A dynamical two dimensional problem of generalized thermoelasticity has been considered to examine the stresses in an isotropic elastic slab. Laplace transform for time variable and exponential Fourier transform for space variable has been applied in the basic equations to form a vector-matrix differential equation which is then solved by eigenvalue approach. Finally numerical computations of the stresses has been made and presented graphically.

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**Nomenclature**

\( \lambda , \mu ' \) - Lame’s constants
\( c_{12}^2 \) - \( (\lambda + 2\mu ') / \rho \)
\( c_2^2 \) - \( \mu ' / \rho \)
\( \beta^2 \) - \( c_1 / c_2 \)
\( u' , v' \) - displacement components along \( x' \) and \( y' \) respectively
\( t' \) - time
\( T' \) - absolute temperature
\( T_0 \) - reference temperature so chosen that \( |(T' - T_0) / T_0| << 1 \)
\( \rho \) - density
\( \sigma_{ij} \) - stress components
\( \tau' \) - relaxation time parameter
\( c_E \) - specific heat at constant strain
\( K \) - thermal conductivity
\( \alpha_t \) - coefficient of linear thermal expansion
\( \gamma = (3\lambda + 2\mu) \alpha_t \)
\( \varepsilon = \gamma^2 T_0 / [\rho c_E (\lambda + 2\mu')] \)
\( \eta = \rho c_E / K \)

**Introduction**

Several problems of generalized thermoelasticity have been solved by Lord and Shulman [1] (L-S) with one relaxation time parameter and Green and Lindsay [2] (G-L) with two relaxation time parameter. In both the theories the conventional Fourier law of heat conduction has been modified to a hyperbolic type of equation which along with the equations of motion of thermoelasticity are considered for the solution of the problem. As such both the theory ensure finite speed of propagation of waves and eliminate automatically the paradox of infinite speed of propagation inherent in both the uncoupled and coupled theories of thermoelasticity vide Chandrasekharaiah [3] and Chandrasekharaiah and Murty [4]. Various problems of both the theories have been investigated and some interesting phenomena, have been revealed. A brief review of this topic have been given by Chandrasekharaiah [5]. Lord and Shulman [1] developed this L-S theory for isotropic media in the absence of heat source. Dhalwal and Sherief [6] extended this theory for anisotropic body in the presence of heat source, Noda et al [7] derived a formulation of generalized thermoelasticity that combines both Lord and Shulman (L-S) and Green and Lindsay (G-L) theories of generalized thermoelasticity for one dimensional problems.
Recently Green and Naghdi [8] developed a new generalized theory of thermoelasticity by including the so called thermal displacement gradient among the independent constitutive variables. An important characteristic feature of this theory does not accommodate dissipation of thermal energy.

In dealing with coupled or generalized thermoelasticity problems, the solution procedure is based on to choose a suitable thermoelastic potential function, but this approach has certain limitations as discussed in Bahar and Hetnarski [9]. Here we prefer to apply the eigenvalue approach as Das and Lahiri [10], Lahiri and Das [11] for the solution of the problem in which the physical quantities involved in the boundary and initial conditions are directly solvable from the governing equations.

In this paper we consider the generalized case of Tauchert and Akoz [12] by considering the corresponding dynamical problem in isotropic medium and solve by eigenvalue approach [Appendix – I]. Finally numerical computations of the stresses are made and presented graphically.

**Formulation of the problem**

We consider a plane strain problem in a homogeneous isotropic infinite slab with thickness $2h$. The origin is located at the middle plane of the slab, $x$-axis along the length and $y$-axis along the height of the slab.

Fig. 1-- Isotropic elastic slab

The non-dimensional displacement equations of motion and heat conduction equation in generalized thermoelasticity [13] are:

\[ T(x,h,t) = f(x,t) = T_0, \tau_{xy} = \tau_{yy} = 0 \]

\[ T(x,-h,t) = g(x,t) = 0, \tau_{xy} = \tau_{yy} = 0 \]
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\[ \beta^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (\beta^2 - 1) \frac{\partial^2 v}{\partial x \partial y} - \beta^2 \frac{\partial T}{\partial x} = \beta^2 \frac{\partial^2 u}{\partial t^2} \]  
(1)

\[ (\beta^2 - 1) \frac{\partial^2 u}{\partial x \partial y} + \beta^2 \frac{\partial^2 v}{\partial y^2} + \frac{\partial^3 v}{\partial x^2} - \beta^2 \frac{\partial T}{\partial y} = \beta^2 \frac{\partial^2 v}{\partial t^2} \]  
(2)

and

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial t} \left[ T + \varepsilon \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) \right] \]  
(3)

