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Unsteady Two Phase Flow in a Doubly Connected Region

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Abstract

The unsteady flow of a fluid particle system in an eccentric catheterized artery is studied. The mathematical model involves the usual assumptions that the arterial segment is straight, arterial wall is rigid and impermeable. Blood is an incompressible Newtonian fluid and flow is fully developed. The flow rate is considered as a periodic function of time. The axial pressure gradient and velocity distribution in the eccentric catheterized artery are obtained as solution of the problem. The method involves mapping the annular domain D in x-y plane to an annular region bounded by concentric circles in ξ - η plane through conformal mapping. The axial pressure gradient as a function of time is estimated.

Keywords—catheterized artery, conformal mapping, eccentric annulus, two phase flow

1. INTRODUCTION

The insertion of a catheter in an artery will increase the frictional resistance to flow through the artery and will modify the pressure distribution. In order to know the catheter – induced errors it is necessary to study Navier Stokes equations governing the flow of blood in the catheterized artery. If the pressure gradient and the diameter of the artery remains unchanged on insertion of the catheter, then the rate of flow of the rate of flow of the blood through the artery would be reduced. The rate of flow, however, is governed by the need of the surrounding tissues for nutrients, such as oxygen. To meet the needs of the tissues, the artery dilates and pressure gradient increases, increasing the rate of flow to the required level. This process is known as auto regulation.

MacDonald [3] studied the pulsatile blood flow in a catheterized artery and obtained theoretical estimates for pressure gradient corrections for catheters which are positioned eccentrically as well as concentrically. Dash et.al. [9] have studied the changed flow pattern in a narrow artery when a catheter is inserted into it and estimated the increases in frictional resistance and non-Newtonian behaviour of blood has been considered by using casson model . Sarkar and Jayaraman [2] have studied the pulsatile flow in a catheterized stenosed artery and estimated the correction in the mean pressure drop along the stenosis due to catheterization. Daripa and Dash [6] have studied the pulsatile flow of blood in a straight eccentric catheterized artery by modeling blood as an incompressible Newtonian fluid, using the fast algorithm for solving Poisson equation inside a circular disk to an annular domain.

Blood has a liquid phase consisting of plasma in which a solid phase of suspended cells and protein particles are present. It is necessary to understand the effect of drag exerted by these particles on the flow. Following Saffmann [7] equations for unsteady flow of an incompressible fluid with inert particles suspended in it is considered in the present study.

Shivakumar and Ji [8] have analytically solved flow of Newtonian fluid in an eccentric annular flow by considering steady fully developed flow. Indira, Venkatachalappa and Siddheshwar [9] have studied the effect of couple stresses on flow in a doubly connected region. The approach of Shivakumar and Ji [8] is modified in the present study to obtain flow through an eccentric catheterized artery.

2. MATHEMATICAL FORMULATION

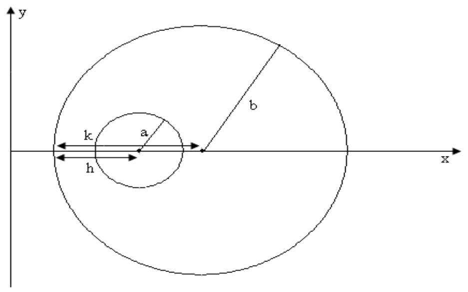

Fig. 1 Physical configuration

The cross section of the artery is shown in figure 1. The annular region Ω is bounded by the circles C_1 and C_2 given by $C_1 : (x-h)^2 + y^2 = a^2$ and $C_2 : (x-k)^2 + y^2 = b^2$, a is the radius of inner tube (catheter), b is the radius of outer tube (artery) a < b, h < k.

The fluid is considered to be incompressible and Newtonian and the flow is assumed to be due to a pulsatile axial pressure gradient $\frac{\partial p}{\partial z}$ with period $\frac{2\pi}{\omega}$. The velocity field is given by $(0 \quad 0 \quad w(x \quad y \quad t))$. It is assumed that the artery and the catheter are infinitely long neglecting end effects. Flow is considered as laminar and fully developed. The governing equation of motion along with the no-slip boundary condition are given by $\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + KN_0(q-w)$ in Ω (1)

