This letter presents chaos synchronization of two different Sprott systems. Chaos synchronization of them by using active control is achieved. Based on the Lyapunov stability theory an adaptive control law is derived such that the two different Sprott systems are to be synchronized. Simulation results are given to demonstrate the effectiveness of the proposed control schemes.

**Keywords:** chaos synchronization, Sprott systems, adaptive synchronization

1. **Introduction**

Since Pecora and Carroll [1] introduced a method to synchronize two identical chaotic systems with different initial conditions, chaos synchronization have attracted a great deal of attention from various fields and has been studied extensively by many researchers during the last decades. Many methods and techniques for handling chaos control and synchronization have been developed, such as PC method[2],OGY method[3],feedback approach[4], adaptive method[5], time-delay feedback approach[6], backstepping design technique[7], etc. However, most of the methods mentioned above synchronize two identical chaotic systems. In fact, in many practical world such as laser array and biological systems, it is hardly the case that every component can be assumed to be identical. As a result, more and more applications of chaos synchronization in secure communication make it much more important to synchronize two different chaotic systems in recent years[8].This work considers the synchronization problems of two different chaotic systems. In paper[9], the author Sprott has offered some simple 3-Dimension autonomous ODEs with few quadratic non-
linearties, which are chaotic by numerical examination. The systems are called Sprott systems by researchers. In these Sprott systems, we select two different chaotic systems and study them. The two systems are Sprott H system and Sprott I system. And they have different topological structure by numerical examination. In this article, we apply two techniques to synchronize Sprott H system and Sprott I system. These two techniques are using active control and adaptive control. Numerical simulations are shown to verify the results.

2. Systems description

The Sprott H system is described by

\[
\begin{align*}
\dot{x} &= ay + z^2 \\
\dot{y} &= x + by \\
\dot{z} &= x - z
\end{align*}
\]  
(1)

which has a chaotic attractor as shown in Fig.1(a) when \( a = -1, b = 0.5 \). The Sprott I system is as follows

\[
\begin{align*}
\dot{x} &= cy \\
\dot{y} &= x + z \\
\dot{z} &= x + y^2 - z
\end{align*}
\]  
(2)

which has a chaotic attractor as shown in Fig.1(b) when \( c = -0.2 \).

![Fig.1 (a) Sprott H chaotic attractor. (b) Sprott I chaotic attractor](image)

3. Chaos synchronization between H and I system using active control

We assume that Sprott H system drives Sprott I system. Therefore, We define the drive and response system as follows
Chaos synchronization

We have introduced the three control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ in (4). The goal is to determine these functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ such that synchronizing between (3) and (4). In order to estimate the control functions, we subtracted (3) from (4). We define the error system as the difference between H system (3) and controlled I system (4). Let us define the state errors between (3) and (4) as

$$e_1 = x_1 - x_1, e_2 = y_2 - y_1, e_3 = z_2 - z_1$$

Subtracting (3) from (4) using the notation (5) yields

$$\left\{ \begin{array}{l}
\dot{e}_1 = cy_2 - ay_1 - z_1^2 + u_1(t) \\
\dot{e}_2 = x_2 - x_1 - by_1 + z_2 + u_2(t) \\
\dot{e}_3 = x_2 - x_1 - (z_2 - z_1) + y^2 + u_3(t)
\end{array} \right.$$  

ie,

$$\left\{ \begin{array}{l}
\dot{e}_1 = ce_2 + (c-a)y_1 - z_1^2 + u_1(t) \\
\dot{e}_2 = e_1 - by_1 + z_2 + u_2(t) \\
\dot{e}_3 = e_1 - e_3 + y^2 + u_3(t)
\end{array} \right.$$  

We define the active control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ as follows

$$\left\{ \begin{array}{l}
u_1(t) = v_1(t) - (c-a)y_1 + z_1^2 \\
v_2(t) = v_2(t) + by_1 - z_2 \\
v_3(t) = v_3(t) - y^2
\end{array} \right.$$  

Hence the error system (7) becomes

$$\left\{ \begin{array}{l}
\dot{e}_1 = ce_2 + v_1(t) \\
\dot{e}_2 = e_1 + v_2(t) \\
\dot{e}_3 = e_1 - e_3 + v_3(t)
\end{array} \right.$$  

The error system (9) to be controlled is a linear system with a control input $v_1(t)$, $v_2(t)$ and $v_3(t)$ as function of error states $e_1$, $e_2$ and $e_3$. As long as these feedback stabilize the system, $e_1$, $e_2$ and $e_3$ converge to zero as time $t$ tends to infinity. This implies that H system and I system are synchronized with feedback control. There are many possible choice for control $v_1(t)$, $v_2(t)$ and $v_3(t)$, We chose
\[
\begin{bmatrix}
v_1(t) \\
v_2(t) \\
v_3(t)
\end{bmatrix} = A \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix},
\]

where \(A\) is a \(3 \times 3\) constant matrix. In order to make the closed loop system will be stable, the proper choice of the elements of the matrix \(A\) is such that the feedback system must have all eigenvalues with negative rear parts. Let the matrix is chosen in the following form

\[
A = \begin{bmatrix}
-1 & -c & 0 \\
-1 & -1 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]

In this particular choice, the closed loop system (9) has the eigenvalues \(-1, -1, -1\). This choice will lead to the error states \(e_1, e_2, e_3\) converge to zero as time \(t\) tends to infinity and hence the synchronization between H system and I system is achieved.

