited mechanical systems [13-18], given by the following differential equation

\[ \ddot{X} - \varepsilon \mu \dot{X} + \varepsilon \alpha X^3 \dot{X} - \varepsilon \gamma X^4 \dot{X} + \omega^2 X - \varepsilon \beta X^3 + \varepsilon \delta X^5 + 2 \varepsilon X \cos(\Omega t) \\
- 2 \varepsilon^2 \beta X^3 \cos(\Omega t) + 2 \varepsilon^2 X^5 \cos(\Omega t) = \varepsilon f [\cos(3t) + \cos(5t) + \ldots] \quad (1) \]

where \( \varepsilon \) is a small perturbation parameter, \( \alpha, \beta, \delta, \gamma \) and \( \mu \) are system parameters. Equation (1) may represent the rolling motion of a ship in the regular beam seas or the longitudinal waves [13-17] or the motion of a single mode of a flexible beam, or plates [18] under combined axial and transverse excitations.

2. PERTURBATION ANALYSIS

A general uniform expression of the analytical solution is of equation (1) is assumed in the form:

\[ X(t; \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k x_k(T_0, T_1) \quad (2) \]
where $T_0 = t$ is a fast time scale associated with changes occurring at the frequencies $\omega$, $\Omega$, and $T_i = \epsilon t$ is a slow time scale associated with modulations in the amplitude and phase caused by the non-linearities of damping and parametric excitation. In addition, we have the time derivatives as:

$$\frac{d}{dt} = D_0 + \epsilon D_1,$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2$$

where $D_j = \frac{\partial}{\partial T_j}$, $j = 0, 1$. From equation (2), we have

$$\dot{X}(t; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k (D_0 + \epsilon D_1)x_k$$  \hspace{1cm} (3)

and

$$\ddot{X}(t; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k (D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2)x_k$$  \hspace{1cm} (4)

Substituting equations (2), (3) and (4) into equation (1), equating the coefficients of the same order of $\epsilon$ in both sides, we get the following set of ordinary differential equations:

$$\epsilon^0 : (D_0^2 + \omega^2) x_0 = 0$$  \hspace{1cm} (5)

$$\epsilon^1 : (D_0^2 + \omega^2) x_1 = -2D_0 D_1 x_0 + \mu(D_0 x_0) - \alpha x_0^2(D_0 x_0) + \gamma x_0^4(D_0 x_0) + \beta x_0^3 - \delta x_0^5 - 2x_0 \cos(\Omega t) + f \cos t + \frac{f \cos 3t}{3} + \frac{f \cos 5t}{5}$$  \hspace{1cm} (6)

$$\epsilon^2 : (D_0^2 + \omega^2) x_2 = -2D_0 D_1 x_1 - D_1^2 x_0 + \mu(D_1 x_1) + \mu(D_0 x_0) - \alpha x_0(x_0D_0 x_0) + x_0 D_0 x_1 + 2x_1 D_0 x_0 + \gamma x_0^4(D_1 x_0) + D_0 x_1 + 4\gamma x_0^4 x_1(D_0 x_0) + 3\beta x_0^5 - 2x_1 \cos \Omega t - 5\delta x_0^5 x_1 + 2\beta x_0^3 \cos \Omega t - 2\beta x_0^3 \cos \Omega t$$  \hspace{1cm} (7)

The general solution of equation (5) can be express in the form:

$$x_0(T_0, T_1) = A(T_1) \exp(i\omega T_0) + \overline{A}(T_1) \exp(-i\omega T_0)$$  \hspace{1cm} (8)

where $A$ and $\overline{A}$ are a complex conjugate functions in $T_1$. Substitution equation (8) into equation (6), we get:

$$(D_0^2 + \omega^2)x_0 = \left[-2i\omega D_0 A - i\mu \omega A - i\alpha \omega A^3 A^2 + 2i\gamma \omega A^4 A^2 + 3\beta A^2 A_0 - 10\delta A^4 A^2 \exp(i\omega T_0) \right] + \left[-i\omega A^3 + 3i\gamma \omega A^4 A + \beta A^3 - 5\delta A^4 A \exp(3i\omega T_0) - A^3[i\gamma \omega - \delta] \exp(5i\omega T_0) \right] - A \exp(i(\Omega + \omega)T_0) + \overline{A} \exp(i(\Omega - \omega)T_0) + \frac{f}{2} \left[\exp(it) \frac{\exp(3it)}{3} + \frac{\exp(5it)}{5} \right] + cc$$  \hspace{1cm} (9)

For a bounded solution of equation (9), the coefficient of the secular terms (the coefficient of $\exp(\pm i\omega T_0)$), should be eliminated, then the first order approximation is given
\[ x_1(T_0, T_1) = A_1 \exp(i \omega T_0) + E_1 \exp(3i \omega T_0) + E_2 \exp(5i \omega T_0) + E_3 \exp(i (\Omega + \omega) T_0) + E_4 \exp(i (\Omega - \omega) T_0) + E_5 \exp(it) + E_6 \exp(3it) + E_7 \exp(5it) + \text{cc} \] (10)

where \( A_1 \) and \( E_i, \ i = 1, 2, ..., 7 \) are complex functions in \( T_1 \).

