Advective–Dispersive Solute Transport
in Inhomogeneous Porous Media

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Abstract

An one-dimensional advective–dispersive equation is solved for constant solute dispersion along non-uniform flow through inhomogeneous medium. Velocity of the flow is considered exponential function of space variable. First order decay term which is proportional to flow velocity is also considered. Analytical solutions are obtained for continuous point sources of uniform and increasing nature in an initially solute free semi-infinite domain by using Laplace transformation.

Keywords: Point source; Inhomogeneity; Pollutant; Groundwater

1. Introduction

Degradation of groundwater quality either caused by point sources such as septic tank, garbage disposal sides, cemeteries, mine spoils and oil spoils or other accidental entry of pollutants into the underground environment or line sources of poor quality of water, like seepage from polluted streams or intrusion of salt water, from oceans. Advective-dispersive equation has many applications like groundwater hydrology, chemical engineering bio-sciences, environmental sciences and petroleum engineering. A list of some exhaustive works are like Bastian and Lapidus (1956), Banks and Ali (1964), Ogata (1970), Al-Niami and Rushton (1977). Most of these works take into account the effects due to adsorption, first order decay, zero order production. Such solutions have been compiled by van Genuchten and Alves (1982) and Lindstrom and Boersma (1989). Logan and Zlotnik (1995), Golz and Dorroh (2001), Smedt (2006) obtained solutions of the convection–diffusion equation with decay and production. Jaiswal et al. (2009) and Kumar et al. (2010) obtained analytical
solutions for temporally and spatially dependent solute dispersion in one dimensional semi-infinite media.

Since analytical solution of hydrodynamic dispersion problems, even in one dimensional may be obtained in limited cases, so numerical solutions of the problems applicable to realistic engineering have been obtained using finite difference and finite element method. But Yates (1990, 1992) developed an analytical solution for describing the transport of dissolved substances in heterogeneous porous media with a space-dependent dispersion relationship. In the present paper, one-dimensional advective–dispersive equation with spatially dependent flow velocity is solved for constant solute dispersion with decay term. Flow velocity is considered exponentially function of space. Continuous point source of uniform nature and increasing nature are considered in an initially solute free semi-infinite domain.

2. Analytical solution

The transport processes of contaminant in porous media results from the advection and dispersion of contaminated fluid in such environment. The contaminants tend to spread due to molecular diffusion and hydrodynamic dispersion. The advective-dispersive equation in one-dimension may be written as

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left( D(x,t) \frac{\partial C}{\partial x} - u(x,t)C \right)$$

(1)

where $C$ is the solute concentration, $x$ is space variable, $t$ is time, $D(x,t)$ is solute dispersion and is called the dispersion coefficient if it is uniform and steady, and $u(x,t)$ is the flow velocity of the medium. Let us write $D(x,t) = D_0$ and $u(x,t) = u_0 \exp(\pm ax)$ and first order decay term which is proportional to flow velocity i.e., $\lambda(x,t) = \lambda_0 \exp(\pm ax)$ where $\lambda_0 = \mp au_0$. The parameter $a$ is inhomogeneity factor and have dimension of inverse of space variable. Inhomogeneity of the medium addressed by dividing the medium into stratified layers and variation in the flow velocity causes by the inhomogeneity of the medium. Eq. (1) may be written as

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left( D_0 \frac{\partial C}{\partial x} - u_0 \exp(\pm ax)C \right) - \lambda_0 \exp(\pm ax)C,$$

(2)

where $D_0$, $u_0$ and $\lambda_0$ are constants. Let us introduce a new independent variable $X$ by following transformation $x = \mp \log X$

(3)
Eq. (2) becomes,
\[ \frac{\partial C}{\partial t} = D_o X^2 \frac{\partial^2 C}{\partial X^2} + D_o X \frac{\partial C}{\partial X} - u_0 X^{-\alpha + 1} \frac{\partial C}{\partial X} \]  
(4)

The thickness between two layers is so small and transport properties being homogeneous in each layer and differing from other layer. The range of inhomogeneity factor in following discussion taken to be account as \( 0 < \alpha \leq 0.01 \) for each layer and after this range the transport properties are changed. There is no any fixed range of the inhomogeneity factor for the stratification. It is depend upon structure of medium. Pickens and Grisak (1981) supposed this factor is \( 0.05 \leq \alpha \leq 0.1 \) but Yates (1990) considered the range of inhomogeneity factor is \( 0 \leq \alpha \leq 2 \). Let the inhomogeneity factor is \( \alpha = 0.01 \). Therefore, \( X^{-\alpha + 1} \) becomes \( X^{-0.01+1} = X^{0.99} \). Now, if 0.99 is considered approximately 1.0. Then the Eq. (4) reduces into,
\[ \frac{\partial C}{\partial t} = D_o X^2 \frac{\partial^2 C}{\partial X^2} - (u_0 - D_o) X \frac{\partial C}{\partial X} \]  
(5)

