Solution of Fractional Order
Rayleigh-Stokes Equations

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Abstract
Fourier sine transform and Laplace transform are used for solving the Stokes’ first problem and the Rayleigh-Stokes problem for a generalized second grade fluid with fractional derivative. Exact solutions for both the velocity and temperature have been achieved. The solutions of the classical problem for both Stokes’ first problem and Rayleigh-Stokes problem have been obtained as limiting cases.

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1. Introduction

Fractional partial differential equations have many applications in applied sciences and engineering. These applications appears in gravitation elastic membrane, electrostatics, fluid flow, steady state, heat conduction and many other topics in both pure and applied mathematics. Typical examples of fractional partial differential equations of the time fractional advection-dispersion equation as in [6,7], fractional diffusion equations as in [15,8,5,9,14], fractional wave equation as in [13]. The Rayleigh-Stokes fractional equations[2].

In this paper we consider Stokes’ first fractional equation for the flat plate and the Rayleigh-Stokes fractional equation. Exact solutions of these equations will be investigated. The Fourier sine transform and fractional Laplace
transform are used for getting exact solutions for these equations. The fractional terms in stokes’ and Rayleigh-Stokes equations are considered as Caputo fractional derivatives.

Basic Definitions:

**Definition 1**: The Riemann-Liouville fractional integral[10,2] of order \( \alpha \) is defined as

\[
I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt
\]

**Definition 2**: The Caputo fractional derivative [10] of order \( n-1 < \alpha \leq n \) is defined as

\[
D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x f^{(n)}(t) \frac{(x-t)^{\alpha-n+1}}{(x-t)^{\alpha-n+1}} \, dt
\]

**Definition 3**: The Laplace integral transform[11,12,4,10], of the function \( f(x) \) is defined as

\[
L(f(x)) = \int_0^\infty f(x) e^{-sx} \, dx
\]

**Definition 4**: The Fourier sine integral transform[4,10,1], of the function \( f(x) \) is defined as

\[
F_e(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\zeta x) f(x) \, dx
\]

2. Solution of Stokes’ first problem:

Consider the Stokes’ first problem for a heated flat plate

\[
\frac{\partial u(x,t)}{\partial t} = \left( v + \alpha D_t^\beta \right) \frac{\partial^2 u(x,t)}{\partial x^2}
\]  

where \( u(x,t) \) is the velocity, \( t \) is the time, \( x \) is the distance and \( v, \alpha \) are constants with respect to \( x \) and \( t \), \( D_t^\beta \) is the Caputo fractional derivative with \( 0 < \beta \leq 2 \). The corresponding initial and boundary condition of Eq.(1), are

\[
u(x,0) = b_0(x) \quad \text{for} \quad x > 0
\]  

\[
\frac{\partial u(x,0)}{\partial t} = b_1(x) \quad \text{for} \quad x > 0
\]  

\[
u(0,t) = U \quad \text{for} \quad t > 0
\]
Moreover, the natural condition
\[ u(x,t), \frac{\partial u(x,t)}{\partial x} \to 0 \quad \text{for} \quad x \to \infty \] (5)
also have to be satisfied.

Employing the non-dimensional quantities
\[ u^* = \frac{u}{U}, x^* = \frac{xU}{v}, t^* = \frac{tU^2}{v}, \eta = \alpha \frac{U^2}{v^2} \] (6)

Eqs. (1) to (5) reduce to dimensionless equations as follows (for brevity the dimensionless mark “*” are omitted here)
\[ \frac{\partial u(x,t)}{\partial t} = \left( 1 + \eta D_t^\beta \right) \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \beta \leq 2 \] (7)
\[ u(x,0) = b_0(x), \quad x > 0 \] (8)
\[ \frac{\partial u(x,0)}{\partial t} = b_1(x), \quad x > 0 \] (9)
\[ u(0,t) = 1, \quad t > 0 \] (10)
\[ u(x,t), \frac{\partial u(x,t)}{\partial x} \to 0 \quad \text{for} \quad x \to \infty \] (11)