Also the non-dimensional stress-components are:

\[ \tau_{xx} = \beta^2 \frac{\partial u}{\partial x} + (\beta^2 - 2) \frac{\partial v}{\partial y} - \beta^2 T \]  
(4a)

\[ \tau_{yy} = (\beta^2 - 2) \frac{\partial u}{\partial y} + \beta^2 \frac{\partial v}{\partial y} - \beta^2 T \]  
(4b)

\[ \tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]  
(4c)

where the non-dimensional variables are:

\[ x = c_1 \eta \xi, \quad y = c_1 \eta \gamma, \quad u = c_1 \eta \nu, \quad v = c_1 \eta \nu, \quad t = c_1 \eta \tau, \quad \tau = c_1 \eta \tau, \quad \tau_{ij} = \tau_{ij} / \mu, \quad T = (T - T_0) / T_0 \]

**Method of Solution**

**Formulation of a Vector-Matrix Differential Equation**

We apply Laplace Transform with respect to time variable \( t \) defined by:

\[ [\tilde{u}, \tilde{v}, \tilde{T}](x, y, p) = \int_0^\infty (u, v, T)e^{-pt} dt \]

and then we apply Fourier Transform with respect to space variable \( x \) defined by:

\[ [\tilde{u}, \tilde{v}, \tilde{T}](\xi, y, p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\tilde{u}, \tilde{v}, \tilde{T})e^{-i\xi} dx \]

On using the above transformations to equations (1) to (4), we get the followings:
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\[- \beta^2 \xi^2 \dd{u}{y} + \frac{d^2 \dd{u}{y}}{dy^2} + i \xi (\beta^2 - 1) \frac{d\dd{v}{y}}{dy} - i \xi \beta^2 T = \beta^2 p^2 \dd{u}{y}\]  \hspace{1cm} (5)

\[i \xi (\beta^2 - 1) \frac{d\dd{u}{y}}{dy} + \beta^2 \frac{d^2 \dd{v}{y}}{dy^2} - \xi^2 \dd{v}{y} - \beta^2 \frac{d\dd{v}{y}}{dy} = \beta^2 p^2 \dd{v}{y}\]  \hspace{1cm} (6)

\[- \xi^2 \dd{T}{y} + \frac{d^2 \dd{T}{y}}{dy^2} = p(1 + \tau p) \left[ \dd{T}{y} + \epsilon \left( i \xi \dd{u}{y} + \frac{d\dd{v}{y}}{dy} \right) \right]\]  \hspace{1cm} (7)

The transformed stress components are as follows:

\[(\dd{\tau}_{xx})_y = i \xi \beta^2 \dd{u}{y} + (\beta^2 - 2) \frac{d\dd{v}{y}}{dy} - \beta^2 \dd{T}{y}\]  \hspace{1cm} (8a)

\[(\dd{\tau}_{xy})_y = i \xi (\beta^2 - 2) \dd{u}{y} + \beta^2 \frac{d\dd{v}{y}}{dy} - \beta^2 \dd{T}{y}\]  \hspace{1cm} (8b)

\[(\dd{\tau}_{xy})_x = \frac{d\dd{u}{y}}{dy} + i \xi \dd{v}{y}\]  \hspace{1cm} (8c)

To obtained equations (5) - (7) we assume that initially at time \(t=0\) the body is at rest and in an undeformed and unstressed state i.e., initially the displacement components \(u, v\) along with their derivatives with respect to \(t\) are zero and maintained at the reference temperature \(f(x)\) at the upper surface and \(g(x)\) on the lower surface so the following initial conditions hold:

\[u(x,y,0) = \frac{\partial u(x,y,0)}{\partial t} = 0, \hspace{0.5cm} v(x,y,0) = \frac{\partial v(x,y,0)}{\partial t} = 0\]

and \[\frac{\partial T(x,y,0)}{\partial t} = 0\]

