Modelling of 3D Steady-State Oscillations of Anisotropic Multilayered Structures Applying the Green’s Functions

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Abstract

Two innovative numerical procedures are suggested for calculating of spatial steady-state oscillations of a infinite layered plate with arbitrary elastic anisotropy of each layer. The procedures are based on the algorithm of construction of the Green’s matrix in Fourier domain for a multilayered structure. This algorithm does not require the inversion of linear systems of large order and allows one to do calculations in the domain of large wave numbers and frequencies. The inverse Fourier transform is defined as a repeated integral in polar coordinates in wave
numbers domain. At first, the inverse Fourier transform is computed by using a contour of integration which deviates from the real axis and encloses the real poles of the transform of the Green’s matrix. To speed up the calculation of a repeated integral, the asymptotic properties of the symbol of the Green’s matrix are used. The second numerical procedure developed is based on the introduction of a complex frequency with small imaginary part, which provides an opportunity to integrate along the real axis. As examples, the complex amplitudes of the displacement on the surface of three different composites for various types of surface loads are computed.

**Keywords:** Three-dimensional elasticity solution, Green’s function, laminated composite plates, wave propagation, oscillating integrals

1 **Introduction**

The analysis of the propagation of elastic waves in multilayered anisotropic media is of major interest for many technical applications, in particular for nondestructive testing. During the last two decades, the researchers’ attention has been increasingly drawn to the study of wave propagation in multilayered media, owing to the increased use of composite materials in engineering applications.

Elastic waves, excited by cracks during the initiation of the crack growth or by an external source, provide significant information on the nature of the defect. A clear understanding of quantitative connections between the waves and their sources is essential for the development of algorithms to detect defects.

Solving direct problems of calculating wave fields excited by surface or internal dynamical sources in an elastic medium is an unavoidable stage in solving inverse problems of detecting hidden heterogeneities. As wave excitation sources for guided wave propagation piezo actuators are frequently used [20, 14, 22]. Obtaining analytical representations for the problem of oscillations excited by a surface load and their scattering by the defects in a multilayered anisotropic medium is in itself a complicated problem. Solution of such problems requires, as a rule, effective numerical methods.

To analyze steady states and transient responses of multilayered anisotropic structures, several accurate and effective methods and approaches are currently in use.

The method most often consists in applying the Fourier transform to the boundary value problem and then invoking Cauchy’s residue theorem to carry out the inverse Fourier transform. The computation of the formal solution for a point, linear or arbitrarily distributed source in terms of integral representations usually implies using the technique of integral transform combined with the methods of calculation of integral representations of the Green’s functions.
In the general case the solution of the problem in Fourier domain can be presented as the triple integral with one time variable and two spatial variables. The reversal of representations requires a numerical calculation of a double integral with respect to wave numbers and a subsequent reversal with respect to frequency. The calculation of such integrals is a most complicated task owing to the presence of singularities and strong oscillations of the integrands at high frequencies and at long distances between the source and the point of observation. The response in the time-domain can be obtained by the inverse Fourier transform, for instance by the method of exponential windows [17, 23, 15, 18].

Thus in paper [5] the double integral with respect to wave numbers is transformed into a one-dimensional integral with the help of contour integration and the use of the residue theorem, afterwards this integral is computed numerically using ordinary quadratures.

In article [23], the one-dimensional integral with respect to wave numbers is calculated applying the modified Clenshaw-Curtis quadrature schemes [26, 27]. The convergence of the integral is achieved by deforming the integration contour to avoid the real poles of the integrand. Another approach to the integration of slowly decreasing oscillating functions is suggested in [21].

In [11] the stiffness method and the modal summation technique are employed [30, 31, 32], and the time response is obtained by the reversal of the Fourier transform with respect to frequency using FFT.

In [32] the Green’s function of displacements under a time-harmonic line load is also obtained by the stiffness method, as well as by the modal expansion. The reversal of the three-dimensional Green’s functions when harmonically loaded by a point source acting in an arbitrary direction for a multilayered isotropic plate is obtained in [4] by a decomposition into a plane and antiplane problems. The steady-state oscillations in the domain of wave numbers are found by means of double sums: the inner sum with respect to normal modes and the outer sum with respect to the propagation directions of plane waves. In [16] the reversal of the Fourier transform in the time-domain is also performed by double fast Fourier transform (DFFT).

Many of the mentioned methods are used and developed in the book [10] for the calculation of steady-state oscillations and transient responses in multilayered anisotropic structures with various load types.

For studying wave propagation in a layered elastic structure two different approaches have been generally employed. One is a wave spectra analysis based on the frequency equation and the other involves fundamental elastodynamic solutions arising from the Green function which correspond to a structure’s dynamic responses due to a concentrated loads [2, 3]. It is known that Green’s functions are essential for applying the boundary element method (BEM) to study the effects of cracks and other flaws in a plate.

Despite a large number of papers discussing wave propagation in multi-
layered media, the majority of results is obtained for two-dimensional problems [17, 14, 13, 6] or for isotropic materials [4, 20]. Numerical results for three-dimensional problems and anisotropic media are still sparse [7, 10], and different modifications of the FEM are frequently used for the calculations [28, 29, 19].

The approach suggested in this paper is based on the classical technique of integral Fourier transforms and integral representations of the Green’s matrix. The spatial steady-state harmonic oscillations of a multilayered structure with arbitrary anisotropy are studied, both for a point source and for distributed sources. In the Fourier domain the displacement vector is defined by the product of symbols of the Green’s matrix and of the surface load vector. The reversal of the two-dimensional Fourier transform is performed by reducing the double integral to a repeated one and integration along a contour which deviates into the complex plane while avoiding the real poles with a subsequent integration with respect to the angle of wave propagation. When the double integral is reduced to the repeated integral, one of the integrals is computed using standard integration programs for the quickly oscillating functions of the NAG package [8], the other can be calculated by means of standard methods of numerical integration. In order to speed up the integration, an interpolation of the symbol of the Green’s matrix on the two-dimensional grid in a polar coordinate system is suggested. This method gives stable results both in the vicinity of the source and at sufficiently far distances from the source. The integration technique is analogous to that presented in [5]. The significant difference is that the real singularities of the symbol of the Green’s matrix are considered in a Cartesian coordinate system in [5] and in a polar coordinate system in the current paper. This provides an opportunity to simplify the calculation and to use asymptotic estimates for the integral. This technique has a particularly simple form in the case of a complex frequency with small imaginary part, which is equivalent to the introduction of a small internal friction of the medium. In this case the complex contour of integration is substituted by a real one.

