An Analytic Study of the $(2 + 1)$-Dimensional Potential Kadomtsev-Petviashvili Equation

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Abstract

In this paper, variational iteration method (VIM) is applied to obtain approximate analytical solution of the $(2 + 1)$-dimensional potential Kadomtsev-Petviashvili equation (PKP) without any discretization. Comparisons with the exact solutions reveal that VIM is very effective and convenient.

Keywords: Variational iteration method (VIM); $(2 + 1)$-dimensional potential Kadomtsev-Petviashvili equation (PKP); Soliton-like solutions

1 Introduction

We consider the $(2+1)$-dimensional potential Kadomtsev-Petviashvili equation (PKP) in the form

$$u_{xt} + \frac{3}{2}u_x u_{xx} + \frac{1}{4}u_{xxxx} + \frac{3}{4}u_{yy} = 0,$$

(1)

where the initial conditions $u(x, 0, t)$ and $u_y(x, 0, t)$ are given. Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either numerically or theoretically. There has recently been much attention
devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models [1, 2, 3].


The variational iteration method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He [8] (see also [9, 10, 11, 12, 13]). The VIM has successfully been applied to many situations. For example, He [9] solved the classical Blasius’ equation using VIM. He [10] used VIM to give approximate solutions for some well-known nonlinear problems. He [11] used VIM to solve autonomous ordinary differential systems. He [12] coupled the iteration method with the perturbation method to solve the well-known Blasius equation. He [13] solved strongly nonlinear equations using VIM. Soliman [14] applied the VIM to solve the KdV-Burger’s and Lax’s seventh-order KdV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa et al. [15]. The VIM has been applied for solving nonlinear differential equations of fractional order by Odibat et al. [16]. Bildik et al. [17] used VIM for solving different types of nonlinear partial differential equations. Dehghan and Tatari [18] employed VIM to solve a Fokker-Planck equation. Wazwaz [19] presented a comparative study between the variational iteration method and Adomian decomposition method. Tamer et al. [20] introduced a modification of VIM. Batiha et al. [21] used VIM to solve the generalized Burgers-Huxley equation. Batiha et al. [22] applied VIM to the generalized Huxley equation. Abbasbandy [23] solved one example of the quadratic Riccati differential equation (with constant coefficient) by He’s variational iteration method by using Adomian’s polynomials.

In this paper, we shall apply VIM to find the approximate analytical solution of PKP equation. Comparisons with the exact solution shall be performed.
2 Variational iteration method

VIM is based on the general Lagrange’s multiplier method [25]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution [24].

To illustrate the basic concepts of VIM, we consider the following nonlinear differential equation:

\[ Lu + Nu = g(x), \quad (2) \]

where \( L \) is a linear operator, \( N \) is a nonlinear operator, and \( g(x) \) is an inhomogeneous term. According to VIM [10, 11, 13, 24], we can construct a correction functional as follows:

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left( L u_n(\tau) + N \tilde{u}_n(\tau) - g(\tau) \right) d\tau, \quad n \geq 0, \quad (3) \]

where \( \lambda \) is a general Lagrangian multiplier [25] which can be identified optimally via the variational theory, the subscript \( n \) denotes the \( n \)th-order approximation, \( \tilde{u}_n \) is considered as a restricted variation [10, 11], i.e. \( \delta \tilde{u}_n = 0 \).

3 Analysis of the PKP equation

In this section, we present the solution of Eq. (1) by means of VIM. To do so, we first construct a correction functional,

\[ u_{n+1} = u_n + \int_0^y \lambda(s) \left[ (u_n)_{ss} + \frac{4}{3} (\tilde{u}_n)_{xt} + 2 (\tilde{u}_n)_x (\tilde{u}_n)_{xx} + \frac{1}{3} (\tilde{u}_n)_{xxxx} \right] ds, \quad (4) \]

where \( \tilde{u}_n \) is considered as restricted variations, which means \( \delta \tilde{u}_n = 0 \). To find the optimal \( \lambda(s) \), we proceed as follows:

\[ \delta u_{n+1} = \delta u_n + \delta \int_0^y \lambda(s) \left[ (u_n)_{ss} + \frac{4}{3} (\tilde{u}_n)_{xt} + 2 (\tilde{u}_n)_x (\tilde{u}_n)_{xx} + \frac{1}{3} (\tilde{u}_n)_{xxxx} \right] ds, \quad (5) \]

and consequently

\[ \delta u_{n+1} = \delta u_n + \delta \int_0^y \lambda(s) [(u_n)_{ss}] ds, \quad (6) \]

which results in

\[ \delta u_{n+1} = \delta u_n (1 - \lambda'(s)) + \delta (u_n)_s \lambda(s) + \int_0^y \delta u_n \lambda''(s) ds = 0. \quad (7) \]
The stationary conditions can be obtained as follows:

\[ 1 - \lambda'(s) = 0 \bigg|_{s=y}, \quad \lambda(s) = 0 \bigg|_{s=y}, \quad \lambda''(s) = 0 \bigg|_{s=y}, \quad (8) \]

