Creep Transition of Transversely Isotropic Thick-Walled Rotating Cylinder

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Abstract

Creep stresses for a transversely isotropic thick-walled rotating cylinder under internal pressure have been obtained by using Seth’s transition theory. Results obtained have been discussed numerically and depicted graphically.

Keywords: Creep, Transition, Transversely Isotropic, Rotating Cylinder.

1 Introduction

Thick-walled circular cylinders are used commonly either as pressure vessels intended for storage in industrial gases or as media transportation of high pressurized fluids. Many authors [1-2,4-8,10-11] have discussed creep of thick-walled cylinders under internal pressure. These authors made the following assumptions:

1. The volume of the material remains constant.
2. The ratios of the principal shear strain rates to the principal shear stresses are equal.
3. Norton’s law holds in the special case, i.e. for uniaxial case.
4. The creep deformation is infinitesimally small.

Transition theory [9] does not require any of the above assumptions and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. It utilizes the concept of generalized principal strain measure and the asymptotic solution at transition points of the governing equations defining the deformed field. It has successfully been applied to a large number of creep problems.
In this paper, we calculated creep stresses and strain rates for a thick-walled circular rotating cylinder under internal pressure by using transition theory.

2 Governing Equations

Consider a thick-walled circular cylinder made of transversely isotropic material of internal and external radii 'a' and 'b' respectively and rotating with an angular velocity 'ω' and pressure 'p' applied at the internal surface.

The displacement components in cylindrical polar co-ordinates are given by

\[ u = r(1 - \beta), \quad v = 0, \quad \omega = dz \]

where \( \beta \) is a function of \( r = \sqrt{x^2 + y^2} \) and \( d \) is a constant.

The Almansi generalized principal components of strain is given as,

\[
ei = \frac{1}{n} \left[ \frac{1}{2} \left(1 - 2e^{n}A_{ij} \right)^{2} d\varepsilon_{ij}^{A} = \frac{1}{n} \left[1 - (1 - 2e^{n})^{n/2} \right] \right]
\]

where \( n \) is the measure and \( e^{A_{ij}} \) is the principal Almansi finite strain components.

Using (1), the generalized components of strain is defined as

\[
e_{rr} = \frac{1}{n} \left[1 - (\beta + r\beta')^{n} \right], \quad e_{\theta\theta} = \frac{1}{n} \left[1 - \beta^{n} \right]
\]

\[
e_{zz} = \frac{1}{n} \left[1 - (1 - d)^{n} \right], \quad e_{r\theta} = e_{\theta r} = e_{zz} = 0
\]

where \( \beta' = \frac{d\beta}{dr} \).

The stress-strain relations for transversely isotropic material are

\[
T_{rr} = C_{11}e_{rr} + (C_{11} - 2C_{66})e_{\theta\theta} + C_{13}e_{zz}
\]

\[
T_{\theta\theta} = (C_{11} - 2C_{66})e_{rr} + C_{11}e_{\theta\theta} + C_{13}e_{zz}
\]

\[
T_{zz} = C_{13}e_{rr} + C_{13}e_{\theta\theta} + C_{33}e_{zz}
\]

\[
T_{r\theta} = T_{\theta r} = T_{r\phi} = 0
\]

where \( C_{ij} \)'s are material constants.

Using equation (2) in (3), we get

\[
T_{rr} = \frac{C_{11}}{n} \left[1 - (\beta + r\beta')^{n} \right] + \left( \frac{C_{11} - 2C_{66}}{n} \right) \left[1 - \beta^{n} \right] + C_{13}e_{zz}
\]

\[
T_{\theta\theta} = \left( \frac{C_{11} - 2C_{66}}{n} \right) \left[1 - (\beta + r\beta')^{n} \right] + \frac{C_{11}}{n} \left[1 - \beta^{n} \right] + C_{13}e_{zz}
\]

\[
T_{zz} = \frac{C_{13}}{n} \left[1 - (\beta + r\beta')^{n} \right] + \frac{C_{13}}{n} \left(1 - \beta^{n} \right) + C_{33}e_{zz}
\]

\[
T_{r\theta} = T_{\theta r} = T_{r\phi} = 0
\]

Equations of equilibrium are all satisfied except
\[
\frac{d}{dr} \left( T_n - T_\infty \right) + \frac{T_n - T_\infty}{r} + \rho r \omega^2 = 0
\]  
(5)

where \( \rho \) is the density of the material.