The algorithm of calculating the Fourier symbol of the Green’s matrix is given. This algorithm allows us not only to do calculations in the domain of high frequencies and large wave numbers, but also to estimate the asymptotic forms of the symbol of the Green’s matrix numerically. The use of the well-known simple asymptotic properties of the symbols of the Green’s matrix can simplify and accelerate the calculations of double integrals with respect to wave numbers.

To provide numerical examples, the displacements on the surface of unidirectional, 8-layer and 24-layer symmetric composites in the near and middle and far zones for various types of surface loads are calculated.

The method of calculating the symbol of the Green’s matrix of the multi-
layered anisotropic plate modified for the extended range of arguments constitutes a novelty of the current article. The numerical procedures developed for the calculation of the inverse Fourier transform contain new elements which substantially speed up the calculation of integrals. The method of contour integration with complex frequency can be simply applied in near and middle fields for the calculation of non-stationary loads.

2 Problem formulation

Let us consider a medium, which consists of homogeneous layers $-\infty \leq x, y \leq \infty, z^{(n+1)} \leq z \leq z^{(n)}, n = \overline{1, N}$. We may assume that $z^{(1)} = 0$; then $z^{(N+1)} = -h$ where $h$ is the thickness of the structure. Throughout this paper, upper indices in brackets are used for numbering the layers.

Harmonic oscillations of the medium are excited by surface mechanical sources acting in a bounded domain $\Omega$ in the plane $z = 0$. They are described by the following equations in which the common multiplier $\exp(-i\omega t)$ is omitted in all cases:

$$\frac{\partial \sigma_{jk}^{(n)}}{\partial x_k} + \rho^{(n)} \omega^2 u_j^{(n)} = 0, \quad (1)$$

$$\sigma_{ji}^{(n)} = C_{kmji}^{(n)} \frac{\partial u_m^{(n)}}{\partial x_k}, \quad (i, j, k, m = 1, 2, 3; \quad n = \overline{1, N}), \quad (2)$$

where $\sigma_{ij}$ is the stress tensor, $u_j^{(n)}$ are components of the displacement vector, $\rho^{(n)}$ is the material’s mass density, $C_{kmji}^{(n)}$ is the elastic tensor, $\omega$ is the angular frequency. In formula (1) and subsequently, the rule of summation over repeated indices is employed unless otherwise mentioned.

The boundary conditions in the plane $z = z^{(1)} = 0$ have the form

$$\sigma_{j3}^{(1)} = 0, \quad (x, y) \in \overline{\Omega}, \quad (j = 1, 2, 3). \quad (3)$$

In the domain $\Omega$ the surface load vector $q$ is specified as follows:

$$\sigma_{j3}^{(1)} = q_j(x, y), \quad (x, y) \in \Omega, \quad (j = 1, 2, 3). \quad (4)$$

The boundary conditions at the base of the bottom layer $z = z^{(N+1)} = -h$ can differ. In what follows, three possible alternatives will be considered.

The composite is connected to the fixed base; hence the mechanical displacements at the base of the bottom layer equals zero:

$$u_1^{(N)} = u_2^{(N)} = u_3^{(N)} = 0. \quad (5)$$

The bottom boundary of the composite is traction-free:

$$\sigma_{13}^{(N)} = \sigma_{23}^{(N)} = \sigma_{33}^{(N)} = 0. \quad (6)$$
The composite lies on the rigid base without friction:
\[ \sigma^{(N)}_{13} = \sigma^{(N)}_{23} = u^{(N)}_3 = 0. \] (7)

The solutions of the boundary problem (1)–(7) can be obtained as Fourier integrals:
\[
u^{(n)}_i(x, y, z) = \frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} K^{(n)}_{ij}(\alpha_1, \alpha_2, z) Q_j(\alpha_1, \alpha_2) \times \\
\times \exp(-i(\alpha_1 x + \alpha_2 y)) d\alpha_1 d\alpha_2, \quad (i, j = 1, 2, 3). \] (8)

Here \( K_{ij} \) is a Fourier transform of the Green’s matrix \( k_{ij} \) with respect to \( x, y \) \( K_{ij} = F_{xy}[k_{ij}] \), and the vector \( Q \) with the elements \( Q_j = F_{xy}[q_j] \) is the Fourier transform of the vector \( q \) given by (4), and \( \Gamma_1, \Gamma_2 \) are the contours of integration (which deviate from the real axes when enclosing the real poles \( K_{ij} \) in accordance with the principle of limiting absorption [1, 24]). The algorithm to derive the solution in the domain of the Fourier transform is described in detail in Appendix. The construction of the Green’s matrix of a multilayered medium (which is described in the Appendix) yields
\[
K^{(n)}_{ij}(\alpha_1, \alpha_2, \omega, z) = \frac{\tilde{K}^{(n)}_{ij}(\alpha_1, \alpha_2, \omega, z)}{\det P(\alpha_1, \alpha_2, \omega)}, \] (9)

where \( P \) is a block matrix consisting of continuous blocks \( P^j_i \) (see Appendix). It is obvious that the functions \( \tilde{K}^{(n)}_{ij} \) are continuous too, being algebraic adjuncts of the corresponding elements \( P \). Hence the poles of the elements of the matrix \( K^{(n)}_{ij} \) satisfy the equation
\[
\det P(\alpha_1, \alpha_2, \omega) = 0 \] (10)

which is also called the dispersion equation. Its roots define the wave numbers of the medium. Using the coordinate transformation
\[
\begin{align*}
\alpha_1 &= \alpha \cos \gamma, \quad \alpha_2 = \alpha \sin \gamma, \\
\gamma &\in [0, 2\pi], \quad \text{Re} \alpha \geq 0, \\
x &= r \cos \varphi, \quad y = r \sin \varphi, \quad z = z, \\
\varphi &\in [0, 2\pi], \quad r \geq 0,
\end{align*}
\] (11)

the dispersion equation is solved for \( \alpha \) as a function of \( \gamma \) and \( \omega \), say \( \alpha = \zeta_m(\gamma, \omega) \) where \( m \) denotes the number of the root. Depending on the boundary conditions at the base of the composite and the frequency, there may exist a finite number of real and a countable number of complex roots for each fixed pair \((\gamma, \omega)\). For each fixed \( \gamma = \gamma_0 \) the real pole with number \( m \) is called regular in case \( d\omega/d\zeta_m(\omega, \gamma_0) > 0 \) and irregular in case \( d\omega/d\zeta_m(\omega, \gamma_0) < 0 \).
In terms of \( \alpha \) and \( \gamma \) and the cylindrical coordinates used in (11), the inverse Fourier transform (8) can be written as

\[
\mathbf{u}^{(n)}(r, \varphi, z) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{\Gamma^+} \mathbf{U}^{(n)}(\alpha, \gamma, z) \exp(-i\alpha r \cos(\gamma - \varphi)) \alpha d\alpha d\gamma, \quad n = 1 \ldots N.
\]  