Again, introducing a new independent variable \( Z \) by the following transformation, \( \log Z = \frac{x}{m} \)  
(6)

The partial differential equation (5) becomes,
\[ \frac{\partial C}{\partial t} = D_o \frac{\partial^2 C}{\partial Z^2} - u_0 \frac{\partial C}{\partial Z} \]  
(7)

Let the domain is initially solute free, it means before introduction of input source the domain is pollution free. An input concentration of uniform nature is assumed at the origin of the domain. The second boundary condition is considered of flux type of homogeneous nature. The initial and boundary conditions for Eq. (1) in a semi-infinite longitudinal domain are
\[ C(x,t) = 0, \ x \geq 0, \ t = 0 \]  
(8)
\[ C(x,t) = C_0, \ x = 0, \ t > 0 \] and \[ \frac{\partial C(x,t)}{\partial x} = 0, \ x \to \infty, t \geq 0 \]  
(9a,b)

The initial and boundary conditions are reduced in terms of new independent variable and applying Laplace transformation to get the analytical solution of advective-dispersive equation for uniform input is,
\[ C(x,t) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x-u_0 t}{2 \sqrt{D_0 t}} \right) + \exp \left( \frac{u_0 x}{D_0} \right) \text{erfc} \left( \frac{x+u_0 t}{2 \sqrt{D_0 t}} \right) \right], \]  
(10)

Due to human and other responsible activities, input condition may not be uniform. It is of increasing nature. For this type of situation Cauchy type boundary condition is applicable, i.e.,
\[ u(x,t)C(x,t) - D(x,t) \frac{\partial C}{\partial x} = u_0 C_0, \ x = 0, \ t > 0 \]  
(11)

Using this condition in the place of (9a) and using all above transformation and
applying Laplace transformation on Eq. (11), the analytical solution of advective-dispersive equation for increasing input is,

\[
C(x,t) = \left[ \frac{1}{2} \text{erfc} \left( \frac{x-u_0t}{2\sqrt{D_0t}} \right) + \frac{u_0}{\pi D_0} \exp \left( -\frac{(x-u_0t)^2}{D_0t} \right) \right] \\
- \frac{1}{2} \left[ 1 + \frac{u_0x}{D_0} + \frac{u_0^2t}{D_0} \right] \exp \left( \frac{u_0x}{D_0} \right) \text{erfc} \left( \frac{x+u_0t}{2\sqrt{D_0t}} \right),
\]

(12)

3. Result and Discussions

The concentration values are evaluated from the analytical solutions described by Eqs. (10) and (12) in a finite domain \(0 \leq x \leq 10\) (km). The other input values are considered as: \(C_0 = 1.0\), \(D_0 = 1.44\) (km\(^2\)/year), \(u_0 = 0.25\) (km/year). Curves are drawn for \(t = 0.4\), \(0.7\) and \(1.0\) (year) in both the figures. In Fig. (1), concentration values are started from one at \(x = 0.0\), which is show the boundary condition of uniform nature and concentration values decreases at a particular position along the domain with increasing time. In Fig. (2), concentration values are increases when time increases at same position and constant far from the origin. When the time is less, the distribution of concentration goes to negative at some position and then increases and becomes constant. This type of nature shows in Fig (2).

Inhomogeneity factor is considered \(a = 0.01\). Advective-dispersive solute transport with variable coefficient reduced into constant coefficients without first order decay term by taking \(a = 0.0\). The comparisons have been done between concentration distribution patterns for variable and constant coefficients of advective-dispersive equation for uniform input at time \(t = 1.0\) (year). Concentration value for constant (shown by circle) and variable coefficient (shown by star) are same at same time. The analytical solution can be used to characterize differences in the transport process relative to the classical convection-dispersion equation which assumes that the hydrodynamic dispersion in the porous medium remains constant.

4. Conclusion

In the present work, one-dimensional advective-dispersive equation is solved with constant dispersion, spatially dependent velocity and first order decay term. Continuous point sources of uniform and increasing type are considered in an initially solute free semi-infinite domain. The degradation is caused due to advective-dispersive transport of pollutants from the point source. Solutions are obtained by using Laplace transformation.
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References


Figures

Fig. 1. Distribution behavior of concentration at different time for uniform input source.

Fig. 2. Distribution behavior of concentration at different time for increasing input source.

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