From equation (8) and (10) into equation (7), we obtain a bounded solution the secular terms should be eliminated, then the second order approximation is given

\[ x_2(T_0, T_1) = A_2 \exp(i \omega T_0) + E_8 \exp(2i \omega T_0) + E_9 \exp(3i \omega T_0) + E_{10} \exp(4i \omega T_0) \]

\[ + E_{11} \exp(5i \omega T_0) + E_{12} \exp(6i \omega T_0) + E_{13} \exp(7i \omega T_0) + E_{14} \exp(8i \omega T_0) \]

\[ + E_{15} \exp(i (\Omega + \omega) T_0) + E_{16} \exp(i (\Omega - \omega) T_0) + E_{17} \exp(i (\Omega + 2\omega) T_0) \]

\[ + E_{18} \exp(i (\Omega - 2\omega) T_0) + E_{19} \exp(i (\Omega + 3\omega) T_0) + E_{20} \exp(i (\Omega - 3\omega) T_0) \]

\[ + E_{21} \exp(i (\Omega + 4\omega) T_0) + E_{22} \exp(i (\Omega - 4\omega) T_0) + E_{23} \exp(i (\Omega + 5\omega) T_0) \]

\[ + E_{24} \exp(i (\Omega - 5\omega) T_0) + E_{25} \exp(i \Omega T_0) + E_{26} \exp(i (1 + \omega) T_0) \]

\[ + E_{27} \exp(i (1 - \omega) T_0) + E_{28} \exp(i (1 + 2\omega) T_0) + E_{29} \exp(i (1 - 2\omega) T_0) \]

\[ + E_{30} \exp(i (1 + 3\omega) T_0) + E_{31} \exp(i (1 - 3\omega) T_0) + E_{32} \exp(i (3 + \omega) T_0) \]

\[ + E_{33} \exp(i (3 - \omega) T_0) + E_{34} \exp(i (3 + 2\omega) T_0) + E_{35} \exp(i (3 - 2\omega) T_0) \]

\[ + E_{36} \exp(i (3 + 3\omega) T_0) + E_{37} \exp(i (3 - 3\omega) T_0) + E_{38} \exp(i (5 + \omega) T_0) \]

\[ + E_{39} \exp(i (5 - \omega) T_0) + E_{40} \exp(i (5 + 2\omega) T_0) + E_{41} \exp(i (5 - 2\omega) T_0) \]

\[ + E_{42} \exp(i (5 + 3\omega) T_0) + E_{43} \exp(i (5 - 3\omega) T_0) + E_{44} + \text{cc} \] (11)

where \( A_2 \) and \( E_i, \ i = 8, 9, ..., 44 \) are complex functions in \( T_1 \). From the above analysis, the general solution of equation (1) is given by:

\[ X(t, \varepsilon) = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) + \varepsilon^2 x_2(T_0, T_1) + O(\varepsilon^3) \]

From the above-proposed solution, many resonance cases can be deduced. The reported resonance cases are:

i- Trivial Resonance Cases: \( \Omega = \omega = 0 \)

ii- Primary Resonance Cases: \( \Omega = \omega \)

iii- Sub-Harmonic Resonance Cases: (Principal parametric resonance \( \Omega = \Omega = 2\omega \))

and \( \Omega \equiv n \omega \ (n = 3, 4, 5, 6) \)

iv- Combined Resonance Cases: \( \Omega \equiv \omega \equiv 1, \Omega \equiv \omega \equiv 2, \Omega \equiv \omega \equiv 3 \)

v- Simultaneous Resonance Cases: any combination of the two or more resonance cases.