(12)

A double improper integral of type (12) can be expressed as a repeated integral

\[
\mathbf{u}^{(n)}(r, \varphi, z) = \frac{1}{4\pi^2} \int_0^{2\pi} d\gamma \int_{\Gamma^+(\gamma)} \mathbf{U}^{(n)}(\alpha, \gamma, z) \exp(-i\alpha r \cos(\gamma - \varphi)) \alpha d\alpha, \quad n = 1 \ldots N,
\]  

(13)

where for each \( \gamma \) the contour \( \Gamma^+(\gamma) \) is chosen in accordance with the principle of limiting absorption [1]. The contour \( \Gamma^+(\gamma) \) encloses all regular poles from below and irregular poles from above.

3 Calculation of the inverse Fourier transform by direct computation of a contour integral

The calculation of the integrals of the inverse Fourier transform (13) causes multiple difficulties. The presence of the poles in the components of the Fourier symbol of the Green’s matrix requires the change of the contour of integration with respect to \( \alpha \): instead of integration along the positive real semiaxis, it becomes necessary to form a contour \( \Gamma^+(\gamma) \) enclosing real poles in the complex plane.

In addition, the computation of the integral (13) is inseparably linked to problems such as the imprropriety of the integral, a strongly oscillating integrand and the complexity of the integrand (requiring a considerable amount of time for its calculation).

The components of the Green’s matrix \( \mathbf{K}^{(n)}_{ij}(\alpha, \gamma, z) \) for the real values of \( \alpha \) are defined [1] by an asymptotic expression for large \( \alpha \rightarrow \infty \):

\[
\mathbf{K}^{(n)}_{ij}(\alpha, \gamma, z) \approx \mathbf{A}^{(n)}_{ij}(\gamma, z), \quad i, j = 1, 2, 3.
\]  

(14)

Hence there exists a sufficiently large value \( R \) such that

\[
\mathbf{u}^{(n)}(r, \varphi, z) = \frac{1}{4\pi^2} \int_0^{2\pi} d\gamma \int_{\Gamma^+(\gamma)} \mathbf{U}^{(n)}(\alpha, \gamma, z) \exp(-i\alpha r \cos(\gamma - \varphi)) \alpha d\alpha = \mathbf{F}^{(n)}_R(r, \varphi, z) + \tilde{\mathbf{F}}^{(n)}_R(r, \varphi, z),
\]  

(15)

\[
\mathbf{F}^{(n)}_R(r, \varphi, z) = \frac{1}{4\pi^2} \int_0^{2\pi} d\gamma \int_{\Gamma^{+R}(\gamma)} \mathbf{U}^{(n)}(\alpha, \gamma, z) \exp(-i\alpha r \cos(\gamma - \varphi)) \alpha d\alpha,
\]  

(16)
where the contour \( \Gamma_{R+}(\gamma) \) coincides with the contour \( \Gamma_+(\gamma) \), but is bounded with respect to \( \alpha \) by the value of \( R \) (for an example of such a contour see Fig. 1). As a result of the asymptotics (14), we find that

\[
\tilde{F}_{R}^{(n)}(r, \varphi, z) = (\tilde{F}_{R1}^{(n)}(r, \varphi, z), \tilde{F}_{R2}^{(n)}(r, \varphi, z), \tilde{F}_{R3}^{(n)}(r, \varphi, z)) \mathbf{T},
\]

\[
\bar{F}_{Ri}^{(n)}(r, \varphi, z) \approx \frac{1}{4\pi^2} \sum_{j=1}^{3} \int_{0}^{2\pi} \int_{0}^{\infty} A_{ij}^{(n)}(\gamma, z) Q_j(\alpha, \gamma) \exp(-i\alpha r \cos(\gamma - \varphi)) d\gamma d\alpha.
\]

(17)

The integral (17) is calculated with respect to the interval of the real axis on which there are no real poles. Due to property (14) which is satisfied for any \( \alpha \) for \( R \) large enough, the expression (17) for \( \tilde{F}_{R}^{(n)}(r, \varphi, z) \) will differ from the exact expression by an arbitrarily small amount depending on the choice of \( R \). Moreover, it can be demonstrated that integral (17) always converges far away from the source too; when \( r \) is large, the contribution of \( \bar{F}_{Ri}^{(n)}(r, \varphi, z) \) into the sum (15) is very small. As an example, let us examine the load set by delta functions \( q_j(x, y) = \delta(x) \delta(y) \), i.e., \( Q_j = 1 \). Then the integrand has the product of \( \alpha \) and \( r \) in its exponent. With the exception of the case \( r = 0 \) it is always possible to choose \( R \) such that the value \( \alpha r \) will be arbitrarily large since \( \alpha > R \) in (17). Thus, the integral with respect to \( \gamma \) of the oscillating function can be calculated by applying the stationary phase method [12]

\[
\bar{F}_{Ri}^{(n)}(r, \varphi, z) \approx \sqrt{\frac{1}{8\pi}} \sum_{j=1}^{3} \left( A_{ij}^{(n)}(\varphi) \exp(i\pi/4) \int_{R}^{\infty} \sqrt{\frac{r}{\alpha}} \exp(-i\alpha r) d\alpha + A_{ij}^{(n)}(\varphi + \pi) \exp(-i\pi/4) \int_{R}^{\infty} \sqrt{\frac{r}{\alpha}} \exp(i\alpha r) d\alpha \right), \quad i = 1, 2, 3.
\]

(18)

The integrals in (18) converge due to the oscillation of the exponent and the multiplier \( 1/\sqrt{\alpha} \). Using integration by parts for the first of the integrals with...
respect to $\alpha$ in (18), we obtain

$$
\int_{R}^{\infty} \sqrt{\frac{1}{r^\alpha}} \exp(-i\alpha r) d\alpha = -\frac{1}{ir\sqrt{rr}} \exp(-iRr) + O(R^{-3/2}),
$$

(19)
i.e. $\tilde{F}_R^{(n)}(r, \varphi, z)$ tends to zero as $r^{-3/2}R^{-1/2}$ with increasing values of $r$ and $R$.