Now we consider the following two cases:

**Case 1:** when \( 0 < \beta \leq 1 \):

Making use of the Fourier sine integral transform and boundary conditions (10),(11). Then Eqs. (7) and (8) leads to
\[ \frac{dU(\zeta,t)}{dt} = -\zeta^2 \left( 1 + \eta D_t^\beta \right) U(\zeta,t) + \sqrt{\frac{2}{\pi}} \zeta \] (12)
\[ U(\zeta,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\zeta x) b_0(x) \, dx = b_0(\zeta) \] (13)

Hence the Laplace transform of Eq.(12) is
\[ \tilde{U}(\zeta,s) = \frac{\sqrt{\frac{2}{\pi}} \zeta}{s + \zeta^2 \eta s^\beta + \zeta^2} + \frac{\zeta^2 \eta b_0(\zeta) s^{\beta-1}}{s + \zeta^2 \eta s^\beta + \zeta^2} \]
\[ = \sqrt{\frac{2}{\pi}} \zeta \sum_{k=0}^{\infty} (-1)^k \zeta^{2(k+1)} \frac{s^{-\beta k - \beta - 1}}{s^{1-\beta} + \eta \zeta^2} + \eta b_0(\zeta) \sum_{k=0}^{\infty} (-1)^k \]
\[ \times \zeta^{2(k+1)} \frac{s^{-\beta k - 1}}{(s^{1-\beta} + \eta \zeta^2)^{k+1}} \]  

Taking the inverse Laplace transform of Eq.(14) and using the relation

\[ L^{-1} \left\{ \frac{n! s^{\lambda - \eta}}{(s^{\lambda} - \zeta)^{n+1}} \right\} = t^{\lambda n + \mu - 1} E_{\lambda, \mu}^{(n)} (\pm \zeta t^{1/\lambda}) \, , \, (Re(s) > |\zeta|^{1/\lambda}) \]  

Then Eq.(14) leads to

\[ U(\zeta, t) = \sqrt{\frac{2}{\pi}} \frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta^{2(k+1)t^{k+1}} E_{1-\beta, 2+\beta k}^{(k)} (-\eta \zeta^2 t^{1-\beta}) + \eta b_0(\zeta) \]

\[ \times \sum_{k=0}^{k} \frac{(-1)^k}{k!} \zeta^{2(k+1)t^{2-\beta}} E_{1-\beta, \beta k - \beta + 2}^{(k)} (-\eta \zeta^2 t^{1-\beta}) \]  

where \( E_{\alpha, \beta}^{(k)} (y) = \frac{d^k}{dy^k} E_{\alpha, \beta} (y) = \sum_{j=k}^{\infty} \frac{(j + k)!y^j}{j! \Gamma (\alpha j + \alpha k + \beta)} \) is the Mittag-Leffler function in two parameters [10].

Now considering the inverse Fourier sine integral transform of Eq.(16). We get

\[ u(x, t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin (\zeta x) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta^{2(k+1)t} \frac{\sqrt{2}}{\zeta} t^k E_{1-\beta, 2+\beta k}^{(k)} (-\eta \zeta^2 t^{1-\beta}) \]

\[ + \eta b_0(\zeta)t^{1-\beta} E_{1-\beta, \beta k - \beta + 2}^{(k)} (-\eta \zeta^2 t^{1-\beta}) d\zeta \]  

which is the exact solution of (7).

**Special cases:**

1. When \( b_0 (\zeta) = 0 \), then Eq.(17) yields

\[ u(x, t) = \frac{2}{\pi} \int_{0}^{\infty} \sin (\zeta x) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta^{2(k+1)t^{k+1}} E_{1-\beta, 2+\beta k}^{(k)} (-\eta \zeta^2 t^{1-\beta}) d\zeta \]  

which is the result obtained by Fang and others [2].