Substituting equation (4) in (5), we get a non-linear differential equation in \( \beta \) as

\[
nPC_{11} \beta^{n+1}(1 + P)^{n+1} \frac{dP}{d\beta} = -nPC_{11} \beta^n (1 + P)^n - (C_{11} - 2C_{66})nP\beta^n + 2C_{66} \left[ 1 - \beta^n (1 + P)^n \right] - 2C_{66} (1 - \beta^n) + \rho nr^2 \omega^2
\]  
(6)

where \( r\beta' = \beta P \)

The transition points of \( \beta \) in equation (6) are \( P \to -1 \) and \( P \to \pm \infty \).

The boundary conditions are,

\[
T_n = -p \quad \text{at} \quad r = a
\]
\[
T_n = 0 \quad \text{at} \quad r = b
\]  
(7)

The resultant force normally applied to the ends of cylinder is,

\[
2\pi \int_a^b rT_dr = \pi a^2 p
\]  
(8)

### 3 Solution through the Principal Stresses

It has been shown that the transition function through the principal stress-difference \([3-4,9-10]\) at the transition point \( P \to -1 \) gives the creep stresses. For finding the creep stresses at the transition point \( P \to -1 \), we define the transition function \( R \) as,

\[
R = T_n - T_\infty = 2C_{66} \frac{n}{\rho} \beta^n \left[ 1 - (1 + P)^n \right]
\]  
(9)

Taking the logarithmic differentiation of equation (9) with respect to \( r' \), and substituting the value of \( \frac{dP}{d\beta} \) from equation (6), we get

\[
\frac{d}{dr} \left( \log R \right) = \frac{1}{rR} \left\{ 2C_{66} P \beta^n \left[ 1 - (1 + P)^n \right] + 2PC_{66} \beta^n (1 + P)^n + 2C_{66} (1 - C_1) P \beta^n - \frac{2}{n} C_{66} C_1 \beta^n \left[ 1 - (1 + P)^n \right] \right\}
\]  
(10)

where \( C_1 = 2C_{66} / C_{11} \)

Now taking the asymptotic value of equation (10) as \( P \to -1 \) and integrating, we get

\[
R = T_n - T_\infty = A_1 r^{-C_1} \left( \frac{2C_{66}}{n} \frac{D^n}{r^n} \right)^2 \exp(f)
\]  
(11)

where asymptotic value of \( \beta \) is \( D/n \) and \( D \) is a constant.
Also \( f = \int \frac{2C_{66}C_1D^a}{r^{n+1}} - C_1 \rho r \omega^2 \) \( dr \) and \( A_i \) is a constant of integration.

From equation (11) and (5), we get

\[ T_{rr} = -A_i \int r^{-\{1+C_1\}} \left( \frac{2C_{66}D^a}{n} \right)^2 \exp(f) dr - \frac{\rho r^2 \omega^2}{2} + A_2 \]  

(12)

where \( A_2 \) is a constant of integration.

From equation (11), we get

\[ T_{\theta\theta} = T_{rr} - A_i r^{-C_1} \left( \frac{2C_{66}D^a}{n} \right)^2 \exp(f) \]  

(13)

The constants \( A_i \) and \( A_2 \) are obtained by using boundary conditions (7) in equation (12), we get

\[ A_1 = \frac{\rho \omega^2}{2} (a^2 - b^2) - p \int_a^b r^{-\{1+C_1\}} \left( \frac{2C_{66}D^a}{n} \right)^2 \exp(f) dr \]

\[ A_2 = \frac{\rho b^2 \omega^2}{2} + \left[ \frac{\rho \omega^2}{2} (a^2 - b^2) - p \right] \int_a^b r^{-\{1+C_1\}} \left( \frac{2C_{66}D^a}{n} \right)^2 \exp(f) dr \]

Substituting the values of \( A_i \) and \( A_2 \) in equation (12) and (13), we get

\[ T_{rr} = \left\{ p - \frac{\rho \omega^2}{2} (a^2 - b^2) \right\} \int_a^b r^{-\{1+C_1\}} \left( \frac{2C_{66}D^a}{n} \right)^2 \exp(f) dr + \frac{\rho \omega^2}{2} (b^2 - r^2) \]  

(14)

\[ T_{\theta\theta} = T_{rr} - \left\{ \frac{\rho \omega^2}{2} (a^2 - b^2) - p \right\} r^{-C_1} \left( \frac{2C_{66}D^a}{n} \right)^2 \exp(f) \]  

(15)