In first approximation, the contribution of the integral (17) to the total sum can be neglected [7], and when necessary, it can be calculated separately and added to the main part of the solution $F_R(r, \varphi, z)$ already obtained. The integral with respect to $\alpha$ in (17) can be computed employing for example the procedure D01ASF of the NAG Fortran Library [9] (integral of an oscillating function on an infinite interval). The significant advantage of the computation of such an integral is that there is no necessity to calculate the Green’s matrix by substituting its values by an asymptotic form (14), which substantially reduces the computation time. Nevertheless, the major task is the calculation of the integral (16) with respect to the finite contour $\Gamma_{R+}(\gamma)$.

The contour of enclosure of the real poles $\Gamma_{R+}(\gamma)$ in the general case depends on the angle $\gamma$; the integrands oscillate strongly and the symbol of the Green’s matrix requires a relatively long time to be calculated. The problems mentioned lead to the calculation process (16) being complicated and requiring a large amount of time. One possible algorithm to calculate the integrals of type (16) is suggested hereafter, although there are other possible methods [23, 5, 26, 21, 32, 4, 16, 18, 7].

The contour presented in Fig. 1 is considered as the contour of integration. The given contour corresponds to the general case and is chosen for each $\gamma$ correspondingly. Given $\gamma$, the contour is characterized by the parameters $R, R_0, R_1, R_2, R_3, d, s$. Here it is generally assumed that $R_0 = R_0(\gamma), R_1 = R_1(\gamma), R_2 = R_2(\gamma), R_3 = R_3(\gamma), d = d(\gamma), s = s(\gamma)$. The parameter $R$ denotes the contour length with respect to $\alpha$ and is specified such that condition (14) is satisfied for $\alpha \geq R$ and is chosen universally for all $\gamma$. The parameters $R_0, R_1$ correspond to the points of deviation of the contour from the real axis into the upper half-plane and are defined so that all real irregular poles (only one such pole $\zeta_1(\gamma)$ is presented in case of Fig. 1) for the chosen $\gamma$ lie within the interval $[R_0, R_1]$. The parameters $R_2, R_3$ for the chosen $\gamma$ correspond to the points of the deviation of the contour into the lower half-plane and all regular real poles lie within the interval $[R_2, R_3]$. The parameter $s$ equals the value of the deviation of the contour from the real axis, and $d$ equals the length of the projection of the transition points between the real axis and the straight lines between points $[(R_0 + d, s)$ and $(R_1, s)]$, $[(R_2 + d, -s)$ and $(R_3, -s)]$ onto the real axis. The intervals of integration are parallel to this projection at distance $s$. The parameter $s$ should be so small that the complex poles neighbouring the real axis lie above the contour of integration. The parameter $d$ is chosen...
with \( d > s \), for instance \( d = 3s \). Nevertheless, it should be mentioned that with the decrease of \( s \) the contour of integration approaches the real poles more and more closely, which results in a quick change of the integrand in their neighbourhood. Therefore, during the computation of the integral (16) with respect to \( \alpha \), it is essential that such an inhomogeneous integration step in the neighbourhood of the real poles is used in order to reach an acceptable level of accuracy.

After the integration contour \( \Gamma_{R^+} \) is chosen, it is possible to apply the D01AKF subroutine \([8]\) of the NAG Fortran Library for the calculation of integral (16). This is a special-purpose subroutine for the computation of integrals of oscillating functions. In order to reduce the vast amount of time needed to solve this problem, a three-stage calculation scheme is suggested. The first stage involves constructing a node grid \((\alpha_k, \gamma_m)\). Here it is necessary to use a non-uniform grid on those parts of the contour close to the real poles (whereas a uniform grid suffices on the other parts of the contour). At the second stage the symbol of the Green’s matrix is calculated and saved on the constructed node grid. At the third stage of integration, the exact values of the matrix are substituted by the values obtained by linear interpolation. After the integration with respect to \( \alpha \), the integral with respect to \( \gamma \) can also be computed applying the D01AKF subroutine \([8]\) since with the increase of \( r \) the integrals with respect to \( \gamma \) will be oscillating too.

For the calculation of integral (16) we could also introduce a complex frequency \( \omega = \omega_R + i\omega_C \), where \( \omega_C > 0 \) is small \( \omega_C \ll \omega_R \). In this case, owing to the principle of limiting absorption \([1]\), all real poles move away from the real axes, which allows us to integrate directly along the real axis. The smaller \( \omega_C \) is chosen, the closer the solution obtained will be to the exact value. However, this approach requires a non-uniform integration grid, since the poles approach the real axis closely. The increase of \( \omega_C \) is related to a decrease in the accuracy for the solution, but the construction of the grid will require far less computer memory and also speeds up the calculations. As follows from the numerical experiments conducted, the recommended value of the imaginary part of frequency can be chosen within the limits \( 0.005 \lesssim \omega_C \omega_R \lesssim 0.05 \).

The introduction of a complex frequency is equivalent to the introduction of a small internal friction, i.e. dissipation. The same effect can be achieved by introducing complex elastic modules.

### 4 Numerical examples

The methods discussed above are applied to solve problems of harmonic oscillations excited in composites by various types of surface loads. The structure of composites under study is presented in Table 1. The elastic modules of carbon-epoxy are obtained from \([25]\), those of graphite-epoxy from \([10]\); they are shown
Table 1: Composite materials

<table>
<thead>
<tr>
<th>N</th>
<th>Stacking sequence</th>
<th>Material</th>
<th>Number of layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>[+45/−45]_6</td>
<td>Carbon-epoxy</td>
<td>24</td>
</tr>
<tr>
<td>II</td>
<td>[+45/−45/0/90]_6</td>
<td>Carbon-epoxy</td>
<td>8</td>
</tr>
<tr>
<td>III</td>
<td>[0]</td>
<td>Graphite-epoxy</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Properties of materials (elastic constants in 10^{11} Pa, density in kg/m^3)

<table>
<thead>
<tr>
<th></th>
<th>C_{11}</th>
<th>C_{12} = C_{13}</th>
<th>C_{22} = C_{33}</th>
<th>C_{23}</th>
<th>C_{44}</th>
<th>C_{55} = C_{66}</th>
<th>\rho</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon-epoxy</td>
<td>1.308</td>
<td>0.05263</td>
<td>0.1299</td>
<td>0.04556</td>
<td>0.0375</td>
<td>0.0597</td>
<td>1578</td>
</tr>
<tr>
<td>Graphite-epoxy</td>
<td>1.6073</td>
<td>0.0644</td>
<td>0.1392</td>
<td>0.0644</td>
<td>0.035</td>
<td>0.0707</td>
<td>1600</td>
</tr>
</tbody>
</table>

in Table 2. The elastic modules and density of the materials are given in 10^{11} Pa and 10^3 kg/m^3, respectively. The magnitudes presented in this article are normalized, and the dimensionless oscillation frequency \( \tilde{\omega} = \omega h/c_T \) is used. The reference velocity for normalization is chosen simply as \( c_T = 1000 \) m/s.