$m\frac{\partial q}{\partial t} = K(w-q)$ in Ω (2)

w = 0 on c_1 and c_2 (3)

where ρ - density, μ - viscosity of the fluid, K- Stokes resistance coefficient, m – mass of the particle . The above equations are non-dimensionalised using b as reference length, P_s as reference pressure gradient, $\frac{P_s b^2}{\mu}$ as reference angular frequency we get $\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \tau\Gamma(q-w)$ (4)

$\frac{\partial q}{\partial t} = \tau(w-q)$ (5)

and w = 0 on c_1 and c_2 (6)

The rate of mass flow is assumed to be of the form $\sum_{j=0}^{n} Q_j e^{i\left(\gamma_j^2 t - \psi_j\right)}$ (7)

The values of γ_j and ψ_j are known. w is expressed as the sum of n+1 component velocities w_j, j = 0,1,2….n. The j^{th} velocity component w_j satisfies equation (3) and also $\iint_R w_j \, dx\, dy = Q_j e^{i\left(\gamma_j^2 t - \psi_j\right)}$ (8)

From (7) we get $w_j = W_j(x, \quad y)e^{i\gamma_j^2 t}$ (9)

and hence $z = \frac{c}{1-\zeta}$ (10)

where Pj is a constant. Using(4), (8) and (9) in (3) we get

$$\frac{\partial^2 W_j}{\partial x^2} + \frac{\partial^2 W_j}{\partial y^2} - i\lambda_j^2 W_j = -P_j \quad (11)$$

where $\lambda^2{}_j = \frac{\tau}{1+i\gamma_j} + 1$, subjected to boundary condition $W_j = 0$ on C_1 and C_2.

$$P_j = \frac{Q_j e^{-i\psi_j}}{\iint\limits_R W_j(x,y)dxdy}, \ \ j = 0,1,2.....n \qquad (12)$$

3. SOLUTION OF THE PROBLEM

Equation (9) is expressed in complex coordinates using $z = x + iy$ and $\bar{z} = x - iy$. Equation (9) takes the form $4\frac{\partial^2 W_j}{\partial z \partial \bar{z}} - i\lambda_j W_j = -P_j$ (13)

w_j is the velocity which is a real valued function. Therefore it is a function of $z\bar{z}$ and $z+\bar{z}$ can be written as $f(z\bar{z}) + g(z+\bar{z})$. Using separation of variables and solving the resulting equations we get

$$W_j = A_j I_0\left(2\gamma_j\sqrt{z\bar{z}}\right) + B_j K_0\left(2\gamma_j\sqrt{z\bar{z}}\right) + C_j \cosh\gamma_j(z+\bar{z}) + D_j \sinh\gamma_j(z+\bar{z}) - \frac{P_j}{4\gamma_j^2} \qquad (14)$$

4. CONFORMAL MAPPING

The mapping $z = \frac{c}{1-\zeta}$ where $z = x + iy$ and $\zeta = \xi + i\eta$ transforms conformally the ring space enclosed by two concentric circles with radius $\left(|\zeta| = \rho^2\right)$, ρ_1 and ρ_2 with $\rho_1 < \rho 2$ where $\rho_1 = \frac{a}{h}$ and $\rho_2 = \frac{1}{k}$ and $c = h - \frac{a}{h^2} = k - \frac{1}{k}$.

W_j becomes $W_j = \frac{-P_j}{4\gamma_j^2}\left[A_j\,\Theta_1(\zeta,\bar{\zeta}) + B_j\Theta_2(\zeta,\bar{\zeta}) + C_j\Theta_3(\zeta,\bar{\zeta}) + D_j\Theta_4(\zeta,\bar{\zeta}) - 1\right]$ (15)

The functions $\Theta_i(\zeta,\bar{\zeta}), i = 0,1,2,3,4$ are given in appendix

5. THE RATE OF FLOW

Using complex form of the Green's theorem $\iint\limits_R \frac{\partial F}{\partial \bar{z}} ds = \frac{1}{2i}\int\limits_{C_2 - C_1} F dz$ we get the components of rate of flow are given by

$$Q_j = \frac{-P_j}{8i\gamma_j^2}\int\limits_{C_2-C_1}\left[\frac{1}{\gamma_j}\sqrt{\frac{\bar{z}}{z}}\left\{AI_1\left(2\gamma_j\sqrt{z\bar{z}}\right) + BK_1\left(2\gamma_j\sqrt{z\bar{z}}\right)\right\}\right]dz + \frac{-P_j}{8i\gamma}\int\limits_{c_2-C_1}\left[\frac{1}{\gamma_j}\left\{C\sinh\gamma_j(z+\bar{z}) + D\cosh\gamma_j(z+\bar{z})\right\} - \bar{z}\right]dz \qquad (16)$$