Equations (5) to (7) can now be written as a vector–matrix differential equation as :

\[\frac{d\dd{v}}{dy} = \dd{A} \dd{v}\]  \hspace{1cm} (9)

where \[\dd{v} = [\dd{u}{y}, \dd{v}{y}, \dd{T}{y}, D \dd{u}{y}, D \dd{v}{y}, D \dd{T}{y}]^T\] (here D indicates differentiation w.r.t. \(y\))
and $\tilde{A} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
C_{41} & 0 & C_{43} & 0 & C_{45} & 0 \\
0 & C_{52} & 0 & C_{54} & 0 & C_{56} \\
0 & C_{61} & 0 & C_{63} & 0 & C_{65} \\
\end{bmatrix}$

where $C_{41} = \beta^2 (\xi^2 + p^2); \ C_{43} = i\xi\beta^2; \ C_{45} = i\xi(1-\beta^2)$

$C_{52} = \frac{\xi^2}{\beta^2} + p^2; \ C_{54} = \frac{i\xi(1-\beta^2)}{\beta^2}; \ C_{56} = 1$

$C_{61} = i\xi\epsilon p(1+\tau p); \ C_{63} = p(1+\tau p) + \xi^2; \ C_{65} = \epsilon p(1+\tau p)$

\begin{equation}
(10)
\end{equation}

**Solution of the Vector-Matrix Differential Equation**

The characteristic equation of the matrix $\tilde{A}$ takes the form:

$\lambda^6 - \lambda^4 (C_{41} + C_{52} + C_{63} + C_{45}C_{54} + C_{56}C_{65}) + \lambda^2 (C_{52}C_{63} + C_{41}C_{52} + C_{41}C_{63} - C_{43}C_{61} + C_{63}C_{45}C_{54} - C_{43}C_{54}C_{65} + C_{41}C_{56}C_{65} - C_{45}C_{61}C_{56}) - C_{41}C_{52}C_{63} + C_{43}C_{54}C_{65} - C_{45}C_{61}C_{56} = 0 \quad (11)$

The roots of characteristic equation (11) which are also the eigenvalues of the matrix $\tilde{A}$ are of the form:

$\lambda = \lambda_1, \ \lambda = \lambda_2 = -\lambda_1, \ \lambda = \lambda_3, \ \lambda = \lambda_4 = -\lambda_3, \ \lambda = \lambda_5, \ \lambda = \lambda_6 = -\lambda_5. \quad (12)$

The eigenvector $\tilde{X} = [x_1, x_2, ..., x_6]^{T}$ of the matrix $\tilde{A}$ corresponding to the eigenvalue $\lambda$ is

\begin{equation}
\tilde{X} = \begin{bmatrix}
\lambda^2 (C_{45}C_{56} + C_{43}) - C_{41}C_{52} \\
\lambda (\lambda^2 C_{56} + C_{54}C_{43} - C_{41}C_{56}) \\
\lambda^4 - \lambda^2 (C_{41} + C_{52} + C_{45}C_{54}) + C_{41}C_{52} \\
\lambda [\lambda^2 (C_{45}C_{56} + C_{43}) - C_{41}C_{52}] \\
\lambda^3 (\lambda^2 C_{56} + C_{54}C_{43} - C_{41}C_{56}) \\
\lambda [\lambda^4 - \lambda^2 (C_{41} + C_{52} + C_{45}C_{54}) + C_{41}C_{52}] \\
\end{bmatrix} \quad (13)
\end{equation}
Using the method as in Das and Bhakta [14] the solution of (9) is:

$$\tilde{v}(\xi, y, p) = \sum_{i=1}^{6} K_i \tilde{X}_i e^{\lambda_i y}$$

where \( \tilde{X}_i \) are the eigen vectors corresponding to the eigenvalues \( \lambda = \lambda_i \) and \( K_i \) are the constants which are to be determined from the boundary conditions. The displacement components \( \tilde{u}_i(\xi, y, p), \tilde{v}_i(\xi, y, p) \) and the temperature \( \tilde{T}_i(\xi, y, p) \) of the vector \( \tilde{v} \) in (9) can now be written as:

$$\tilde{u}_i(\xi, y, p) = \sum_{i=1}^{6} K_i x_{ii} e^{\lambda_i y}$$

(15)

$$\tilde{v}_i(\xi, y, p) = \sum_{i=1}^{6} K_i x_{ii} e^{\lambda_i y}$$

(16)

$$\tilde{T}_i(\xi, y, p) = \sum_{i=1}^{6} K_i x_{ii} e^{\lambda_i y}$$

(17)

where \( x_{ii} = \left[ x_{i} \right]_{\lambda=\lambda_j} \) are determined from (13).