For the method of contour integration which is studied in this article the knowledge of the location of the poles of \( \alpha \) along the axis for the chosen values \( \gamma \) and \( \omega \) plays an essential role. The location of the poles depending on \( \omega \) and \( \gamma \) can be represented as dispersion curves. Thus, figure 2 shows the dispersion curves for composite III from Table 1 when \( \gamma = \pi/3 \) in the frequency range \( 0 < \omega h/c_T \leq 9.26 \) (a) and depending on \( \gamma \) when \( \omega h/c_T = 9.25 \) (b). Figure 2 represents the case of irregular poles appearing in the range \( 8.72 \leq \omega h/c_T \leq 9.26 \) (due the effect that one of the modes decreases with an increase of frequency i.e., \( d\omega/d\zeta < 0 \)). Hereafter we will present the calculation of the displacements both in the case of presence and absence of irregular poles.

The first type of surface excitations under consideration is given by a concentrated vertical load located at the origin:

\[
q_1(x, y) = 0, \quad q_2(x, y) = 0, \quad q_3(x, y) = A\delta(x)\delta(y).
\]

The coefficient \( A \) in this case and hereafter corresponds to the required amplitude of a surface load. The Fourier transform of (20) can be written as

\[
Q_1(\alpha, \gamma) = Q_2(\alpha, \gamma) = 0, \quad Q_3(\alpha, \gamma) = A.
\]

The second type of surface excitations is presented by a vertical load which is evenly distributed in the circle with radius \( L \), i.e.,

\[
q_1(x, y) = 0, \quad q_2(x, y) = 0, \quad q_3(x, y) = A, \quad x^2 + y^2 < L^2.
\]
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The Fourier transform of (22) has the form of

\[ Q_1(\alpha, \gamma) = Q_2(\alpha, \gamma) = 0, \quad Q_3(\alpha, \gamma) = 2\pi LAJ_1(\alpha L) \alpha. \]  

(23)

The third type of surface load is given by an unevenly distributed vertical load in the circle with radius 1, namely

\[ q_1(x, y) = 0, \quad q_2(x, y) = 0, \quad q_3(x, y) = A \frac{2}{\pi} (1 - r^2), \quad r < 1. \]  

(24)

The Fourier transform of (24) leads to

\[ Q_1(\alpha, \gamma) = Q_2(\alpha, \gamma) = 0, \quad Q_3(\alpha, \gamma) = 8A \frac{J_2(\alpha)}{\alpha^2}. \]  

(25)

In (22), (24) the functions \( J_1(\alpha L) \) and \( J_2(\alpha) \) are the Bessel functions of order 1 and 2 respectively.

The graphs of the resulting displacements in case of a concentrated load (20), which are calculated by the method of contour integration, are shown in Fig. 3. The graphs are calculated along the \( x \)-axis (\( \varphi = 0 \)) in range of \( 0 \leq r/h \leq 25 \) for harmonic vibrations excited by a vertical concentrated force on the surface (20) of composites I (Figs. 3a, b and c). The dimensionless frequency of oscillations is chosen to be \( \omega h/c_T = 1 \). The parameter \( R \) hereafter is chosen to be \( R = 60 \). Figs. 3a, b and c demonstrate that the oscillations on the surface subside with increasing distance from the source. At the coordinate origin there is a singularity due to the fact that a point source is assumed. The accuracy of finding solutions nearby to the concentrated source also decreases as a result of the necessity of increasing the parameter \( R \).
Figure 3: Normalized dimensionless displacements (real (−) and imaginary (−⋅−) parts) on the surface caused by a concentrated vertical load. Composite I, frequency $\omega h/c_T = 1$ for $\varphi = 0$ $u_x(r)/(hA)$ (a), $u_y(r)/(hA)$ (b), $u_z(r)/(hA)$ (c). Composite III, $u_z(r)/(hA)$ for frequency $\omega h/c_T = 9.25$, $\varphi = 0$ (d), $\omega h/c_T = 9.25$, $\varphi = \pi/6$ (e), $\omega h/c_T = 21.02$, $\varphi = \pi/6$ (f).
Figure 4: Values of averaged relative errors of calculation of displacements (26) in case of a distributed load of type (22) when $\omega h/c_T = 1$ and $\varphi = 0$. Composite I. $\delta(u_x(r))$ (a), $\delta(u_y(r))$ (b), $\delta(u_z(r))$ (c). Composite II. $\delta(u_x(r))$ (d), $\delta(u_y(r))$ (e), $\delta(u_z(r))$ (f). The solid line denotes the results when $\omega_C \omega_R = 0.01$, the broken line denotes the results when $\omega_C \omega_R = 0.005$. 
In order to assess the influence of introducing the complex frequency \( \tilde{\omega} = (\omega_R h + \omega_C h \cdot i)/c_T \), we compare calculations for frequencies \( \tilde{\omega} = 1 + 0.01 \cdot i \) and \( \tilde{\omega} = 1 + 0.005 \cdot i \), corresponding to the relations \( \omega_C \omega_R = 0.005 \) and \( \omega_C \omega_R = 0.01 \). Figure 4 shows averaged relative errors of the absolute values of separate components of the displacement for both composites when \( \varphi = 0, z = 0 \), calculated according to the formulas

\[
\delta u_j = |u_j - \tilde{u}_j|/\bar{u}_j, \quad \bar{u}_j = \frac{1}{r - r_0} \int_{r_0}^{r} |u_j| dr, \quad j = 1, 2, 3, \quad (26)
\]

where \( u_j \) are the values of the \( j \)-th component of the displacement vector for the real frequency \( \omega h/c_T = 1 \), where \( \tilde{u}_j \) are the values calculated for the complex frequency and \( \bar{u}_j \) is the amplitude of \( u_j \) averaged on the interval \((r_0, r)\). The solid line denotes the results for \( \omega_C \omega_R = 0.01 \), the dotted line for \( \omega_C \omega_R = 0.005 \). As can be seen from these figures, the error does not exceed 10\% for \( \omega_C \omega_R = 0.01 \) and 5\% for \( \omega_C \omega_R = 0.005 \). In addition, we calculated the averaged error for the considered range of \( r/h \in (0, 20) \) according to the formula

\[
\hat{\delta}_j = \frac{1}{r - r_0} \int_{r_0}^{r} \delta u_j dr. \quad (27)
\]

It is 5.2\% in composite I for \( u_x \), 5.1\% for \( u_y \), 5.3\% for \( u_z \) in case of \( \omega_C \omega_R = 0.01 \) and 2.7\% for all components when \( \omega_C \omega_R = 0.005 \). In composite II similar calculations give the averaged errors 5.2\%, 5\%, 5.2\% in case \( \omega_C \omega_R = 0.01 \) and 2.7\%, 2.6\%, 2.7\% in case \( \omega_C \omega_R = 0.005 \), respectively. It can easily be noticed that the decrease of the imaginary part of frequency by half allows reducing the errors almost by a factor of two. However, its further decrease leads to an increase in the amount of calculations.

