Study of Existence of Linear and Nonlinear Wave in 

a Non-Darcian Fluid Saturated Porous Medium

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Abstract. Study of linear and nonlinear wave propagation in a heterogeneous fluid saturated porous medium is studied using both numerical and analytical techniques considering Darcy-Lapwood-Forchheimer model. This model leads to a system of nonlinear partial differential equations which are then reduced to two nonlinear ordinary differential equations with the help of an intermediate integral. In general, we observe that both linear and nonlinear travelling waves are non periodic. These waves are found to be periodic only when they propagate either horizontally or vertically for small porous parameter $\sigma$. Similar behaviour of waves is observed even in the Darcy regime. For a general reduced system there exist two singularities but in particular we have shown that for horizontally propagating waves or in the absence of quadratic drag, there exists only one singular point. The system of equations are linearised about this singular point and the nature of the waves is studied. The observations made in this paper can help to select a suitable material in material processing for specific end application and also help in better understanding of medical imaging techniques.

Keyword: nonlinear wave, heterogeneous fluid, porous medium, Darcy-Lapwood-Forchheimer model, travelling waves, periodic waves
1 INTRODUCTION

In geophysics the earth's crust has been modelled as a two phase medium consisting of a material having net work of pores which in turn saturated by an incompressible viscous fluid (Biot 1956). The contraction and expansion of flow in these pores produce waves, known as travelling waves. In addition, understanding of the nature of these waves in porous layer is also important in many diverse fields in science, engineering and technology ranging from agriculture, petroleum, coastal, harbour engineering, contaminant transport in geological media, soil science, biomechanics, construction technology, chemical and ceramic engineering, metallurgy to food technology. Some of these problems involve natural porous media like rocks, sand, loose soil and so on and others involve synthetic porous material particularly in industrial and biomedical engineering applications. In these materials the propagation of travelling waves play significant role in improving the mobility of fluid in porous material. Because of these importance Dersiewiz (1960a,b,1962) had extended the work of Biot to study the propagation of these waves in liquid filled porous layer. Using sonic pulse techniques, some observational data in the propagation of these waves in fluid saturated porous layer were obtained by Fatt (1959) and Wyllie et.al. (1958). Later Yew and Jogie (1976) have made a comparative study of experimental results of Fatt (1959) and Wyllie et.al (1958) with the theoretical results predicted by Biot (1956a, b, 1962).

The works so far mentioned above have been concerned with propagation of elastic surface waves in a porous layer. Works of Venkatachallappa and Sekar (1984) are concerned with quantitative study of propagation of internal gravity waves in a flow through porous layer and a recent review of Chwang and Chan (1998) is concerned with propagation of surface waves past a porous structure.

However, the study of propagation of travelling waves through porous medium is very sparse and their study is the main object of this chapter. The theoretical study of such wave motion through a porous medium depends on using a general theory of porous media flow whose physical phenomenon is extremely complicated. Therefore in the study of wave motion Chwang and Chan (1998) mostly observed that the porous medium obeys Darcy's law. For wave motion across a porous medium Sollit and Chan (1972) gave modification to Darcy law. The contraction and expansion of flows in porous media generate travelling waves.

The effect of curvature of the flow in such porous media play a significant role on the generation of such waves. Because of the importance of such wave motion through porous medium we investigate in this present chapter both linear and nonlinear travelling waves in a stratified fluid saturated porous layer. The required basic equations incorporating the curvature effect are the Darcy-Lapwood-Forchheimer model. Although Lapwood term $\rho(\overline{q} \cdot \nabla)\overline{q}$ and Forchheimer term...
both represent inertial acceleration we have retained the Lapwood term to check our results with those in the absence of porous media as \( k \to \infty \).