From equation (8) using (15)-(17) we can get the stresses in Laplace-Fourier domain

$$\left( \tilde{\tau}_{xx} \right)_i = \sum_{j=1}^{6} K_j e^{\lambda_j y} \left[ i \xi (\beta^2 - 2) \left( \lambda_j \right)^2 - C_{41} C_{52} \right] + \beta^2 \left( \lambda_j \right)^2 \left( C_{41} C_{52} + C_{54} C_{43} - C_{43} C_{54} \right)$$

(18)

$$\left( \tilde{\tau}_{yy} \right)_i = \sum_{j=1}^{6} K_j e^{\lambda_j y} \left[ \xi (\beta^2 - 2) \left( \lambda_j \right)^2 - C_{41} C_{52} \right] + \beta^2 \left( \lambda_j \right)^2 \left( C_{41} C_{52} + C_{54} C_{43} - C_{43} C_{54} \right)$$

(19)

$$\left( \tilde{\tau}_{xy} \right)_i = \sum_{j=1}^{6} K_j e^{\lambda_j y} \left[ \lambda_j \left( C_{45} C_{56} + C_{43} \right) - C_{43} C_{52} + i \xi (\lambda_j \left( C_{56} + C_{43} \right) - C_{43} C_{54} - C_{41} C_{52}) \right]$$

(20)

The temperature and the stresses in the Laplace transform domain will be of the form:
\[\left[\overline{T}, \overline{\tau}_{xx}, \overline{\tau}_{yy}\right](x, y, p) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \left[\overline{T}, (\overline{\tau}_{xx})_1, (\overline{\tau}_{yy})_1\right] \cos(\xi x) d\xi\]  

\[\overline{\tau}_{yy}(x, y, p) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} [(\overline{\tau}_{yy})_1] \sin(\xi x) d\xi\]  

where, \(\overline{T}_1, (\overline{\tau}_{xx})_1, (\overline{\tau}_{yy})_1\) are even function of \(\xi\) and \((\overline{\tau}_{yy})_1\) is an odd function of \(\xi\) and the Laplace inversion is made by the method of Bellman et al [15].

**Boundary Conditions**

We assume that the surfaces of the slab are stress-free

i.e. \(\tau_{yy} = \tau_{yy} = 0\) at \(y = \pm h\)  

We further assume the temperature distribution over the upper and lower surfaces of the slab as follows:

\(T(x, h, t) = f(x, t) = T_0\); \(|x| < a\)

\(T(x, -h, t) = g(x, t) = 0\)  

Taking the Laplace-Fourier transform to equation (24) as described earlier, we get

\[\overline{T}_1(\xi, h, p) = \sqrt{\frac{2}{\pi}} T_0 \sin(a \xi) \frac{p \xi}{\xi}\]  

Numerical Solution

The Laplace-Fourier inversion for temperature and the stress components are very complex in nature, so we calculate those by some efficient computer programs. First, we invert the Fourier transform using the formulae (21) and (22) by seven point Gauss quadrature formula. For numerical inversion of Laplace transform, we follow the method of Bellman et al [15] which will provide us the values of the variables at the times \(t = t_i; i = 1(1)7\), where \(t_i\)’s are the roots of the Legendre polynomial of degree seven.
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For our numerical computation we have chosen Aluminum-epoxy for which the following data have been used:

\[ \beta = 2.45 ; \quad \varepsilon = 0.073 ; \quad \tau = 0.02 \]

The results for stresses for different values of time viz \( t_2 = 2.04612 \), \( t_4 = 0.693147 \) & \( t_6 = 0.138382 \) for \( x = 0 \) (Fig. 2-7) and \( x = a \) (Fig. 8-10) are shown. We mainly observe the nature of the stresses near the middle plane \( y = 0 \) for different ratios of \( h/a \) (viz 1/5, 1, 5).