With the increase of \( r \), a tendency for the relative error to increase appears, since the introduction of the internal friction leads to a faster decrease of the solution at infinity. With the decrease of the imaginary part \( \omega_C \), a larger number of calculations is required, as the poles of the integrand will lie too closely to the real axis, causing a necessity to increase the number of nodes for the numerical integration.

Furthermore, the displacements are also calculated at higher frequencies. So Figs. 3d and e present the values of the vertical component of the displacement in the vicinity of the source in directions \( \varphi = 0 \) (d) and \( \varphi = \pi/6 \) (e) caused in composite III by a concentrated vertical load, calculated by the method of contour integration. There is an irregular pole at the frequency \( \omega h/c_T = 9.25 \) shown in Figs. 3d and e; therefore in this case we introduced, for convenience of calculations, a complex frequency with a small imaginary part \( \tilde{\omega} = 9.25 + 0.005 \cdot i \), i.e. \( \omega_C \omega_R \approx 0.05 \), and the integration is performed
accordingly with respect to the real axis. In Fig. (3f) there is a representation of a vertical component of the displacement calculated in composite III in direction $\varphi = \pi/6$ in case of harmonic oscillations caused by a concentrated vertical load at the frequency $\omega h/c_T = 21.02$ at which there exist 12 regular modes.

For the axisymmetrically distributed load of type (22) when $L/h = 1$, numerical calculations are performed for the frequency $\omega h/c_T = 1$ and for $\varphi = \pi/6$ for composites I, II. The calculated displacements are shown in Fig. 5 (Figs. 5a, b and c for composite I, Figs. 5d, e and f for composite II). In addition, we calculated the displacements in the direction $\varphi = \pi/6$ for both composites in case of a higher frequency $\omega h/c_T = 7.78$ for the uniformly distributed load of type (22) (Fig. 6). Here contour integration along the real axis is used, and the complex frequency $\omega h/c_T = 7.78 + 0.005 \cdot i$ is introduced. As can be seen, for a higher frequency the type of wave propagation becomes substantially more complicated, and the results for composite I and for composite II (Figs. 6a, b and c) and for composite II (Figs. 6d, e and f) differ markedly.

5 Conclusion

This work is dedicated to the development of two numerical procedures for the calculation of steady-state spatial oscillations of a multilayered anisotropic structures with arbitrary elastic anisotropy and arbitrary spatial orientation of each layer. The authors proposed two alternatives of calculating a double integral with respect to wave numbers which is given as a double integral in polar coordinates. First, the integration contour deviates from the real axis while enclosing the real poles of the symbol of the Green’s matrix; second, a small imaginary part of the frequency is introduced, which leads to the shifting of the poles from the real axis and provides an opportunity to integrate directly along the real axis. This made it possible to use standard algorithms and programs of calculating integrals of oscillating functions to evaluate the inner integral. On the other hand, it is sufficient to use standard methods of numerical integration in order to compute the external integral. At a long distance from the source, when the oscillations of the function which is integrated with respect to the first variable are increasing, the integration algorithm for oscillating functions can be applied repeatedly to compute the external integral.

The numerical procedures developed are based on the algorithm of constructing the Fourier symbol of the Green’s matrix for a multilayered anisotropic structure. The asymptotic properties of the symbol of the Green’s matrix are used in order to evaluate improper integrals and accelerate the calculation process. For the evaluation of the asymptotic forms of the elements of the symbol of the Green’s matrix, a modified algorithm is suggested which allows
one to do calculations in the domain of large wave numbers and frequencies.

Furthermore, to speed up the calculation of the steady-state oscillations, a two-stage calculation scheme is developed which involves applying linear interpolation of the previously calculated elements of the symbol of the Green’s matrix on a two-dimensional non-uniform node grid. The given method reduced the amount of calculations substantially, since during the integration the most time is spent precisely on the calculation of the symbol of the Green’s matrix, especially when there is a large number of heterogeneous layers.

In the calculations, three types of composites (unidirectional, 8-layer and 24-layer), four types of concentrated and distributed sources (a point source, a uniform circular source, an nonuniform circular source, a rectangular source with certain peculiarities along the perimeter) for four dimensionless frequencies \( \omega h/c_T = 1, 7.78, 9.25, 21.02 \) are considered. The complex amplitudes of the displacements on the surface of the structure are given. The goal of these calculations was not the analysis of concrete composites, but the demonstration of the capabilities of this method.

The influence of the complex frequency component on the solution accuracy is estimated practically which points to the fact that this algorithm, which allows a simpler implementation, can be applied for practical calculations. In this case there is no necessity to search for the real poles of the symbol of the Green’s matrix. To calculate the transient response in near and middle fields, it is more appropriate to use a numerical procedure involving the introduction of a complex frequency, since such a procedure does not require any previous analysis of the dispersion surfaces within a large frequency range.

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Appendix. The Fourier symbol of the Green’s matrix for a multilayered structure

In monography [1] the authors developed a stable algorithm to compute the Green’s matrix of a multilayered anisotropic half-space in Fourier domain. The algorithm developed does not comprise growing exponents at any calculation stage. The method described below is a generalization of the given algorithm for the case of the multilayered anisotropic plate.