## 2 MATHEMATICAL FORMULATIONS

We consider a Cartesian coordinate system with x-axis in the horizontal plane and z-axis along the vertical direction. We study the possible wave propagation in an incompressible stratified viscous heterogeneous fluid in a porous medium using Darcy-Lapwood-Forchheimer equation. The basic equations of this system are

\[
\nabla \cdot \vec{q} = 0, 
\]

(1)

\[
\frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla) \rho = 0, 
\]

(2)

\[
\rho \left[ \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla p + \frac{\rho \ddot{g}}{k} - \frac{\rho \bar{c}_b}{\sqrt{k}} |\vec{q}| \vec{q}. 
\]

(3)

where \( \vec{q} \) is the fluid velocity, \( p \) the hydrodynamic pressure, \( \rho \) the density, \( t \) the time, \( \ddot{g} \) the acceleration due to gravity, \( \mu \) viscosity of fluid, \( k \) permeability of the medium, and \( c_b \) is the quadratic drag.

Since the porous medium considered here is unbounded i.e. boundary effects are negligible so that Brinkman correction to Darcy equation is neglected. We assume the basic fluid to be at rest (i.e. quiescent) while the mass density \( \rho_0(z) \) and viscosity \( \mu_0(z) \) vary exponentially with \( z \) and are given by

\[
\rho_0(z) = \rho_c \exp(-z/H), \quad \mu_0(z) = \mu_c \exp(-z/H), \quad (H > 0) 
\]

(4)

where \( H \) is the scale height and \( \rho_c \) and \( \mu_c \) are the reference density and viscosity at \( z = 0 \). From (3) we obtain the pressure \( p_0(z) \) in the basic state as

\[
p_0(z) = p_c \exp(-z/H), 
\]

(5)

where \( p_c = gH\rho_c \). We superpose the arbitrary disturbances upon the above equilibrium configuration and making the resulting equations (1) to (3) dimensionless using \( H, \ (g/H)^{\frac{1}{2}}, \ (gH)^{\frac{1}{2}}, \ p_c \exp(-z/H) \) and \( \rho_c \exp(-z/H) \), as the scales for length, time, velocity, pressure and density respectively and obtain:

\[
\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{w}}{\partial \hat{z}} = 0, 
\]

(6)
\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} - \rho w = 0, \tag{7}
\]
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\sigma^2}{R} u + \sigma \beta u \sqrt{(u^2 + w^2)} = 0, \tag{8}
\]
\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\sigma^2}{R} w + \sigma \beta w \sqrt{(u^2 + w^2)} = 0, \tag{9}
\]
where \(\nu = \frac{\mu}{\rho_c}\) is the kinematic viscosity, \(R = \frac{UH}{\mu_c}\) is the Reynolds number, \(\sigma = \frac{H}{\sqrt{k}}\) is the porous parameter.

### 3 TRAVELLING WAVE SOLUTIONS

Equations (6) - (9) are highly non-linear and we seek the special travelling wave solutions in the form
\[
u = u(\phi), w = w(\phi), \rho = \rho(\phi), p = p(\phi), \tag{10}
\]
where \(\phi = \phi(x, y, z, t)\) is the phase function. The form of \(\phi\) is obtained below. Our object here is to show that the equations (6) to (9) admit solutions of the form (10) subject to the initial condition,
\[
u = w = 0, p = \rho = 1 \text{ at time } t = 0. \tag{11}
\]
Now equations (6) to (9) transforming to phase-plane using (10) can be written as
\[
u u_x + w_x = 0, \tag{12}
\]
\[
u p_x + (u_x + w_x) \rho_x - \rho w = 0, \tag{13}
\]
\[
u u_t + (u_x + w_x) u_x + \frac{1}{\rho} p_x + u \left( \frac{\sigma^2}{R} + \sigma \beta \sqrt{(u^2 + w^2)} \right) = 0, \tag{14}
\]
\[
u w_t + (u_x + w_x) w_x + \frac{1}{\rho} p_x + (1 - \frac{p}{\rho}) + w \left( \frac{\sigma^2}{R} + \sigma \beta \sqrt{(u^2 + w^2)} \right) = 0, \tag{15}
\]
where the suffixes denote the partial derivatives. Equations (12) - (15) can be treated as a linear inhomogeneous system of equations for the derivatives \(\phi_x, \phi_y\) and \(\phi_z\). We note that the determinant of the coefficient matrix does not vanish except for trivial solutions and therefore we obtain
\[
u \phi_x = \phi_y(\phi), \phi_y = \phi_z(\phi), \phi_z = \phi_x(\phi). \tag{16}
\]
From the compatibility of these partial derivatives we obtain
\[
u \phi = \frac{x}{\lambda_1} + \frac{y}{\lambda_2} - t. \tag{17}
\]
where $\lambda_1$ and $\lambda_3$ are arbitrary constants, which can be regarded as wave lengths in x and z directions respectively. Using (17), equations (12) to (15) can be written as