Using the conformal mapping , completing the integration and simplifying we get

$$Q_j = \frac{P_j \pi c^2}{4\gamma_j^2}\left[A_j\Gamma_{1j} + B_j\Gamma_{2j} + C_j\Gamma_{3j} + D_j\Gamma_{4j} - \frac{\rho_2^2}{\left(1-\rho_2^2\right)^2} + \frac{\rho_2^2}{\left(1-\rho_2^2\right)^2}\right] \tag{17}$$

Constants are given in the appendix.

6. RESULTS AND DISCUSSIONS

The flow modeling in this study involves the usual assumptions that the arterial segment is straight, the arterial wall is rigid and impermeable, the artery and the catheter are infinitely long and the pressure sensing device is mounted on the catheter at a large distance upstream from the tip of the catheter.

The flux (flow rate) Q(t) is considered to be a prescribed periodic function of time following MacDonald [4] as $Q = \sum_{j=0}^{6} Q_j \cos(\omega_j t_A - \varphi_j)$ where Q_j and φ_j are given by Table 1.

Figures 2-3 show the plot of pressure gradient vs different values of eccentricity, drag parameter and different values of catheter radius ‘a’.

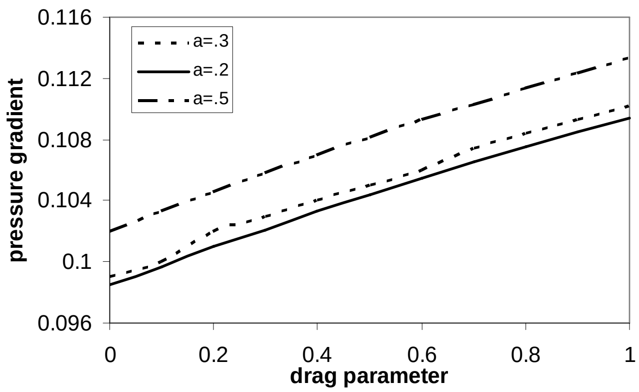

Fig. 2 Plot of pressure gradient vs drag parameter

The pressure gradient changes considerably by the introduction of catheter and increases with increase in radius of catheter. The pressure gradient decreases considerably with increase in eccentricity for a fixed value of a.

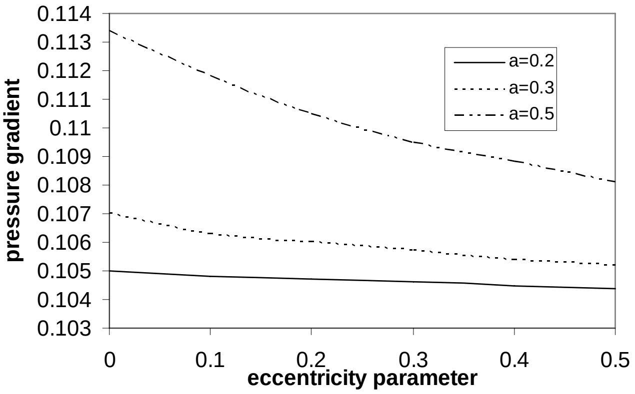

Fig. 3 Plot of pressure gradient vs eccentricity

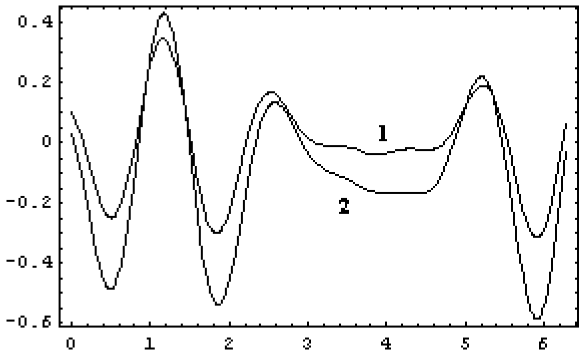

Fig. 4 Plot of rate of flow and pressure pulse for a = 0.3 and eccentricity ε = 0.5 (1) Q(t), (2) P(t)

The effect of drag parameter is to increase the pressure gradient in order to maintain a constant rate of flow. Figure 4 shows a comparison of pressure pulse with that of rate of flow.