The Lagrange multipliers, therefore, can be identified as

\[ \lambda(s) = s - y, \quad (9) \]

and the iteration formula is given as

\[ u_{n+1} = u_n + \int_0^y (s - y) \left( (u_n)_{ss} + \frac{4}{3}(u_n)_{xt} + 2(u_n)_x(u_n)_{xx} + \frac{1}{3}(u_n)_{xxxx} \right) ds, \quad (10) \]

4 Applications

We first consider the PKP equation (1) with initial conditions

\[ u(x, 0, t) = 1 + 2k\alpha \tanh[k(\alpha x - ct)], \quad u_y(x, 0, t) = 2\alpha \beta k^2 \text{sech}^2[k(\alpha x - ct)], \quad (11) \]

where \( c = (k^2 \alpha^3 + \frac{3\beta^3}{4\alpha}). \)

The exact solution for Eq. (1) was found to be [4]:

\[ u(x, y, t) = 1 + 2k\alpha \tanh \left[ k \left( \alpha x + \beta y - \left( k^2 \alpha^3 + \frac{3\beta^3}{4\alpha} \right) t \right) \right], \quad (12) \]

We can take an initial approximation as

\[ u_0 = u(x, 0, t) + yu_y(x, 0, t), \quad (13) \]

\[ = 1 + 2k\alpha \tanh[k(\alpha x - ct)] + 2\alpha \beta k^2 y \text{sech}^2[k(\alpha x - ct)], \quad (14) \]

where \( c = (k^2 \alpha^3 + \frac{3\beta^3}{4\alpha}). \)

The first iterate is easily obtained from (10) and is given by:

\[ u[1] = 1 + 2kam + 2\alpha \beta k^2 y^2 - 2\alpha^2 \beta k^4 n^2 m(8\alpha^3 \beta k^4 n^2 m^2 \]

\[ - 4\alpha^3 \beta k^4 n^2 (1 - m^2)y^4 + 1/3(8y\alpha^2 \beta k^3 n^2 m(8\alpha^3 \beta k^4 n^2 m^2 \]

\[ - 4\alpha^3 \beta k^4 n^2 (1 - m^2)) + 32/3\alpha^2 \beta k^4 n^2 m^2(-k^2 \alpha^3 - 3/4\beta^3/\alpha) \]

\[ + 4k^2 \alpha^2 (1 - m^2)(8\alpha^3 \beta k^4 n^2 m^2 - 4\alpha^3 \beta k^4 n^2 (1 - m^2)) \]

\[ - 80/3\alpha^5 \beta k^6 n^2 m^2 (1 - m^2) - 16/3\alpha^2 \beta k^4 n^2 (1 - m^2)(-k^2 \alpha^3 - 3/4\beta^3/\alpha) \]

\[ + 32/3\alpha^5 \beta k^6 n^2 m^4 + 32/3\alpha^5 \beta k^6 n^2 (1 - m^2)y^3 \]

\[ + 1/2(-y(32/3\alpha^2 \beta k^4 n^2 m^2(-k^2 \alpha^3 - 3/4\beta^3/\alpha) \]

\[ + 4k^2 \alpha^2 (1 - m^2)(8\alpha^3 \beta k^4 n^2 m^2 - 4\alpha^3 \beta k^4 n^2 (1 - m^2)) \]
The first iterate is easily obtained from (10) and is given by:

\[ u_0 = u(x, 0, t) + y u_y(x, 0, t) \]
\[ = 1 + 2k \alpha \tan[k(\alpha x - ct)] + 2\alpha \beta k^2 \sec^2[k(\alpha x - ct)] \]  

The first iterate is easily obtained from (10) and is given by:

\[ u[1] = 1 + 2k \alpha m + 2\alpha \beta k^2 y n^2 \]  
\[ + 2\alpha^2 \beta k^3 n^2 m (8\alpha^3 \beta k^4 n^2 m^2) \]
\[ + 4\alpha^3 \beta k^4 n^2 (1 + m^2) y^4 + 1/3 (-8y^2 \beta k^3 n^2 m (8\alpha^3 \beta k^4 n^2 m^2) \]