\[ \frac{u_\phi}{\lambda_1} + \frac{w_\phi}{\lambda_3} = 0, \] (18)

\[ E\rho_\phi - \rho w = 0, \] (19)

\[ Eu_\phi + \frac{1}{\rho\lambda_1} p_\phi + u(\sigma_1 + \sigma c_b \sqrt{(u^2 + w^2)}) = 0, \] (20)

\[ Ew_\phi + \frac{1}{\rho\lambda_3} p_\phi + w(\sigma_1 + \sigma c_b \sqrt{(u^2 + w^2)}) - \frac{p}{\rho} + 1 = 0, \] (21)

where $E = \frac{u}{\lambda_1} + \frac{w}{\lambda_3} - 1$ and $\sigma_1 = \frac{\sigma^2}{R}$. Integrating (18) and using the initial conditions (11) we get

\[ \frac{u}{\lambda_1} + \frac{w}{\lambda_3} = 0. \] (22)

This is a first integral of the system. Solving equations (18) - (21) for $u_\phi, w_\phi, \rho_\phi$ and $p_\phi$ and using (22) we get

\[ u_\phi = \frac{(K - 1)}{n\lambda_1\lambda_3} + u(\sigma_1 + u\sigma c_b \lambda_3 \sqrt{n}), \] (23)

\[ w_\phi = -\frac{(K - 1)}{n\lambda_3^2} + w(\sigma_1 + w\sigma c_b \lambda_1 \sqrt{n}), \] (24)

\[ K_\phi = \frac{(K - 1)}{n\lambda_3} + wK, \] (25)

where

\[ K = \frac{p}{\rho}, n = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_3^2}, \text{ (total wave number).} \]

In the limit $\sigma \to 0$ (i.e absence of porous media) equations (24) and (25) are similar to the equations obtained by Seshadri and Sachdev (1977) in the incompressible limit $\gamma \to \infty$, and the equations obtained by Venkatachalappa et al (1991) in the incompressible and non-rotating limits, that is, $\gamma \to \infty$ and $\Omega \to 0$.

The singular points of the equation (24) and (25) are

\[ i)w = 0, K = 1, \] (26)

\[ ii)w = \frac{1 - K_\beta}{n\lambda_3 K_\beta} K = \frac{\lambda_1^2}{2} \left[-a_1 \pm \sqrt{a_1^2 - 4a_2}\right], \] (27)

where
The singular point (26) describes the equilibrium state with zero velocity and exponential density stratification. This singular point is the same as the one obtained by Seshadri and Sachdev (1977) for compressible fluid and by Venkatachalappa et al (1991) for compressible rotating fluid. We further note that in the limit \( \sigma \to 0 \) the singular point given by (27) coincides with singular point of Seshadri and Sachdev (1977) in the incompressible limit, \( \gamma \to \infty \). In the case of waves in compressible fluid there exists two singularities in contrast to only one in incompressible fluid. In the case of incompressible fluid in a porous medium, the two singularities prevail as in the case of compressible fluid. Physically this can be attributed to contraction and expansion of waves in a porous medium which is analogous to the situation in compressible fluids.