As the insertion of the catheter in the artery in a rate of mass flow with reduced amplitude. To maintain the rate of mass flow the pressure gradient increase with the increase in catheter radius, as the rate of flow of blood through an artery is governed by the need of surrounding tissues for nutrients such as oxygen.

7. CONCLUSION

The results show that as eccentricity $\varepsilon \rightarrow 0$ the amplitude of pressure is high and reduces with increase in eccentricity. The effect of increase in drag parameter is to increase the pressure gradient.

Appendix

$$\Gamma_{1j} = \sum_{n=1}^{\infty} \frac{(\gamma_j c)^{2n}}{(n!)^2} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \alpha(n,m,k)\left\{\rho_2^{2m+2k} - \rho_1^{2m+2k}\right\};\ \Gamma_{2j}^{+} = \log\left[\frac{1-\rho_2^2}{1-\rho_1^2}\right]$$

$$\Gamma'_{2j} = \sum_{n=1}^{\infty} \frac{(\gamma_j c)^{2n}}{(n!)^2} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \alpha(n,m,k)\left\{\begin{matrix}\rho_2^{2m}(1+\rho_2^2)log(1-\rho_2^2)\\ -\rho_1^{2m}(1+\rho_1^2)log(1-\rho_1^2)\end{matrix}\right\}$$

$$\Gamma_{2j}^{*} = \sum_{n=1}^{\infty} \frac{(\gamma_j c)^{2n}}{(n!)^2} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha(n,m,k)}{l+1}\left\{\rho_2^{2m+2k} - \rho_1^{2m+2k}\right\}$$

$$\Gamma_{2j}^{**} = \sum_{n=1}^{\infty} \frac{(\gamma_j c)^{2n}}{(n!)^2} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left[\begin{matrix}\frac{\alpha(n,m,k)}{1+1}\psi_{1j}(n+1)\\ \times\left\{\rho_2^{2m+2k} - \rho_1^{2m+2k}\right\}\end{matrix}\right];\Gamma_{2j} = \Gamma_{2j}^{+} + \Gamma'_{2j} + \Gamma_{2j}^{*} + \Gamma_{2j}^{**}$$

$$\Gamma_{3j} = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \beta_{2j}(l,n,m,k)\left(\rho_2^{2m+2k} - \rho_1^{2m+2k}\right) \quad ;$$

$$\Gamma_{4j} = 1 + \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \beta_{2j}(l,n,m,k)\left(\rho_2^{2m+2k} - \rho_1^{2m+2k}\right) \qquad A_j = \frac{F_{21j} - F_{22j}}{F_{11j}F_{22j} - F_{12j}F_{21j}} \quad ;$$

$$B_j = \frac{F_{12j} - F_{11j}}{F_{11j}F_{22j} - F_{12j}F_{21j}};C_j = \frac{F_{42j} - F_{41j}}{F_{31j}F_{42j} - F_{32j}F_{41j}};D_j = \frac{F_{31j} - F_{32j}}{F_{31j}F_{42j} - F_{32j}F_{41j}}$$

$$F_{31j} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \beta_{1j}(l,n,m,0)\rho_1^{2m};\ F_{32j} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \beta_{1j}(l,n,m,0)\rho_2^{2m}$$

$$F_{41j} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \beta_{2j}(l,n,m,0)\rho_1^{2m};\ F_{42j} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \beta_{2j}(l,n,m,0)\rho_2^{2m}$$

$$\beta_{1j}(l,n,m,k) = \frac{(\gamma_j c)^{2l+2n}}{(l+n)!\,l!}\left\{\begin{matrix}\varphi(l+2n,m)\varphi(l,m+k)+\\ \varphi(2n+l,k+m)\varphi(l,m)\end{matrix}\right\} \quad ;$$

$$\beta_{1j}(l,n,m,k) = \frac{(\gamma_j c)^{2l+2n}}{(l+n)!\,l!}\left\{\begin{matrix}\varphi(l+2n,m)\varphi(l,m+k)+\\ \varphi(2n+l,k+m)\varphi(l,m)\end{matrix}\right\}$$

$$F_{22j} = -0.5772 + \sum_{n=1}^{\infty} \frac{(\gamma_j c)^{2n}}{(n!)^2}\left[\psi_{1j}(n)\sum_{m=0}^{\infty}\{\varphi(n,m)\}^2\rho_2^{2m} - \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{\alpha(n,m,k)}{k}\rho_1^{2m+2k}\right]$$

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