2. When \( b_0 (\zeta) = 0, \beta = 1 \), then Eq.(17) becomes

\[ u(x, t) = 1 - \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin (\zeta x)}{\zeta} \exp \left( \frac{-\zeta^2}{1 + \eta \zeta^2} t \right) d\zeta \]
which is the result obtained by Fetacau and Corina [3].

**Case 2**: when $1 < \beta \leq 2$

In the same way of case 1. Using the Fourier sine integral transform and boundary conditions (10), (11), Eqs.(7), (8), leads to

$$\frac{dU(\zeta, t)}{dt} = -\zeta^2 \left( 1 + \eta D^\beta_t \right) U(\zeta, t) + \sqrt{\frac{2}{\pi}}\zeta, \quad 1 < \beta \leq 2 \quad (20)$$

$$U(\zeta, 0) = \frac{\sqrt{2}}{\pi} \int_0^\infty \sin (\zeta x) b_0(x) \, dx = b_0(\zeta) \quad (21)$$

$$U'(\zeta, 0) = \frac{\sqrt{2}}{\pi} \int_0^\infty \sin (\zeta x) b_1(x) \, dx = b_1(\zeta) \quad (22)$$

The fractional Laplace transform of Eq.(20) subject to the initial conditions (21), (22) is

$$\tilde{U}(\zeta, s) = \frac{\sqrt{2}}{\pi} \zeta \frac{\sqrt{2}}{s (\zeta^2 \eta s^\beta + s + \zeta^2)} + \frac{\zeta^2 \eta b_0(\zeta)}{s \zeta^2 \eta s^\beta + s + \zeta^2} + \frac{\zeta^2 \eta b_1(\zeta)}{s \zeta^2 \eta s^\beta + s + \zeta^2}$$

$$= \frac{\sqrt{2}}{\pi} \zeta \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\eta} \right)^{k+1} \frac{s^{-k-2}}{(s^{\beta-1} + \frac{1}{\zeta^2 \eta})^{k+1}}$$

$$+ \eta b_0(\zeta) \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\eta} \right)^{k+1} \frac{s^{\beta-1-k}}{(s^{\beta-1} + \frac{1}{\zeta^2 \eta})^{k+1}}$$

$$+ \eta b_1(\zeta) \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\eta} \right)^{k+1} \frac{s^{(\beta-1)-(k+1)}}{(s^{\beta-1} + \frac{1}{\zeta^2 \eta})^{k+1}} \quad (23)$$

Now taking the Laplace inverse integral transform and inverse Fourier sine integral transform respectively of Eq.(23), we get the exact solution of Eq.(7) as

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin (\zeta x) \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{k!} \right)^{k+1} t^{\beta k-1}$$

$$\times \left[ \sqrt{\frac{2}{\pi}} \zeta t^{\beta+1} E_{\beta-1,\beta+1}^{(k)} \left( -\frac{1}{\zeta^2 \eta} t^{\beta-1} \right) + \eta b_0(\zeta) \right. \right.$$

$$\left. \times E_{\beta-1,k}^{(k)} \left( -\frac{1}{\zeta^2 \eta} t^{\beta-1} \right) + \eta b_1(\zeta) t^2 E_{\beta-1,k+1}^{(k)} \left( -\frac{1}{\zeta^2 \eta} t^{\beta-1} \right) \right] \, d\zeta \quad (24)$$
Special cases:

1. When $\beta = 2$, then Eq.(24) leads to

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\zeta x) \sum_{k=0}^\infty \frac{(-1)^k}{k!} \left(\frac{1}{\eta}\right)^{k+1} t^{2k-1} \times \left[ \sqrt{\frac{2}{\pi}} \zeta t \right] \sum_{k=0}^\infty \frac{(-1)^k}{\zeta^2 \eta} b_0(\zeta) + \eta b_1(\zeta) t^2 E_{1,k+1}^{(k)} \left(\frac{1}{\zeta^2 \eta} t\right) \right] d\zeta$$

(25)