### 3.1 LINEAR ANALYSIS

The linearised form of (24) and (25) about the singular point, (26), are

\[
a_1 = \frac{\sigma_1}{\lambda_3} - \frac{\sigma \lambda}{\lambda^2 2 n}, \quad a_2 = \frac{\sigma \lambda}{\lambda^2 2 n} \cdot
\]

(28)

The singular point (26) describes the equilibrium state with zero velocity and exponential density stratification. This singular point is the same as the one obtained by Seshadri and Sachdev (1977) for compressible fluid and by Venkatachalappa et al (1991) for compressible rotating fluid. We further note that in the limit \( \sigma \to 0 \) the singular point given by (27) coincides with singular point of Seshadri and Sachdev (1977) in the incompressible limit, \( \gamma \to \infty \). In the case of waves in compressible fluid there exists two singularities in contrast to only one in incompressible fluid. In the case of incompressible fluid in a porous medium, the two singularities prevail as in the case of compressible fluid. Physically this can be attributed to contraction and expansion of waves in a porous medium which is analogous to the situation in compressible fluids.

The linearised form of (24) and (25) about the singular point, (26), are

\[
\frac{dw'}{\lambda^2} = -\frac{K'}{n \lambda^2} + \sigma_1 w',
\]

(29)

\[
\frac{dK'}{\lambda^2} = \frac{K'}{n \lambda^2} + w'.
\]

(30)

where the quadratic drag term (i.e. Forchheimer term) is absent. The equations (29) and (30) in the phase-plane can be written as

\[
\frac{dw'}{dK'} = \frac{-K'}{n \lambda^2} + w' - \sigma_1 \frac{w'}{\lambda^2}.
\]

(31)

Here the primed quantities are perturbations over the equilibrium state (26) and are assumed to be very small. Development of the solution about the singularity \( w = 0 \) and \( K = 1 \) may be useful in patching up the numerical solution of the original nonlinear equation (24) to (25). However, we note that analytical solutions to linear equations (29) and (30) can be obtained by using a method of ordinary differential equation as explained below.

We assume a solution of (29) and (30) in the form \( e^{(\lambda \phi)} \) where \( \lambda \) satisfies the characteristic equation

\[
\lambda^2 - (\sigma_1 + \frac{1}{n \lambda^2}) \lambda + \left( \sigma_1 \frac{1}{n \lambda^2} + \frac{1}{n \lambda^3} \right) = 0.
\]

(32)
Here the quantity \((\sigma_i + \frac{1}{n\lambda_3})\) represents the resistance of the medium damping the waves. We note that in the absence of a porous medium the vertical propagation dampens the travelling waves in incompressible fluid. In the case of travelling waves in an incompressible fluid through a porous medium both the resistance offered by the particles of a porous medium and vertical propagation dampen the waves. We note that the linear equation (32) has a pair of pure imaginary roots when
\[
\sigma_i + \frac{1}{n\lambda_3} = 0. \tag{33}
\]
Physically this implies that there is no damping effect and the waves execute simple harmonic motion with a frequency \(\sqrt{\left(\frac{\sigma_i}{n\lambda_3} + \frac{1}{n\lambda_1^2}\right)}\). This will be true only when \(\lambda_3 < 0\). The solution of (31) is
\[
\log \left[ K^2 \left(\frac{w'^2}{K^2} + 2c_2 \frac{w'}{K} + \frac{1}{n\lambda_1^2}\right) \right] + \frac{c_3}{c_1} \tan^{-1} \left[ \frac{w'}{K} + c_2 \right] + c = 0, \tag{34}
\]
where
\[
c_1 = \left[ \frac{1}{n\lambda_2^2} - \frac{1}{4} \left(\frac{1}{n\lambda_3} - \sigma_i\right)^2 \right], \tag{35}
\]
\[
c_2 = \frac{1}{2n\lambda_3} \frac{\sigma_i}{2}, \tag{36}
\]
\[
c_3 = \frac{1}{2n\lambda_3} \frac{\sigma_i}{2}. \tag{37}
\]
The phase curves of (29) and (30) are depicted in Fig. 1a-1b for different values of \(\sigma_i\). In the remaining part of this section we consider the special cases namely \(\lambda_3 \to \infty\) and \(\sigma_i \to 0\).
i) The Case $\lambda_3 \to \infty$ $(\sigma \neq 0)$