The axial stress is obtained from equation (4) as

\[ T_{zz} = \frac{C_{13}}{2(C_{11} - C_{66})} (T_{rr} + T_{\theta\theta}) + \frac{C_{33}(C_{11} - C_{66}) - C_{13}^2}{C_{11} - C_{66}} e_{zz} \]  

(16)

Using equation (16) in the end condition (8), the axial strain is given as
Creep transition

\[ K_{e_{zz}} = \frac{a^2 p}{(b^2 - a^2)} - \frac{\rho \omega^2 C_{13}}{4(C_{11} - C_{66}) (b^2 - a^2)} \left[b^4 - a^2 (2b^2 - a^2)\right] + \left(\frac{1}{b^2 - a^2}\right)\frac{C_{13} \rho \omega^2}{4(C_{11} - C_{66})} \left[b^4 - a^2 (2b^2 - a^2)\right] - \frac{2C_{13}}{(b^2 - a^2)(C_{11} - C_{66})} \int_a^b r T_{zz} dr \]  

(17)

Substituting equation (17) in equation (16), we get

\[ T_{zz} = \frac{C_{13}}{2(C_{11} - C_{66})} \left[T_{zz} + T_{zz}^0\right] + \frac{p}{\left(\frac{b}{a}\right)^2 - 1} - \frac{C_{13}}{(b^2 - a^2)(C_{11} - C_{66})} \int_a^b r T_{zz} + T_{zz}^0 dr \]  

(18)

Now we introduce the following non-dimensional quantities

\[ R = \frac{r}{b}, \quad R_0 = \frac{a}{b}, \quad \sigma_r = \frac{T_{zz}}{C_{66}}, \quad \sigma_\theta = \frac{T_{zz}^0}{C_{66}}, \quad \sigma_z = \frac{T_{zz}}{C_{66}}, \quad \Omega^2 = \frac{\rho b^2 \omega^2}{C_{66}} \]

The transitional stresses in non-dimensional state are

\[ \sigma_r = \frac{\frac{1}{2} \rho \omega^2 \left(R_0^2 - 1\right) \int_{R_0}^R R^{-4(1+C_1)} \left[\frac{2C_{66}}{n R^n} \left(\frac{D}{b}\right)^n\right] \exp(f_1) dR}{\frac{\Omega^2}{2} \left(1 - R^2\right)} \]  

(19)

\[ \sigma_\theta = \sigma_r \frac{\left(\frac{\Omega^2}{2} (R_0^2 - 1) - \frac{p}{C_{66}} \left[R_{C_1}^R \left[\frac{2C_{66}}{n R^n} \left(\frac{D}{b}\right)^n\right] \exp(f_1)\right]\right)}{\frac{\Omega^2}{2} \left(1 - R^2\right)} \]  

(20)

\[ \sigma_z = \frac{C_{13}}{C_{11}(2 - C_1)} \left[\sigma_r + \sigma_\theta\right] + \frac{p}{C_{66}} \left[\frac{R_0^2}{1 - R_0^2}\right] - \frac{2C_{13}}{C_{11}(2 - C_1)(1 - R_0^2)} \int_{R_0}^R \frac{R (\sigma_r + \sigma_\theta) dR}{R^n b^n} \]  

(21)

where \( f_1 = \int_{R_0}^R \frac{2C_{66}}{n R^n} \left(\frac{D}{b}\right)^n dR \)
If the angular speed becomes zero, then the transitional stresses become

\[
\sigma_r = \frac{-p}{C_{66}} \left[ R^{-\gamma} \left( \frac{D}{b} \right)^n \right]^2 \exp(f_2) dR
\]

(22)

\[
\sigma_\theta = \sigma_r + \frac{1}{R_{0}^{(1+C_1)}} \left[ \frac{R_{0}^{2}}{nR^*} \left( \frac{D}{b} \right)^n \right]^2 \exp(f_2) dR
\]

(23)

\[
\sigma_z = \frac{C_{13}}{C_{11}(2-C_1)} \left[ \sigma_r + \sigma_\theta \right] + \frac{p}{C_{66}} \left( \frac{R_0^2}{1-R_0^2} \right) - \frac{2C_{13}}{C_{11}(2-C_1)(1-R_0^2)} \int_{R_0}^{1} R(\sigma_r + \sigma_\theta) dR
\]

(24)

where

\[
f_2 = \int \frac{2C_{66}C_1}{R^*} \left( \frac{D}{b} \right)^n dR
\]

4 Isotropic case

For an elastically isotropic material,

\[
C_{11} = C_{22} = C_{33} = \lambda + 2\mu, \quad C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = C_{11} - 2C_{66} = \lambda
\]