2. When $\beta = 2$, $b_0(\zeta) = 0$, $b_1(\zeta) = 0$, then Eq.(24) becomes

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\zeta x)}{\zeta} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \left(\frac{1}{\eta}\right)^{k+1} t^{2k+1} E_{1,k+2}^{(k)} \left(\frac{1}{\zeta^2 \eta} t\right) d\zeta$$

(26)

3. The energy fractional equation:

The time-fractional energy equation, when the Fourier’s law of heat conduction is considered may be written in the form

$$\frac{k}{c \rho} \frac{\partial^2 \theta(x,t)}{\partial x^2} + \frac{v}{c} \left[ \frac{\partial u(x,t)}{\partial x} \right]^2 + \frac{r(x,t)}{\rho c} = \frac{\partial^\gamma \theta(x,t)}{\partial t^\gamma}$$

(27)

where $r(x,t)$ is the radiant heating, which is neglected in this paper, $c$ is the specific heat and $k$ is the conductivity which is assumed to be constant and $0 < \gamma \leq 1$.

The corresponding initial and boundary conditions of Eq.(27) are

$$\theta(x,0) = a_0(x), \quad \text{for} \quad x > 0$$

(28)

$$\theta(0,t) = T_0, \quad \text{for} \quad t \geq 0$$

(29)

Moreover, the natural condition

$$\theta(x,t), \frac{\partial \theta(x,t)}{\partial x} \to 0, \quad \text{for} \quad x \to \infty$$

(30)

also have to be satisfied.
Applying the non-dimensional quantities

\[ \theta^* = \frac{\theta}{T_0}, \quad v^* = \frac{u}{U}, \quad x^* = \frac{xU}{v}, \quad t^* = \frac{tU^2}{v}, \quad \lambda = \frac{U^2}{cT_0}, \quad \text{Pr} = \frac{c\mu}{k} \quad (31) \]

Eqs. (28), (29) and (30) can be reduced to non-dimensional equations as follows

\[ \frac{1}{\text{Pr}} \frac{\partial^2 \theta (x, t)}{\partial x^2} + \lambda \left[ \frac{\partial u (x, t)}{\partial x} \right]^2 = \frac{\partial^\gamma \theta (x, t)}{\partial t^\gamma} \quad (32) \]

\[ \theta (x, 0) = a_0 (x), \quad \text{for} \quad x > 0 \quad (33) \]

\[ \theta (0, t) = 1, \quad \text{for} \quad t \geq 0 \quad (34) \]

\[ \theta (x, t), \frac{\partial \theta (x, t)}{\partial x} \rightarrow 0, \quad \text{for} \quad x \rightarrow \infty \quad (35) \]

Letting \( g (x, t) = \lambda \left[ \frac{\partial u (x, t)}{\partial x} \right]^2 \), then Eq. (32) can be rewritten as

\[ \frac{1}{\text{Pr}} \frac{\partial^2 \theta (x, t)}{\partial x^2} + g (x, t) = \frac{\partial^\gamma \theta (x, t)}{\partial t^\gamma}, \quad 0 < \gamma \leq 1 \quad (36) \]

Applying Fourier integral sine transform to Eqs. (36) and (33), we get

\[ \frac{\partial^\gamma \theta (\zeta, t)}{\partial t^\gamma} + \frac{1}{\text{Pr}} \zeta^2 \theta (\zeta, t) = \sqrt{\frac{2}{\pi}} \frac{1}{\text{Pr}^{\gamma}} \zeta + g (x, t) \quad (37) \]

\[ \theta (\zeta, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin (\zeta x) a_0 (x) \, dx = a_0 (\zeta) \quad (38) \]

Using initial condition (38) for getting fractional Laplace transform of Eq. (37) as

\[ \tilde{\theta} (\zeta, t) = \sqrt{\frac{2}{\pi}} \frac{1}{\text{Pr}^{\gamma}} \frac{\zeta}{s^\gamma + \frac{\zeta^2}{\text{Pr}^{\gamma}}} + \tilde{g} (\zeta, s) + \frac{a_0 (\zeta) s^{\gamma-1}}{s^\gamma + \frac{c^2}{\text{Pr}^{\gamma}}} \quad (39) \]