This pertains to the case of waves propagating only in the x-direction. Hence when we consider only horizontally propagating linear waves, equations (29) - (30) become

$$w'_{,\phi} = -K' + \sigma_1 w', \quad (38)$$
$$K'_{,\phi} = w', \quad (39)$$

which in the phase-plane becomes

$$\frac{dw'}{dK} = -K' + \sigma_1 w'. \quad (40)$$

In the limit, $\lambda_3 \to \infty$ the characteristic equation (34) takes the form

$$\lambda^2 - \sigma_1 \lambda + 1 = 0. \quad (41)$$

The solution of (40) is

$$\log K' = \frac{1}{2} \log \left[ \frac{w'}{K'} - \frac{\sigma_1}{2} \right] + \left(1 - \frac{\sigma_1^2}{4}\right) - \frac{\sigma_1}{2} \tan^{-1} \left[ \frac{w'}{K'} - \frac{\sigma_1}{2} \right]. \quad (42)$$

Here $\sigma_1$ represents damping phenomena. The roots of (41) are

$$\lambda = \frac{\sigma_1 \pm \sqrt{\sigma_1^2 - 4}}{2}$$

These roots are real and distinct if $\sigma_1 > 2$ and are equal if $\sigma_1 = 2$ which represents resonance effect. The roots are complex if $\sigma_1 < 2$. They are pure imaginary (i.e., no dampening) when $\sigma_1 = 0$. Thus we have simple periodic type of solutions only when $\sigma_1 = 0$. Therefore, in general, horizontally propagating waves are not periodic in contrast to the cases studied by Seshadri and Sachdev (1977) for compressible medium and Venkatachalappa et al (1991) for compressible rotating medium. The phase curves for (38) and (39) are depicted in Fig.2a and 2b for different values of $\sigma_1$. 

![Phase curves](image-url)
ii) The Case $\sigma \to 0$ ($\lambda_3$ is finite)

This corresponds to the case of waves in an incompressible heterogeneous fluid in the absence of a porous medium. From equations (29) and (30) we get

$$\frac{dw'}{dK} = \left( \frac{\lambda_3}{\lambda_3^2} \right) \left( \frac{K'}{K' + w'n\lambda_3} \right).$$

(43)

The characteristic equation (34) for this case becomes

$$\lambda^2 - \left( \frac{1}{n\lambda_3} \right) \lambda + \frac{1}{n\lambda_3^2} = 0.$$  

(44)

This quadratic equation has purely imaginary roots if $\lambda_3 \to \infty$. Therefore horizontally propagating waves are periodic only in the fluid without porous layer. In the limit $\lambda_3 \to \infty$ and $\sigma \to 0$, phase curves are given by

$$\frac{w^2}{2} + \frac{K^2}{2} = \text{constant.}$$

(45)

Equation (45) represents circles. Thus linear waves are periodic when they propagate horizontally and the propagation behaviour is independent of drag and horizontal wave length.

3.2 NONLINEAR ANALYSIS

The nonlinear system is governed by ordinary differential equations (24) - (25). The solution of these equations is difficult to obtain analytically. Therefore, we solve them numerically and the phase curves are plotted in Fig. 3a and 3b. In general, we observe that the phase curves are not closed implying non-periodic solutions. In this case also we study the propagation of travelling waves in the following limiting cases. ($\lambda_3 \to \infty$, $\sigma \to 0$ and $c_b \to 0$).

i) The Case $\sigma \neq 0$

The case $\lambda_3 \to \infty$ implies only horizontally propagating waves where the system of equations (23) and (24) reduce to the equation

$$\frac{dw}{dK} = \frac{(1-K) + w(\sigma_1 + \sigma_2w)}{wK}.$$  

(46)

The only singular point of (46) is

$$K = 1, w = 0.$$  

(47)