Figure 1: Comparison between exact solution and the numerical results for \( u(x, y, t) \) by means of 1-iterate VIM solution when \( \alpha = 1, \beta = 0.1, k = 0.1 \) and \( c^\prime = k^2 \alpha^3 + \frac{3\beta^3}{4\alpha} \)

\[ -80/3\alpha^5 \beta k^6 n^2 m^2 (1 - m^2) - 16/3\alpha^2 \beta k^4 n^2 (1 - m^2) (-k^2 \alpha^3 - 3/4 \beta^3 / \alpha) \]
\[ + 32/3\alpha^5 \beta k^6 n^2 m^4 + 32/3\alpha^5 \beta k^6 n^2 (1 - m^2)^2) \]
\[ -16/3\alpha^3 \alpha^2 m (1 - m^2)(-k^2 \alpha^3 - 3/4 \beta^3 / \alpha) - 16/3k^5 \alpha^5 (1 - m^2)^2 m \]
\[ -16/3k^5 \alpha^5 m^3 (1 - m^2)^2 y^2 - y^2 (-16/3k^3 \alpha^2 m (1 - m^2)(-k^2 \alpha^3 - 3/4 \beta^3 / \alpha) \]
\[ -16/3k^5 \alpha^5 (1 - m^2)^2 m - 16/3k^5 \alpha^5 m^3 (1 - m^2) \]
Table 1: Absolute errors between the 1-iterate of VIM and the exact solutions (17) when $\alpha = 1, \beta = 0.1, k = 0.1$ and $c = k^2\alpha^3 + \frac{3\beta^3}{4\alpha}$

<table>
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<tr>
<th>$x_i/y_i$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
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<td>0.1</td>
<td>$1.88111E-08$</td>
<td>$7.76727E-08$</td>
<td>$1.80227E-07$</td>
<td>$3.30118E-07$</td>
<td>$5.30987E-07$</td>
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<td>$1.50595E-07$</td>
<td>$3.44314E-07$</td>
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<td>0.3</td>
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<td>$2.23718E-07$</td>
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<td>0.4</td>
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<td>0.5</td>
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<td>$8.40199E-07$</td>
<td>$1.50356E-06$</td>
<td>$2.36474E-06$</td>
</tr>
</tbody>
</table>

$+4\alpha^3\beta k^4n^2(1 + m^2) + 32/3\alpha^2\beta k^4n^2m^2c$

$+4k^2\alpha^2(1 + m^2)(8\alpha^3\beta k^4n^2m^2 + 4\alpha^3\beta k^4n^2(1 + m^2))$

$+272/3\alpha^5\beta k^6n^2m^2(1 + m^2) + 16/3\alpha^2\beta k^4n^2(1 + m^2)c + 32/3\alpha^5\beta k^6n^2m^4$

$+32/3\alpha^5\beta k^6n^2(1 + m^2)^2)y^3 + 1/2(-y(32/3\alpha^2\beta k^4n^2m^2c$

$+4k^2\alpha^2(1 + m^2)(8\alpha^3\beta k^4n^2m^2\alpha^3\beta k^4n^2(1 + m^2))$

$+272/3\alpha^5\beta k^6n^2m^2(1 + m^2) + 16/3\alpha^2\beta k^4n^2c + 32/3\alpha^5\beta k^6n^2m^4$

$+32/3\alpha^5\beta k^6n^2(1 + m^2)^2 + 16/3k^3\alpha^2m(1 + m^2)c + 80/3k^5\alpha^5(1 + m^2)^2m$

$+16/3k^5\alpha^5m^3(1 + m^2)y^2 - y^2(16/3k^3\alpha^2m(1 + m^2)c + 80/3k^5\alpha^5(1 + m^2)^2m$

$+16/3k^5\alpha^5m^3(1 + m^2))$

where $c = (k^2\alpha^3 + \frac{3\beta^3}{4\alpha}), m = \tan[k(\alpha x - ct)], n = \sec[k(\alpha x - ct)]$.

Table 1 shows the comparison between the 1-iterate of VIM and the exact solution (17).

5 Conclusions

In this paper, the variation iteration method (VIM) has been successfully employed to obtain the approximate analytical solutions of the (2+1)-dimensional potential Kadomtsev-Petviashvili equation (PKP). The method has been applied directly without using linearization or any restrictive assumptions. Comparisons with the exact solution reveal that VIM is very effective and convenient. It may be concluded that VIM is very powerful and efficient in finding analytical as well as numerical solutions for a wide class of linear and non-linear differential equations. VIM provides more realistic series solutions that converge very rapidly in real physical problems.
References


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