Applying the inversion Laplace Fourier transform, we get the exact solution of Eq. (36) as

\[ \theta (x, t) = \frac{2}{\pi} \int_0^\infty \frac{\zeta}{\text{Pr}} \sin (\zeta x) t^\gamma E_{\gamma, \gamma-1} \left( -\frac{\zeta^2}{\text{Pr}^{\gamma}} t^\gamma \right) d\zeta \]

\[ + \sqrt{\frac{2}{\pi}} \int_0^\infty \sin (\zeta x) g (\zeta, t) t^{\gamma-1} E_{\gamma, \gamma} \left( -\frac{\zeta^2}{\text{Pr}^{\gamma}} t^\gamma \right) d\zeta \]
\[ + \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin (\zeta x) a_0 (\zeta) E_{\gamma,1} \left( -\frac{\zeta^2}{Pr} t^{\gamma} \right) d\zeta \]  \hspace{1cm} (40)

**Special cases:**

1. When \( \gamma = 1, a_0 (\zeta) = 0 \), then Eq.(40) yields

\[ \theta (x, t) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\zeta}{Pr} \sin (\zeta x) \exp \left( -\frac{\zeta^2}{Pr} t \right) \]

\[ \times \int_{0}^{t} \left[ \sqrt{\frac{2}{\pi}} \frac{1}{Pr} \zeta + g (\zeta, \tau) \exp \left( -\frac{\zeta^2}{Pr} \tau \right) \right] d\tau d\zeta \]  \hspace{1cm} (41)

which is the result obtained by Fang and others [2].

2. When \( \gamma = 1, g (\zeta, t) = 0 \), then from Eq.(40) we get

\[ \theta (x, t) = 1 - \text{erf} \left( \frac{x}{2 \sqrt{\frac{t}{Pr}}} \right) \]  \hspace{1cm} (42)

which is the result obtained also by Fetacau and Corina [3].

**4. Solution of the Rayleigh-Stokes problem:**

The Rayleigh-Stokes equation involving a time -fractional derivative is written as

\[ \frac{\partial u (x, z, t)}{\partial t} = \left( \nu + \alpha D_t^\beta \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) u (x, z, t) \]  \hspace{1cm} (43)

where \( u (x, z, t) \) is the velocity in the \( xz \)-plane, \( 0 < \beta \leq 2 \), and \( \nu, \alpha \) are constant with respect to \( x \) and \( t \).

The corresponding initial and boundary conditions are

\[ u (x, z, 0) = b_0 (x, z), \quad \text{for} \quad x > 0, z > 0 \]  \hspace{1cm} (44)

\[ \frac{\partial u (x, z, 0)}{\partial t} = b_1 (x, z), \quad \text{for} \quad x > 0, z > 0 \]  \hspace{1cm} (45)

\[ u (0, z, t) = u (x, 0, t) = U, \quad \text{for} \quad t > 0 \]  \hspace{1cm} (46)

The natural condition

\[ u (x, z, t), \frac{\partial u (x, z, t)}{\partial x}, \frac{\partial u (x, z, t)}{\partial z} \rightarrow 0 \quad \text{for} \quad x^2 + z^2 \rightarrow \infty \]  \hspace{1cm} (47)
Fractional order Rayleigh-Stokes equations

have to be satisfied too.