Fig. 2 – Distribution of \( \tau_{xx} \) for \( x = 0 \) and \( h/a = 1 \)
Fig. 3 – Distribution of $\tau_{yy}$ for $x = 0$ and $h/a = 1$.

Fig. 4 – Distribution of $\tau_{xx}$ for $x = 0$ and $h/a = 1/5$. 
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Fig. 5 – Distribution of $\tau_{yy}$ for $x = 0$ and $h/a = 1/5$

Fig. 6 – Distribution of $\tau_{xx}$ for $x = 0$ and $h/a = 5$
Fig. 7 – Distribution of $\tau_{yy}$ for $x = 0$ and $h/a = 5$

Fig. 8 – Distribution of $\tau_{xx}$ for $x = a$ & $h/a = 1$
Fig. 9 -- Distribution of $\tau_{yy}$ for $x = a$ & $h/a = 1$

Fig. 10 -- Distribution of $\tau_{xy}$ for $x = a$ & $h/a = 1$
Conclusion

From the figures it is observed that,

i) the absolute values of the stresses have a larger value for time $t_2$ in comparison with $t_4$ and $t_6$. The dispersion of the curves (for all cases) is greater for $t_2$ than that of $t_4$ and $t_6$. The stresses have closer values for $t_4$ and $t_6$.

ii) the absolute values of $\tau_{yy}$ are greater than that of $\tau_{xx}$ for all cases and for $x=a$ the absolute values of $\tau_{xy}$ is much more smaller than $\tau_{xx}$ and $\tau_{yy}$ (fig. 8,9,10).

iii) for all values of the ratios ($h/a$) the absolute maximum values of the stresses occur near the heated boundary.

iv) the magnitude of all stresses diminish as $y<0$.

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Appendix

Consider the vector matrix differential equation of the form:
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\[
\frac{d\vec{v}}{dx} = \tilde{A}\vec{v}
\]

where \( \vec{v} = [v_1, v_2, \ldots, v_n]^T \) and \( \tilde{A} = (a_{ij}) \); \( i, j = 1, 2, \ldots, n \). \( \textbf{(b)} \) ; are real vector and matrix respectively.

Let \( \tilde{A} = V \Lambda V^{-1} \) \( \textbf{(c)} \)

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}
\]

where \( \Lambda \) is a diagonal matrix whose elements \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the distinct eigenvalues of \( \tilde{A} \). Let \( [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n] \) be the eigenvectors of \( \tilde{A} \) corresponding to \( \lambda_1, \lambda_2, \ldots, \lambda_n \) respectively, and

\[
V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n] = (x_{ij}) \text{ (say)}; \quad i, j = 1, 2, \ldots, n.
\] \( \textbf{(d)} \)

Substituting (c) in (a) and premultiplying by \( V^{-1} \), we get,

\[
V^{-1}\frac{d\vec{v}}{dx} = \Lambda \quad \text{or} \quad \frac{d}{dx} (V^{-1}\vec{v}) = \Lambda (V^{-1}\vec{v})
\]

Let we define \( \vec{y} = V^{-1}\tilde{A} \) \( \textbf{(e)} \)

Now we need to solve the equations

\[
\frac{d\vec{y}}{dx} = \Lambda\vec{y} \quad \textbf{(f)}
\]

This is a set of \( n \) decoupled differential equations. Consider the \( r \)th equation, which is typical

\[
\frac{dy_r}{dx} = \lambda_r y_r
\]

The solution is \( y_r = C_r e^{\lambda_r x}, \quad r = 1, 2, \ldots, n \) \( \textbf{(g)} \)

where \( C_r \) are scalars to be determined from the initial conditions.

Since from (e), \( \vec{v} = V\vec{y} \), we write
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\[ \tilde{v} = \sum_{r=1}^{n} V_r \gamma_r \]

which can be explicitly written as,

\[
\begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_n
\end{bmatrix} =
\begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nn}
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_n
\end{bmatrix}
\]

Substituting (g) in (h) we get the complete solution of (a) in the form

\[ v_r = c_1 x_{1r} e^{\lambda_1 x} + c_2 x_{2r} e^{\lambda_2 x} + \cdots + c_n x_{nr} e^{\lambda_n x} \]

\[ r = 1, 2, \ldots, n \]

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