For incompressible rotating cylinder under internal pressure, the stresses are

\[
\sigma_r = \left\{ \frac{-p}{C_{66}} + \frac{\Omega^2}{2} \left( R_0^2 - 1 \right) \right\} \left[ R^{-\gamma} \left( \frac{D}{b} \right)^n \right]^2 \exp(f_3) dR + \frac{\Omega^2}{2} \left( 1 - R^2 \right)
\]

(25)

\[
\sigma_\theta = \sigma_r - \left\{ \frac{\Omega^2}{2} \left( R_0^2 - 1 \right) - \frac{p}{C_{66}} \right\} \left[ \frac{2C_{66}}{nR^*} \left( \frac{D}{b} \right)^n \right]^2 \exp(f_3) dR
\]

(26)

\[
\sigma_z = \left( \frac{1}{2} \right) \left[ \sigma_r + \sigma_\theta \right] + \frac{p}{C_{66}} \left( \frac{R_0^2}{1-R_0^2} \right) - \frac{1}{1-R_0^2} \int_{R_0}^{1} R(\sigma_r + \sigma_\theta) dR
\]

(27)

where \( f_3 = 0 \)

The stress-strain rate relationship can be given as,
\[ \dot{\varepsilon}_{rr} = -\frac{(A - 2C_{66})}{H} \Theta + 2 \frac{(A - C_{66})}{H} T_{rr} + \frac{T_{zz}}{H} \left( A - 2C_{66} - \frac{2C_{13}C_{66}}{C_{33}} \right) \] (28)

\[ \dot{\varepsilon}_{\theta\theta} = \frac{2(A - C_{66})}{H} T_{\theta\theta} - \frac{(A - 2C_{66})}{H} \Theta + \frac{T_{zz}}{H} \left( A - 2C_{66} - \frac{2C_{13}C_{66}}{C_{33}} \right) \] (29)

\[ \dot{\varepsilon}_z = -\frac{2C_{13}C_{66}}{HC_{33}} \Theta + \frac{T_{zz}}{C_{33}} \left[ \frac{C_{11} - C_{66} + 2C_{13}}{4(A - C_{66})} \right] \] (30)

where \( \dot{\varepsilon}_{rr}, \dot{\varepsilon}_{\theta\theta}, \dot{\varepsilon}_z \) is the strain rate tensor with respect to flow parameter \( t \) and \( \Theta = T_{rr} + T_{\theta\theta} + T_{zz} \), \( H = 4C_{66}(A - C_{66}) \), \( A = C_{11} - \frac{C_{13}^2}{C_{33}} \).

Differentiating equation (2) with respect to \( r \), we get

\[ \dot{\varepsilon}_{\theta\theta} = -\beta n^{-1} \dot{\beta} \] (31)

For Swainger measure \((n = 1)\) we have from equation (31)

\[ \dot{\varepsilon}_{\theta\theta} = -\dot{\beta} \] (32)

The transition value of equation (9) as \((P \to -1)\) gives

\[ \beta = \left[ \frac{n}{2C_{66}} \left\{ T_{rr} - T_{\theta\theta} \right\} \right]^{1/n} \] (33)

Using equations (31)-(33) in equations (28)-(30), we get

\[ \dot{\varepsilon}_{rr} = \chi \left[ \frac{\sigma_r}{\eta} \left( \alpha - \frac{2C_{13}C_{66}}{C_{11}C_{33}} \right) - \frac{\alpha}{\eta} (\sigma_r + \sigma_\theta + \sigma_z) + \frac{\sigma_r}{2} \right] \] (34)

\[ \dot{\varepsilon}_{\theta\theta} = \chi \left[ \frac{\sigma_\theta}{2} \eta (\sigma_r + \sigma_\theta + \sigma_z) + \frac{\sigma_z}{\eta} \left( \alpha - \frac{2C_{13}C_{66}}{C_{11}C_{33}} \right) \right] \] (34)

\[ \dot{\varepsilon}_z = \chi \left[ -\frac{2C_{13}C_{66}}{C_{11}C_{33}} \eta (\sigma_r + \sigma_\theta + \sigma_z) + \frac{\sigma_\theta}{C_{33}} \left( 1 - \frac{C_{66}}{C_{11}} + \frac{2C_{13}}{C_{11}} \right) \right] \] (34)

where \( \eta = 4 \left( 1 - \frac{C_{13}^2}{C_{11}C_{33}} \right) \), \( \alpha = 1 - \frac{C_{13}^2}{C_{11}C_{33}} - \frac{2C_{66}}{C_{11}} \), \( \beta = 1 - \frac{C_{13}^2}{C_{11}C_{33}} \), \( \chi = \left[ \frac{n}{2} (\sigma_r - \sigma_\theta) \right]^{1-1} \).