### 4. Adaptive chaos synchronization between H system and I system with unknown uncertain parameters

In order to observe the synchronization behavior in H system and I system. We assume that H system drives the I system. Therefore, we define the drive and response systems as follows

\[
\begin{align*}
\dot{x}_1 &= ay_1 + z_1^2 \\
\dot{y}_1 &= x_1 + by_1 \\
\dot{z}_1 &= x_1 - z_1
\end{align*}
\]

and

\[
\begin{align*}
\dot{x}_2 &= cy_2 + u_1(t) \\
\dot{y}_2 &= x_2 + z_2 + u_2(t) \\
\dot{z}_2 &= x_2 + y_2^2 - z_2 + u_3(t)
\end{align*}
\]

where \(a, b\) and \(c\) are constant parameter, \(u = [u_1(t), u_2(t), u_3(t)]\) is the controller we introduced EQ.(11). The control is to be determined for the purpose of control I system to be H system with all unknown uncertain parameters. Subtracting EQ. (10) from EQ. (11), yields error dynamical system between EQ. (10) and EQ. (11).

\[
\begin{align*}
\dot{e}_1 &= ce_2 + (c - a)y_1 - z_1^2 + u_1(t) \\
\dot{e}_2 &= e_1 - by_1 + z_2 + u_2(t) \\
\dot{e}_3 &= e_1 - e_3 + y_2^2 + u_3(t)
\end{align*}
\]

Where
Let $a$, $b$ and $c$ are unknown uncertain parameters, we can choose lyapunov function for (12) as follows

$$V(e_i, e_j, \tilde{a}, \tilde{b}, \tilde{c}) = \frac{1}{2}(e_i^2 + e_j^2 + \tilde{a}^2, \tilde{b}^2, \tilde{c}^2)$$

Where $\tilde{a} = a - \hat{a}, \tilde{b} = b - \hat{b}, \tilde{c} = c - \hat{c}$, and $\hat{a}, \hat{b}, \hat{c}$ are estimate values of unknown parameters $a, b, c$ respectively, and require

$$\frac{dV}{dt} = -(e_i^2 + e_j^2 + e_k^2).$$

We chose

$$\begin{cases} u_i(t) = -e_i - \hat{c}(y_i + z_i) + \tilde{a}y_i + z_i^2 \\ u_j(t) = -e_j - \hat{b}y_j - z_j \\ u_k(t) = -e_k - y_k^2 \\ \hat{a} = -y_i e_i \\ \hat{b} = -y_j e_j \\ \hat{c} = e_i e_j + y_i e_i \\ \end{cases}$$

With this choice, time derivative of $v$ along the solution of the error dynamical system (12) is given by

$$\frac{dV}{dt} = -(e_i^2 + e_j^2 + e_k^2)$$

It is clear that $V$ is positive definite and $dV/dt$ is negative definite in the neighborhood of the zero solution for the system (12). Therefore, the states $x_2, y_2$ and $z_2$ of response system (11) and the states $x_1, y_1$ and $z_1$ of drive system (10) are globally synchronize asymptotically, i.e., $\lim_{t \to \infty} \|e_{(i)}\| = 0, \forall a, b, c \in R$. Where $e_{(i)} = [e_i, e_j, e_k]^T$, Hence the chaos synchronization between H system and I system is achieved under the controller and parameters estimation update law (13).

5. Simulation results

In this section, numerical simulations are given to verify the method proposed. The numerical simulations are carried out using MATLAB. Fourth order Runge-Kutta integration method is used to solve the systems of differential equations. In addition, a time step of 0.001 is employed. We select the parameters of two systems as $a = -1, b = 0.5, c = -0.2$, so that each of Sprott H and I system exhibits chaotic behavior. The initial states of systems (3) and (4) are

$x_1(0) = 0.5, y_1(0) = 0.5, z_1(0) = -0.5$, and $x_2(0) = 0.1, y_2(0) = -0.1, z_2(0) = -0.1$.

The diagram of the Sprott I system controlled to be Sprott H system is shown in Fig.2: (a) – (d). (a) shows the time series of signals $x_1$ and $x_2$. (b) shows the signal
$y_1$ and $y_2$. (c) shows the signal $z_1$ and $z_2$. (d) shows the synchronization errors $e_1, e_2$ and $e_3$ with time $t$.

The initial values of the parameters $\hat{a}, \hat{b}, \hat{c}$, are zero. The initial states of the drive and response systems (10) and (11) are $x_i(0) = 0.5, y_i(0) = 0.5, z_i(0) = -0.5$ and $x_i(0) = 0.1, y_i(0) = -0.1, z_i(0) = -0.1$ respectively, and the initial states of the error system (13) are $e_1(0) = -0.4, e_2(0) = -0.6, e_3(0) = -0.4$. The diagram of synchronization is shown in Fig. 3. (a) shows the time series of signals $x_1$ and $x_2$. (b) shows the signal $y_1$ and $y_2$. (c) shows the Signal $z_1$ and $z_2$. (d) shows dynamics of errors states $(e_1, e_2, e_3)$ for H system and I system with time $t$.

Fig. 2 The diagram of the Sprott I system controlled to be Sprott H system by using active control

Fig. 3 The diagram of the Sprott I system controlled to be Sprott H system by adaptive control


References


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