Here, we investigate the stability at principal parametric resonance case where \( \Omega = 2\omega \). We introduce a detuning parameter \( \sigma \) such that,

\[ \Omega - \omega = \omega + \varepsilon \sigma \Rightarrow \Omega = 2\omega + \varepsilon \sigma \] (12)

From equation (9) the secular terms are eliminated, using equation (12) and dividing by \( \exp(i\omega T_0) \), we get the solvability condition as:

\[ 2i \omega D_1 A - i \mu B A + i \alpha \omega A T_0 - 2i \omega \gamma A^2 T_0 - 3\beta A^2 T_0 + 10\delta A T_0^2 + A \exp(i \varepsilon \sigma T_0) = 0 \] (13)

Substituting the polar form \( A = \frac{1}{2} a(T_1) \exp(i \beta(T_1)) \) into equation (13), and separating real and imaginary parts we get the following equations:
Stability at principal resonance

\[ \omega a \left( \frac{\theta' - \sigma}{2} \right) - \frac{3}{8} \beta a^3 + \frac{5}{8} \delta a^5 + \frac{a}{2} \cos \theta = 0 \]  
(14)

\[ \omega a' - \frac{\mu \omega}{2} a + \frac{\alpha}{8} a^3 - \frac{\gamma}{8} \omega a^5 + \frac{a}{2} \sin \theta = 0 \]  
(15)

where, \( \theta = \sigma T - 2 \beta_1 \).

For steady-state solutions, \( a' = \theta' = 0 \), and equations (14) and (15) becomes:

\[ 4 \omega a \sigma + 3 \beta a^3 - 5 \delta a^5 = 4 a \cos \theta \]  
(16)

\[ 4 \mu \omega a - \alpha a + \gamma \omega a^5 = 4 a \sin \theta \]  
(17)

From equations (16) and (17) the frequency response equation is given by:

\[ 16 \omega^2 \sigma^2 + (24 \beta^2 a^2 - 40 \omega \delta a^4) \sigma + (25 \delta^2 + \gamma^2 \omega^2) a^8 - (30 \beta \delta + 2 \alpha \gamma \omega) a^6 \\
+ (9 \beta^2 + \alpha^2 - 8 \alpha \mu \omega) a^4 - 8 \alpha \mu \omega^2 a^2 + 16 \omega^2 \mu^2 - 16 = 0 \]  
(18)

### 3. STABILITY ANALYSIS

To determine the stability of the fixed points, one lets

\[ a = a_0 + a_1 \quad \text{and} \quad \theta = \theta_0 + \theta_1 \]  
(19)

where \( a_0 \) and \( \theta_0 \) correspond to a linear solutions and \( a_1 \) and \( \theta_1 \) are perturbations which are assumed to be small compared with \( a_0 \) and \( \theta_0 \). Substituting from equation (19) into equations (14) and (15), we have

\[ (a'_0 + a'_1) - \frac{\mu}{2} (a_0 + a_1) + \frac{\alpha}{8} (a_0 + a_1)^3 - \frac{\gamma}{16} (a_0 + a_1)^5 + \frac{1}{2 \omega} (a_0 + a_1) \sin (\theta_0 + \theta_1) = 0 \]  
(20)

\[ \frac{1}{2} (\sigma - \sigma'_0 - \sigma'_1) (a_0 + a_1) + \frac{3 \beta \omega}{8} (a_0 + a_1)^3 - \frac{5 \delta a^5}{16} (a_0 + a_1)^5 - \frac{1}{2 \omega} (a_0 + a_1) \cos (\theta_0 + \theta_1) = 0 \]  
(21)

Since \( a_1 \) and \( \theta_1 \) are very small, \( a_0 \) and \( \theta_0 \) are solution of equations (14) and (15), then equations (20) and (21), can be written as:

\[ a'_0 - \frac{\mu}{2} a_1 + \frac{3 \alpha}{8} a_0 a^3_1 - \frac{5 \gamma}{16} a^5_0 a_1 + \frac{1}{2 \omega} a_0 \cos \theta_0) \theta_1 + \frac{1}{2 \omega} \sin \theta_0) a_1 = 0 \]  
(22)

\[ \frac{1}{2} (\sigma - \sigma'_0) a_0 - \frac{1}{2} a_0 \sigma'_1 + \frac{9 \beta \omega}{8} a^3_1 + \frac{25 \delta a^5}{16} a^3_0 a_1 + \frac{1}{2 \omega} a_0 \sin \theta_0) \theta_1 - \frac{1}{2 \omega} \cos \theta_0) a_1 = 0 \]  
(23)