For isotropic materials, strain rates (34) becomes

\[ \dot{\varepsilon}_{rr} = \chi \left[ -\frac{\alpha}{\eta} (\sigma_r + \sigma_\theta + \sigma_z) + \frac{\sigma_r}{2} \left( \alpha - C(1-2C) \right) \right] \] (35)

\[ \dot{\varepsilon}_{\theta\theta} = \chi \left[ \frac{\sigma_\theta}{2} \eta (\sigma_r + \sigma_\theta + \sigma_z) + \frac{\sigma_z}{\eta} \left( \alpha - C(1-2C) \right) \right] \] (35)

\[ \dot{\varepsilon}_z = \chi \left[ -C(1-2C)(\sigma_r + \sigma_\theta + \sigma_z) + \frac{\sigma_zC_{66}}{C_{33}\eta} \left( 1 - \frac{C_{66}}{C_{11}} + \frac{2C_{13}}{C_{11}} \right) \right] \] (35)
5 Numerical Illustration and Discussion

For calculating the stresses and strain rate distribution based on the above analysis, the following values of measure $n$, $D$, pressure $P_1$ and angular velocity $\Omega$ have been taken as:

\[ \Omega^2 = 0.5 \quad n = 1, 1/3 \text{ and } 1/7 \quad (i.e. \ N = 1, 3 \text{ and } 7) \]
\[ P_1 = \frac{P}{C_{66}} = 0.5, 1.5 \quad D = 1 \]

Elastic constants $C_{ij}$ for transversely isotropic material (Magnesium) and isotropic material (Brass) have been given in table 1.

| Table 1: Elastic Constants $C_{ij}$ (in terms of $10^{10} \text{ N/m}^2$) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                         | $C_{44}$        | $C_{11}$        | $C_{12}$        | $C_{13}$        | $C_{33}$        |
| TIM (Mg)                | 1.64            | 5.97            | 2.62            | 2.17            | 6.17            |
| IM (Brass)              | 1.0             | 3.0             | 1.0             | 1.0             | 3.0             |

Curves have been drawn in figures 1-2 between the stresses and radii ratio $(R = r/b)$ for transversely isotropic / isotropic material for different angular velocity.

It can be seen from figure 1 that without rotation, circumferential stress is maximum at the internal surface for transversely isotropic / isotropic circular cylinder under internal pressure for measure $n = 1$ and $n = 1/3$ while for measure $n = 1/7$, the circumferential stress is maximum at external surface. With the increase in angular speed, it can be seen from figure 2 that circumferential stress increases at internal surface for transversely isotropic circular cylinder under internal pressure as compared to isotropic circular cylinder. With the increase in measure, circumferential stress decreases at internal surface for transversely isotropic / isotropic circular cylinder under internal pressure.

In figures 3-4, curves have been drawn between strain rates and radii ratio $(R = r/b)$. With the increase in angular speed, the creep rate has large value at the internal surface as compared to cylinder made of transversely isotropic material for measure $n = 1/3$ and these values further increases at the internal surface with the increase in pressure. For measure $n = 1/7$ (i.e. $N = 7$), the creep rates have lesser value at the internal surface as compared to measure $n = 1/3$ (i.e. $N = 3$). The value of creep rates decreases with the increase in strain.

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**Figure 1:** Creep Stresses in a Thick-walled Rotating Cylinder along the Radius (R) for different measure of N (=1/n) and angular speed $\left(\Omega^2 = 0\right)$

- $N=1$
- $N=3$
- $N=7$

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**Figure 2:** Creep Stresses in a Thick-walled Rotating Cylinder along the Radius (R) for different measure of N (=1/n) and angular speed $\left(\Omega^2 = 5\right)$

- $N=1$
- $N=3$
- $N=7$

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Material Isotropic ........................................

Material Isotropically Transverse ..........................
Figure 3: Strain Rates Distribution in a Thick-walled Rotating Cylinder along the Radius (R) for different measure of N (=1/n) and angular speed $\Omega^2 = 0$

- N=1
- N=3
- N=7

Figure 4: Strain Rates Distribution in a Thick-walled Rotating Cylinder along the Radius (R) for different measure of N (=1/n) and angular speed $\Omega^2 = 5$

- N=1
- N=3
- N=7
References


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