Using the non-dimensional quantities and \( z^* = \frac{z U}{\nu} \). Eqs.(43), (44), (45) and (46) reduce to dimensionless equations as follows (for brevity the dimensionless mark "\( \ast \)" are omitted here).

\[
\frac{\partial u(x, z, t)}{\partial t} = (1 + \eta D^\beta_t) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) u(x, z, t), \quad 0 < \beta \leq 2 \quad (48)
\]

\[
u(0, z, t) = u(x, 0, t) = 1, \quad \text{for} \quad t > 0 \quad (49)
\]

Now we will take \( 0 < \beta \leq 2 \) as two parts

**Case 1:** When \( 0 < \beta \leq 1 \):

Applying the Fourier sine and fractional Laplace with respect to \( x-z \) and \( t \) respectively to above equations, we get

\[
\tilde{U}(\zeta, \xi, s) = \frac{2}{\pi \zeta \xi s [s + (\zeta^2 + \xi^2) \eta s^\beta + \zeta^2]^2} + \frac{(\zeta^2 + \xi^2) \eta b_0(\zeta, \xi) s^{\beta-1}}{s + (\zeta^2 + \xi^2) \eta s^\beta + (\zeta^2 + \xi^2)} \quad (50)
\]

where \( b_0(\zeta, \xi) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin(\zeta x) \sin(\xi z) u(x, z, 0) \, dx \, dz \)

We can rewrite Eq.(50) as

\[
\tilde{U}(\zeta, \xi, s) = \sum_{k=0}^{\infty} \frac{(-1)^k (\zeta^2 + \xi^2)^{k+1}}{s^{\beta k - \beta - 1} (s^\beta - \eta (\zeta^2 + \xi^2))^{k+1}} \quad (51)
\]

Taking the inverse Laplace transform and Fourier sine transform respectively of Eq.(51), we leads to

\[
u(x, z, t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin(\zeta x) \sin(\xi z) \sum_{k=0}^{\infty} \frac{(-1)^k (\zeta^2 + \xi^2)^{k+1}}{k! (s^{\beta k - \beta - 1} (s^\beta - \eta (\zeta^2 + \xi^2))^{k+1})} t
\]

\[
\times \left[ \frac{2}{\pi} t^k E_{1-\beta,2+\beta k}^{(k)} (-\eta (\zeta^2 + \xi^2) t^{1-\beta}) + \eta b_0(\zeta, \xi) t^{1-\beta} E_{1-\beta,\beta k-\beta+2}^{(k)} (-\eta (\zeta^2 + \xi^2) t^{1-\beta}) \right] d\zeta d\xi \quad (52)
\]

**Special cases:**
1. When \( b_0 (\zeta, \xi) = 0 \), then Eq. (51) yields
\[
\begin{align*}
    u(x, z, t) &= 4 \pi^2 \int_0^\infty \int_0^\infty \sin(\zeta x) \sin(\xi z) \sum_{k=0}^\infty \frac{(-1)^k}{k!} \left( \zeta^2 + \xi^2 \right)^{k+1} \\
    &\times t^{k+1} E_{1-\beta_2+\beta_1 k} \left(-\eta \left( \zeta^2 + \xi^2 \right) t^{1-\beta} \right) d\zeta d\xi 
\end{align*}
\]
which is the result obtained by Fang and others [2].

2. When \( \beta = 1 \) and \( b_0 (\zeta, \xi) = 0 \), then Eq. (52) leads to
\[
\begin{align*}
    u(x, z, t) &= 1 - 4 \pi^2 \int_0^\infty \frac{\sin(\zeta x)}{\zeta} \int_0^\infty \frac{\sin(\xi z)}{\xi} \exp \left( -\frac{\zeta^2 + \xi^2}{1 + \eta (\zeta^2 + \xi^2)} \right) d\zeta d\xi 
\end{align*}
\]
which is the result obtained also by Fetacau and Corina [3].