Let us consider a vector \( \mathbf{U} = (U_1, U_2, U_3, U'_1, U'_2, U'_3)^T \) where \( U'_j = dU_j/dz \).
Figure 5: Normalized dimensionless displacements (real (−) and imaginary (−−) parts) on the surface caused by an axisymmetric vertical load of type (22) for $\omega h/c_T = 1$, $L/h = 1$ and $\varphi = 0$. Composite I. $u_x(r)/(hA)$ (a), $u_y(r)/(hA)$ (b), $u_z(r)/(hA)$ (c). Composite II. $u_x(r)/(hA)$ (d), $u_y(r)/(hA)$ (e), $u_z(r)/(hA)$ (f).
Figure 6: Normalized dimensionless displacements (real (−) and imaginary (−−) parts) on the surface caused by an axisymmetric vertical load of type (22) for $\omega h/c_T = 7.78 + 0.005 \cdot i$, $L/h = 1$ and $\varphi = \pi/6$. Composite I: $u_x(r)/(hA)$ (a), $u_y(r)/(hA)$ (b), $u_z(r)/(hA)$ (c). Composite II: $u_x(r)/(hA)$ (d), $u_y(r)/(hA)$ (e), $u_z(r)/(hA)$ (f).
In each layer the required vector $U^{(n)}$ can be presented as follows:

$$U^{(n)} = \sum_{s=1}^{6} f^{(n)}_s p^{(n)}_s e^{\lambda^{(n)}_s z_n}, \quad n = \overline{1,N},$$

(28)

where $\lambda^{(n)}$, $p^{(n)}$ are the eigenvalues and eigenvectors of the following system for the $n$-th layer and $t^{(n)}$ are the unknown vectors specified hereafter:

$$(M^{(n)} - \lambda I)p^{(n)} = 0, \quad M = \begin{pmatrix} O & \cdots & I \\ \cdots & \cdots & \cdots \\ (J^{(n)})^{-1} A^{(n)} & i (J^{(n)})^{-1} B^{(n)} \end{pmatrix}.$$  

(29)

Here $I$, $O$ are the identity and null matrices, respectively, whereas $A^{(n)}$, $B^{(n)}$, $J^{(n)}$ are the symmetric matrices

$$A^{(n)}_{js} = -C_{irs} \alpha_s \alpha_i + \delta_{js} \rho^{(n)} \omega^2, \quad B^{(n)}_{jk} = -\alpha_i (C^{(n)}_{ki3} + C^{(n)}_{ki3}), \quad J^{(n)}_{jk} = C^{(n)}_{3k3j},$$

$r, i, j = 1, 2; \quad j, k, s, m = 1, 2, 3.$

(30)

On the interfaces between layers the continuity conditions for the displacements and stresses can be written in matrix form as follows:

$$R^{(n)} U^{(n)}(z^{(n+1)}) = R^{(n+1)} U^{(n+1)}(z^{(n+1)}),$$

$$U_j^{(n)}(z^{(n+1)}) = U_j^{(n+1)}(z^{(n+1)}), \quad j = 1, 2, 3; \quad n = \overline{1,N-1},$$

$$R^{(n)}_{jk} = -i \alpha_m c^{(n)}_{kmj3}, \quad R^{(n)}_{j(k+3)} = C^{(n)}_{k3j3}, \quad j, k = 1, 2, 3; \quad m = 1, 2.$$  

(31)

With respect to the unknown vectors $t^{(n)}$, equations (31) lead to the linear algebraic equations

$$C^{(n)}(z^{(n+1)}) t^{(n)} - C^{(n+1)}(z^{(n+1)}) t^{(n+1)} = 0, \quad n = \overline{1,N-1}$$

(32)

with matrices $C^{(n)}(z)$ defined as follows:

$$C^{(n)}(z) = \begin{pmatrix} R^{(n)} H^{(n)} \\ H^{(n)} \end{pmatrix} E^{(n)}(z) = C^{(n)}(0) E^{(n)}(z),$$

(33)

where $H^{(n)}$ is a matrix constructed from the eigenvectors $p^{(n)}_{ij}$ of systems (29), namely

$$H_{ij}^{(n)} = p^{(n)}_{ij}, \quad i, j = 1, 6; \quad H_{ij}^{(n)} = \hat{H}_{ij}^{(n)} = p^{(n)}_{ij}, \quad i = 1, 2, 3; \quad j = 1, 6.$$  

(34)

where $p^{(n)}_{ij}$ is the component $i$ of the eigenvector with number $j$ for layer $n$. In formula (33), the expression $E^{(n)}(z)$ is a diagonal matrix with the elements

$$C^{(n)}(0) = \begin{pmatrix} \varepsilon^{(n)}_i \\ R^{(n)} H^{(n)} \end{pmatrix}, \quad t^{(n)} = \begin{pmatrix} t_1^{(n)} \\ \vdots \\ t_6^{(n)} \end{pmatrix}^T.$$  

(35)
If the load acting on the surface is given by Dirac function, the system for \( t^{(1)} \) takes the following form:

\[
R^{(1)} H^{(1)} t^{(1)} = e_k, \quad e_k = \{\delta_{1k}, \delta_{2k}, \delta_{3k}\}^T, \quad k = 1, 2, 3. \tag{36}
\]

Relations (32, 36) form a system with a block-diagonal matrix \( P \), namely

\[
PT = E_k, \quad P_{1,nm}^{(1)} = C_{nm}^{(1)} (z^{(1)}) , \quad P_{N+1,nm}^{(N)} = C_{nm}^{(N)} (z^{(N+1)}) , \quad n = 1, 2, 3, \quad m = 1, 6, \tag{37}
\]

where

\[
P = \begin{pmatrix}
P_1^{(1)} & -P_2^{(2)} & 0 & \cdots & 0 \\
0 & P_1^{(1)} & -P_2^{(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & P_{N-1}^{(N-1)} & -P_N^{(N)} \\
0 & 0 & \cdots & 0 & P_{N+1}^{(N+1)} \\
\end{pmatrix}, \quad T = \begin{pmatrix}
t^{(1)} \\
t^{(2)} \\
\vdots \\
t^{(N-1)} \\
t^{(N)}
\end{pmatrix}, \quad E_k = \begin{pmatrix}
e_k \\
0 \\
\vdots \\
0
\end{pmatrix}. \tag{38}
\]

In formula (35) for the matrix \( C^{(N)} (z^{(N+1)}) \), the matrix \( R^{(n)} \) will be substituted by the matrix \( \tilde{R}^{(N)} \) corresponding to the chosen boundary conditions (5, 6 or 7):

\[
a) \ \tilde{R}_{ij}^{(N)} = \delta_{ij}, \quad b) \ \tilde{R}_{ij}^{(N)} = R_{ij}^{(N)}, \quad c) \ \tilde{R}_{ij}^{(N)} = R_{ij}^{(N)}, \quad \tilde{R}_{3j}^{(N)} = \delta_{3j}, \quad i = 1, 2. \tag{39}
\]

Relations (37, 38) allow us to define \( t^{(n)} \) by \( t^{(n+1)} \) via

\[
t^{(n)} = C^{(n)} (z^{(n+1)})^{-1} C^{(n+1)} (z^{(n+1)}) t^{(n+1)} \tag{40}
\]

and sequentially all \( t^{(n)} \) by \( t^{(N)} \). In particular for \( t^{(1)} \)

\[
t^{(1)} = (C^{(1)} (z^{(2)})^{-1} C^{(2)} (z^{(2)}) (C^{(2)} (z^{(3)})^{-1} C^{(3)} (z^{(3)}) \ldots \ldots (C^{(N-1)} (z^{(N)})^{-1} C^{(N)} (z^{(N)}) t^{(N)} \equiv \tilde{D} t^{(N)}. \tag{41}
\]