Using equations (16) and (17), we get

\[ a'_0 = \Gamma_1 a_0 - \Gamma_2 \theta_1 = 0 \]  
(24)

\[ \theta'_0 = \Gamma_3 a_0 + \Gamma_4 \theta_1 = 0 \]  
(25)

where,

\[ \Gamma_1 = \frac{a_0^2}{4} (\gamma a_0^2 - \alpha) \quad \Gamma_2 = - \frac{a_0}{2} (\sigma + 3 \beta \omega a_0^2 - \frac{5 \delta}{8} a_0^4) \quad \Gamma_3 = \frac{\omega a_0}{2} (3 \beta - 5 \delta a_0^2) \quad \text{and} \]

\[ \Gamma_4 = (\mu - \frac{\alpha}{4} a_0^2 + \frac{\gamma}{8} a_0^4) \]

Equations (24) and (25), can written in matrix form
The eigen equation of the above system is
\[
(\lambda - \Gamma_1)(\lambda - \Gamma_4) - \Gamma_2\Gamma_3 = 0
\]
i.e.
\[
\lambda^2 - (\Gamma_1 + \Gamma_4)\lambda + \Gamma_1\Gamma_4 - \Gamma_2\Gamma_3 = 0
\]
Equation (27) has the following eigenvalues:
\[
\lambda = \frac{1}{2}[(\Gamma_1 + \Gamma_4) \pm \sqrt{(\Gamma_1 + \Gamma_4)^2 + 4(\Gamma_2\Gamma_3 - \Gamma_1\Gamma_4)}]
\]
And hence the fixed points are unstable if and only if \( \Gamma_2\Gamma_3 - \Gamma_1\Gamma_4 > 0 \), otherwise they are stable.

4. RESULTS AND DISCUSSION

The harmonic response of the one-degree-of-freedom non-linear system under both multi-parametric and external excitations is studied. The solution of this system is determined up to and including the second order approximation by applying the multiple time scale perturbation. The steady state solution and its stability are determined and representative numerical results are included. The stability zone and effects of the different parameters are discussed using frequency response equation. The stability of the numerical solution is studied also using the phase-plane method. Some of the resulting resonance cases are confirmed applying well-known numerical techniques. The effects of the different parameters on the vibrating system behavior are investigated and discussed.

4.1) Frequency Response Curves

The frequency response equation (18) is a non-linear algebraic equation in the amplitude \( a \). This equation is solved numerically and the results are shown in Figs.1-7, which representing the variation of the amplitude \( a \) against the detuning parameter \( \sigma \) for the given values of other parameters. In all Figures, the solid line represents stable solutions and the dotted line represents the unstable solutions. Figure 1 represent the frequency response curve for principle parametric resonance for the parameters, \( \alpha = 0.6, \beta = 0.03, \gamma = 0.02, \omega = 0.8, \mu = 0.01, \delta = 0.06 \). In Fig.1, the steady state amplitude has a continuous curve, which bent to the right. So, it has hardening effect and multi-valued solutions with jump phenomena. The continuous curve has unstable and stable solutions. For decreasing value of the parameter \( \alpha \), we show that the region of stability is decreased and the response amplitude, region of multi-valued solutions are decreased, as illustrated in Fig. 2(a). For increasing value of \( \alpha \), the stability region and response amplitude are increased and decreased respectively as shown in Fig. 2(b). The multi-valued
region disappear for increasing $\alpha$. As $\beta$ is increased, the response amplitude and the stability zone are increase as shown in Fig. 3(a) and 3(b).

Fig. 1. Variation of the amplitude of the response with the detuning parameter $\sigma$ for $\alpha = 0.6, \beta = 0.03, \gamma = 0.02, \delta = 0.06, \mu = 0.01, \omega_i = 0.8$.

Fig. 2. Variation of the amplitude of the response with the detuning parameter $\sigma$ for a) $\alpha = 0.06$ b) $\alpha = 1.2$. 

Fig. 3. Variation of the amplitude of the response with the detuning parameter $\sigma$ for
a) $\beta = 0.3$  
 b) $\beta = 0.6$

Fig. 4. Variation of the amplitude of the response with the detuning parameter $\sigma$ for
a) $\gamma = 0.7$  
 b) $\gamma = 1.2$

Fig. 5. Variation of the amplitude of the response with the detuning parameter $\sigma$ for
a) $\omega = 0.4$  
 b) $\omega = 0.2$
For large value of $\beta$ ($\beta = 0.6$) the continuous curve is bent to the left and have softening phenomena. Fig. 4(a) and 4(b) shows that for increasing value of $\gamma$, the region of stability is decreased and the multi-valued region disappears. Also, the response amplitude is decreased. Fig. 5(a) and 5(b) shows that, for decreasing value of the natural frequency $\omega$ the region of stability is increase. Also for small value of $\omega$ ($\omega = 0.2$), the multi-valued region disappears. As $\mu$ is increased the region of stability is decreased and the response amplitude is increased as shown in Fig. 6(a). The zone of stability is increased for negative value of $\mu$ as shown in Fig. 6(b). For increasing value of parameter $\delta$ the region of stability and the response amplitude are decrease as shown in Fig. 7(a). As $\delta$ takes negative value the curve is bend to the left and have softening phenomena as illustrated in Fig. 7(b). The region of stability and the response amplitude are decrease.
4.2) Numerical Solutions