**Case 2**: When \( 1 < \beta \leq 2 \):

As in Case 1, applying the Fourier sine and fractional Laplace transforms with respect to \( x, z \) and \( t \) respectively to Eqs. (48), (44) and (45) by using the conditions (47) and (49), we get
\[
\begin{align*}
    \tilde{U}(\zeta, \xi, s) &= \frac{2 (\zeta^2 + \xi^2)}{\pi \zeta \xi [s (\zeta^2 + \xi^2) \eta s^\beta + s + \zeta^2]} + \frac{(\zeta^2 + \xi^2) \eta b_0 (\zeta, \xi) s^{\beta-1}}{(\zeta^2 + \xi^2) \eta s^\beta + s + (\zeta^2 + \xi^2)} \\
    &+ \frac{(\zeta^2 + \xi^2) \eta b_1 (\zeta, \xi) s^{\beta-2}}{(\zeta^2 + \xi^2) \eta s^\beta + s + (\zeta^2 + \xi^2)}
\end{align*}
\]
where
\[
\begin{align*}
    b_0 (\zeta, \xi) &= \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin(\zeta x) \sin(\xi z) u(x, z, 0) \, dx \, dz \\
    b_1 (\zeta, \xi) &= \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin(\zeta x) \sin(\xi z) \frac{\partial u(x, z, 0)}{\partial t} \, dx \, dz
\end{align*}
\]
Eq. (55) can be written as
\[
\begin{align*}
    \tilde{U}(\zeta, \xi, s) &= \frac{2}{\pi \zeta \xi} \sum_{k=0}^\infty (-1)^k \left( \frac{1}{\eta} \right)^{k+1} \frac{s^{-k-2}}{\left( s^{\beta-1} + \frac{1}{\eta (\zeta^2 + \xi^2)} \right)^{k+1}} \\
    &+ \eta b_0 (\zeta, \xi) \sum_{k=0}^\infty (-1)^k \left( \frac{1}{\eta} \right)^{k+1} \frac{s^{\beta-1-k}}{\left( s^{\beta-1} + \frac{1}{\eta (\zeta^2 + \xi^2)} \right)^{k+1}}
\end{align*}
\]
\[ + \eta b_1 (\zeta, \xi) \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\eta} \right)^{k+1} s^{(\beta-1)-(k+1)} \left( \frac{1}{\eta(\zeta^2+\xi^2)} \right)^{k+1} \]  

To get the exact solution of Eq.(58), we consider the inverse Laplace and the inverse Fourier sine transforms respectively of that equation which yields

\[ u (x, z, t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin (\zeta x) \sin (\xi z) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{1}{\eta} \right)^{k+1} t^{\beta k-1} \]

\[ \times \left[ \frac{2}{\pi \xi} t^{\beta+1} E^{(k)}_{\beta-1, \beta+k+1} \left( -\frac{1}{\eta(\zeta^2+\xi^2)} t^{\beta-1} \right) + \eta b_0 (\zeta, \xi) \right] d\zeta d\xi \]  

Now we will take \( \beta = 2 \), Eq.(59) yields

\[ u (x, z, t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin (\zeta x) \sin (\xi z) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{1}{\eta} \right)^{k+1} t^{2k-1} \]

\[ \times \left[ \frac{2}{\pi \xi} t^2 E^{(k)}_{1, k+2} \left( -\frac{1}{\eta(\zeta^2+\xi^2)} t \right) + \eta b_1 (\zeta, \xi) \right] d\zeta d\xi \]  

The time-fractional energy equation in \( xz \) plane is written as

\[ \frac{k}{c \rho} \left[ \frac{\partial^2 \theta (x, z, t)}{\partial x^2} + \frac{\partial^2 \theta (x, z, t)}{\partial z^2} \right] + \frac{\nu}{c} f (x, z, t) + \frac{r (x, z, t)}{\rho c} = \frac{\partial^{\gamma} \theta (x, z, t)}{\partial t^{\gamma}} \]  

where \( f (x, z, t) = \left[ \frac{\partial u (x, z, t)}{\partial x} \right]^2 + \left[ \frac{\partial u (x, z, t)}{\partial z} \right]^2 \) is a known function as soon as the velocity field \( u (x, z, t) \) is prescribed, \( r (x, z, t) \) is the radiant heating, which is neglected.