In order to find \( t^{(N)} \) we substitute (41) into (36) and, with regard to the last equation (37), obtain a system

\[
\begin{pmatrix}
P_{1}^{(1)} \tilde{D} P_{N+1}^{(N)} \\
P_{N+1}^{(N)}
\end{pmatrix}
= \begin{pmatrix}
e_k \\
0
\end{pmatrix}, \quad (k = 1, 2, 3). \tag{42}
\]

After solving (42), other \( t^{(n)} \) are specified from relations (40). This is the stage at which the basic part of the algorithm of calculating the matrix \( K \) is completed. Further transformations are required for the extension of the range of arguments \( \alpha, \omega \) in which the calculations of the matrix \( K \) remain
correct. The exponent indices $\lambda^{(n)}$ entering in the matrices $C, E$ (29) grow with increasing values of $\alpha, \omega$ and lead to a quick exponentional overflow during computer calculations. The requirement of such transformations often appears during direct calculations of contour integrals, evaluation of asymptotic forms etc.

Let us introduce new unknown values $\tilde{S}^{(n)}$ via

$$t^{(n)} = G \left( z^{(n+1)} \right) \tilde{S}^{(n)}, \tag{43}$$

where $G$ is the diagonal matrix given by

$$G_{ii}(z^{(n+1)}) = e^{(\hat{\lambda}^{(n)} - \lambda^{(n)}) z^{(n+1)}}, \quad i = 1, 6. \tag{44}$$

In the last formula, $\hat{\lambda}^{(n)}$ is the eigenvalue of system (29) for the $n$-th layer with the largest real part. This substitution leads to the removal of exponents from the matrices $C^{(n)}$ given by

$$
\begin{align*}
C^{(n)}(z^{(n+1)}) t^{(n)} &= e^{\hat{\lambda}^{(n)} z^{(n+1)}} C^{(n)}(0) \tilde{S}^{(n)}, \\
C^{(n+1)}(z^{(n+1)}) t^{(n+1)} &= e^{\hat{\lambda}^{(n+1)} z^{(n+1)}} C^{(n+1)}(0) \tilde{E}^{(n+1)} \tilde{S}^{(n+1)},
\end{align*}
\tag{45}$$

where $\tilde{E}^{(n+1)}$ is a diagonal matrix with the elements

$$\tilde{E}_{ii}^{(n+1)} = e^{\left(\hat{\lambda}^{(n+1)} - \lambda^{(n+1)}\right) \left(z^{(n+2)} - z^{(n+1)}\right)}, \quad i = 1, 6. \tag{46}$$

Since $\text{Re} \lambda^{(n+1)} \geq \text{Re} \lambda_j^{(n+1)}$ and $z^{(n+2)} - z^{(n+1)} < 0$, the elements $\tilde{E}^{(n+1)}$ are less than one in modulus.

Below we describe an algorithm to construct the symbol of the Green’s matrix for a multilayered structure analogous to that in [1]. The algorithm is derived on the basis of the substitution mentioned above. It allows us to avoid growing exponents at all calculation stages.

1. $D_n = \left( C^{(n)}(0) \right)^{-1} C^{(n+1)}(0) \tilde{E}^{(n+1)}$,  $n = \overline{1, N - 1}$,  $N \geq 2$,
2. $D = D_1 D_2 \ldots D_{N-1}$,
3. $S_N = M^{-1} \left( \begin{array}{c} e_k \\ 0 \end{array} \right) = \left( \begin{array}{c} P_1^{(1)} G \left( z^{(2)} \right) D \\ P_N^{(N)} \end{array} \right)^{-1} \left( \begin{array}{c} e_k \\ 0 \end{array} \right)$,  $k = 1, 2, 3$,
4. $S_n = D_n S_{n+1}$,  $n = \overline{N - 1, 1}$,
5. $Y^{(n)}(z) = H^{(n)} Y^{(n)}(z) S_n$. \tag{47}

Here $Y^{(n)}$ is a diagonal matrix

$$
\begin{align*}
Y_{ii}^{(n)} &= e^{\lambda^{(n)} z + (\hat{\lambda}^{(n)} - \lambda^{(n)}) z^{(n+1)} - \sum_n}, \quad i = 1, 6 \\
\Sigma_n &= \sum_{k=1}^{n-1} (\hat{\lambda}^{(k+1)} - \lambda^{(k)}) z^{(k+1)}, \quad \Sigma_1 = 0, \\
S_n &= S_n e^{\Sigma_n}.
\end{align*}
\tag{48}$$
The elements of the matrices $H^{(n)}$, $G$ are given by formulas (34) and (44), respectively. The last formula (47) can be written in the following way:

$$U^{(n)}(z) = H^{(n)} Y^{(n)}(z) D_n \cdots D_{N-1} M^{-1} \begin{pmatrix} e_k \\ 0 \end{pmatrix}. \quad (49)$$

The column of the Green’s matrix corresponds to the vector $\begin{pmatrix} e_k \\ 0 \end{pmatrix}$; hence for the layer with number $n$ we obtain

$$K(z) = H^{(n)} Y^{(n)}(z) D_n D_{n+1} \cdots D_{N-1} M^{-1} \begin{pmatrix} I \\ O \end{pmatrix}, \quad z^{(n+1)} \leq z \leq z^{(n)}, \quad (50)$$

where $I$, $O$ are the identity and null matrices, respectively.

Let us define the matrix $\tilde{D}_n$ by

$$\tilde{D}_n = \tilde{Y}^{(n)} D_n D_{n+1} \cdots D_{N-1} M^{-1} \begin{pmatrix} I \\ O \end{pmatrix}, \quad (51)$$

where $\tilde{Y}^{(n)}$ is a diagonal matrix with the elements

$$\tilde{Y}_{ii}^{(n)} = e^{(\bar{\lambda}^{(n)} - \lambda^{(n)}) z_{n+1} - \Sigma_n}, \quad i = 1, 6. \quad (52)$$

Here $\Sigma_n$ is given by relations (48). Then formula (50) obtains its final form

$$K(z) = K^{(n)}(z) = H^{(n)} E^{(n)}(z) \tilde{D}_n, \quad z^{(n+1)} \leq z \leq z^{(n)}. \quad (53)$$

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