The behavior of the given system has been studied applying Runge-Kutta fourth order method. The numerical solution and its stability are obtained as shown in Fig. 8. From this figure, it can be notice that the steady state amplitude is about 10 times that of the excitation amplitude $f$ and the phase-plane shows a fine limit cycle.

4.2.1) Effects study of different parameters

**Damping coefficient** $\mu$; The steady state amplitude is monotonic increasing function in damping coefficient $\mu$, with tuned oscillations as shown in Fig. 9a. This is attributed to the negative sign of the damping term. It represents an excitation force rather than a restoring one.

**Non-linear parameter** $\alpha$; For small values of $\alpha$, the oscillations are tuned and the system shows the beating phenomenon as shown in Fig. 9b.

**Non-linear parameter** $\beta$; From Fig. 9c, we have the steady state amplitude is a monotonic increasing function in the non-linear coefficient $\beta$. The increase is due to the soft spring effects.

**Non-linear parameter** $\delta$; From Fig. 9d, the steady state amplitude is a monotonic decreasing excitation amplitude $\delta$. The decrease is due to the hard spring effects.

![Fig. 8](image)

**Fig. 8.** Numerical solution of the time response at selected values (basic case):

$$\mu = 0.04, \alpha = 0.8, \gamma = 0.05, \beta = 0.05, \delta = 0.3, \omega = 2.0, \Omega = 3.5, f = 0.05$$
4.2.2) Resonance Cases

a) *primary resonance case*: $\Omega \cong \omega$

From Fig. 10a, it can be seen that at primary the steady state amplitude is increased to about 200% of the basic case in Fig. 8.

b) *Sub-harmonic (Principal parametric) resonance case*: $\Omega \cong 2\omega$

From Fig. 10b, it can be seen that the steady state amplitude is increased to about 240% of the basic case in Fig. 8.

c) *Combined resonance cases*:

For the case $\Omega \cong 2\omega$ and $\Omega - \omega \cong 3$, it can be shown that from Fig. 10c, the steady state amplitude is increased to about 180% of the basic case in Fig. 8. But for the case $\Omega \cong 2\omega$ and $\Omega - \omega \cong 5$, the steady state amplitude is increased to about 240% of the basic case in Fig. 8, as shown in Fig 10d. The latter two cases have small values of the natural frequency, which leads to the increase of the steady state amplitudes due to the system-reduced stiffness.

It is clear that all investigated resonance cases are harmful to the system. This denotes that the system needs to control the oscillations, which will be dealt with in a future work.
Y. A. Amer and M. Sayed

Fig. 10. Some resonance cases

a) Primary resonance $\Omega \cong \omega$

b) Sub-harmonic (Principle) resonance $\Omega \cong 2\omega$

c) Simultaneous resonance $\Omega \cong 2\omega$ and $\Omega - \omega \cong 3.0$

d) Simultaneous resonance $\Omega \cong 2\omega$ and $\Omega - \omega \cong 5.0$

5. CONCLUSIONS

The harmonic response of the one-degree-of-freedom non-linear system under multi-parametric and external excitation forces is studied. The solution of this system up to and including the second order approximation is determined applying the multiple time scale perturbation. The steady state solution and its stability are determined and representative numerical results, which are in excellent agreement with references [10] and [3-5] are reported. The main results can be concluded as following:

1- The frequency response equation (18) is solved numerically to get the stable and unstable solutions.

2- The stability zone increased as the parameter $\beta$ and natural frequency $\omega$ decreased. Moreover, this zone decreased as the parameters $\gamma, \mu$ increased.

3- The multi-valued region disappears for increasing parameters $\gamma, \alpha$ and for decreasing natural frequency $\omega$. 
4- The steady state amplitude is a monotonic increasing function for increasing of the parameter $\beta$, and a monotonic decreasing for increasing $\gamma$.
5- The continuous curve is bent to the left and have softening phenomena for increasing parameter $\beta$ and for negative value of $\delta$.
6- All resonance cases lead the system to increased amplitude of about 200% of the basic case. This large value needs to be controlled for safe system operation.

REFERENCES


Received: September, 2010