The corresponding initial and boundary conditions of Eq.(29) are

\[ \theta (x, z, 0) = a_0 (x, z), \quad \text{for} \quad x > 0, z > 0 \]  

\[ \theta (0, z, t) = \theta (x, 0, t) = T_0, \quad \text{for} \quad t > 0 \]  

Moreover, the natural condition

\[ \theta (x, z, t), \frac{\partial \theta (x, z, t)}{\partial x}, \frac{\partial \theta (x, z, t)}{\partial z} \rightarrow 0, \quad \text{for} \quad x^2 + z^2 \rightarrow \infty \]  

also have to be satisfied.
Using the non-dimensional quantities \((31)\), and \(z^* = \frac{zU}{v}\), Eqs.\((61)\), \((62)\), \((63)\) and \((64)\) reduce to dimensionless equations as follows (for brevity the dimensionless mark \(*\) are omitted here).

\[
\frac{1}{Pr} \left[ \frac{\partial^2 \theta(x, z, t)}{\partial x^2} + \frac{\partial^2 \theta(x, z, t)}{\partial z^2} \right] + \lambda \left\{ \left[ \frac{\partial u(x, z, t)}{\partial x} \right]^2 + \left[ \frac{\partial u(x, z, t)}{\partial z} \right]^2 \right\} = \frac{\partial^\gamma \theta(x, z, t)}{\partial t^\gamma} \tag{65}
\]

where \(0 < \gamma \leq 1\).

\[
\theta(x, z, 0) = a_0(x, z), \quad \text{for} \quad x > 0 \tag{66}
\]

\[
\theta(0, z, t) = \theta(x, 0, t) = 1, \quad \text{for} \quad t > 0 \tag{67}
\]

\[
\theta(x, z, t), \frac{\partial \theta(x, z, t)}{\partial x}, \frac{\partial \theta(x, z, t)}{\partial z} \to 0, \quad \text{for} \quad x^2 + z^2 \to \infty \tag{68}
\]

Letting \(g(x, z, t) = \lambda \left\{ \left[ \frac{\partial u(x, z, t)}{\partial x} \right]^2 + \left[ \frac{\partial u(x, z, t)}{\partial z} \right]^2 \right\}\), then Eq.\((65)\) becomes

\[
\frac{1}{Pr} \left[ \frac{\partial^2 \theta(x, z, t)}{\partial x^2} + \frac{\partial^2 \theta(x, z, t)}{\partial z^2} \right] + g(x, z, t) = \frac{\partial^\gamma \theta(x, z, t)}{\partial t^\gamma} \tag{69}
\]

By following the same steps as in section 3, we get the exact solution of Eq.\((69)\) as

\[
\theta(x, z, t) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \frac{\zeta^2 + \xi^2}{Pr} \sin(\zeta x) \sin(\xi z) t^\gamma E_{\gamma, \gamma-1} \left( -\frac{\zeta^2 + \xi^2}{Pr} t^{\gamma} \right) d\zeta d\xi
\]

\[
+ \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin(\zeta x) \sin(\xi z) g(\zeta, \xi, t) t^{\gamma-1} E_{\gamma, \gamma} \left( -\frac{\zeta^2 + \xi^2}{Pr} t^{\gamma} \right) d\zeta d\xi
\]

\[
+ \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin(\zeta x) \sin(\xi z) a_0(\zeta, \xi) E_{\gamma, 1} \left( -\frac{\zeta^2 + \xi^2}{Pr} t^{\gamma} \right) d\zeta d\xi \tag{70}
\]

5. Conclusion

In this paper, we have presented some results about Stokes’ first problem, the Rayleigh-Stokes problem and energy equation. Exact solutions of these equations are obtained by using the Fourier sine integral transform and
fractional Laplace transform. The Caputo fractional derivative is considered in both Stokes’ first problem and Rayleigh-Stokes problem as time derivatives, where the order of the fractional derivative is considered as $0 < \beta \leq 2$ and $0 < \gamma \leq 1$. Special cases have been considered in the cases $\beta = 1, \beta = 2$, and $\gamma = 1$.

References:

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