The equations (46) and (47) coincide with those of Venkatachalappa et al (1991) studied for compressible rotating fluid and those of Seshadri and Sachdev (1977) studied for non-rotating system in the absence of porous medium. Seshadri and Sachdev (1977) have obtained two singular points, one is the same as (47) obtained
for the present problem and their second singular point coincides with the first in the incompressible limit, $\gamma \to \infty$. Phase curves are depicted in Fig. 3a and 3b for $\lambda_3$ finite and $\lambda_3 \to \infty$, we observe that the singular point is spiral.

\[ \frac{dw}{dK} = \frac{\lambda_3 (1 - K)}{\lambda_3^2 (K (wn\lambda_3 + 1) - 1)} . \]  

(48)

The only singular point of (48) is $K = 1$, $w = 0$.

\[ w^2 = \log K - K + 1. \]  

(50)

ii) The Case $\sigma \to 0$

In the absence of porous media, equations (24) and (25) become

In otherwords the singularity in the absence or presence of porous media remains the same. In the limit $\lambda_3 \to \infty$, equations (48) can be easily integrated to give

Phase curves are plotted in Fig. 4a, 4b using (48) for $\lambda_3$ finite and $\lambda_3 \to \infty$ in $(w, K)$ plane.
We now show that equations (24) and (25) for the limit $\sigma \to 0$ are equivalent to equation (1) obtained by Odulo et al (1977) for other physical systems. Eliminating $K$ between equations (24) and (25) we obtain

$$w_{\phi\phi} + \frac{1}{n\lambda^2} w - \frac{1}{n\lambda} w_{\phi} - ww_{\phi} = 0. \quad (51)$$

This equation can be transformed to

$$y'' + 2\alpha y' + yy' + y = 0, \quad (52)$$

where

$$w = \frac{1}{\sqrt{n\lambda^2}} y, \; \phi = -n\lambda^2 \tau, \; \alpha = \frac{1}{2\lambda_1\lambda_2\sqrt{n}}$$

and the primes in (52) denote differentiation with respect to $\tau$. This equation represents an one-dimensional oscillator with a nonlinear friction term in which the coefficient of friction depends linearly on the unknown function. This equation was also obtained by Odulo et al (1977) in the study of nonlinear stationary waves in three other systems in the ocean namely, internal waves, Rossby waves and topographic Rossby waves, where parameter $\alpha$ represents stratification, rotation or topography.

iii) The Case $c_b \to 0 \; (\sigma \neq 0)$

When we consider that the quadratic drag is negligible, that is, the limit $c_b \to 0$, system (24) - (25) reduce to a system of two ordinary differential equations

$$w_{\phi} = \frac{(1-K)}{n\lambda^2} + w\sigma, \quad (53)$$

$$K_{\phi} = \frac{(K-1)}{n\lambda^2} + wK. \quad (54)$$

The singular points of this system are $w = 0$ and $K = 1$. Thus it is clear that in the limit $c_b \to 0$, the phase curves are not closed for both linear and nonlinear cases. Hence only horizontally propagating waves are periodic when the medium is nonporous. For finite $\lambda_3$ we solve nonlinear equations numerically by using Runge-Kutta-Gill method. The phase curves are plotted in Fig. 5a,b for both $\lambda_3$ finite and infinite. From the figure we observe that the both the trajectories are not closed and hence the solutions are not periodic.
4. DISCUSSIONS

The propagation of linear and nonlinear travelling waves in heterogeneous fluid saturated porous medium have been investigated both analytically and numerically using phase-plane analysis. The suitable phase function transforms the system of PDEs to autonomous system of ODEs which are useful to study the singularities in phase-space. The linear analysis reveals that there exists only one singular point and no periodic solutions due to permeability and resistance. However in absence of resistance wave propagation is periodic. In case of nonlinear study there exist two singularities which are same as in compressible fluids and are spiral points. In general no periodic waves exists for finite or infinite horizontal wave length.

REFERENCES


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