Adv. Studies Theor. Phys., Vol. 2, 2008, no. 15, 741 - 749

Homogeneous and Isotropic Expanding Universe with Unconventional Role of Torsion

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Abstract

An isotropic homogeneous cosmological model with torsion is considered. Torsion is treated as a parameter of the theory that, unconventionally, is not varied in deriving the field equations from the Einstein-Hilbert action. The model is discussed in cases that have counterparts in Standard Cosmology. It is shown that inflationary phases in the time evolution are possible. Origin of torsion is no further justified. However, if considered like here, it gives an energy like contribution, different from the conventional matter and radiation contribution, that exists also in the empty (open) space-time.

Physics and Astronomy Classification Scheme: 04.50; 98.80.Jk; 04.20.Cv

Keywords: Homogeneous isotropic Universe; Einstein-Hilbert action; Torsion; Cosmological Model; Inflation

1 Introduction

The assumption of homogeneity and isotropy of the Universe is the generally admitted principle for the construction of a cosmological model. Of course inhomogeneous cosmological models are as well of interest not only in relation to observational data, but also in connection to the problem of their stability (See e.g. [2, 10]). Once applied to general relativity the principle translates into the request of a maximally symmetric metric that in turn leads to the

Robertson-Walker metric [19]. If the cosmological model is extended to include torsion, the cosmological principle implies that the torsion tensor has a special form depending on two functions of the time [18] (See also [3, 6]). The inclusion of torsion in general relativity is a condition whose request goes back to the Einstein-Cartan-Sciama-Kibble theory. (For a review see [8]; for its spinor formulation see [14]). Accordingly the field equations are derived from the Einstein-Hilbert action by varying the metric and the torsion independently [4, 8] (See also [17]). This situation has been recently re-considered and it has been shown that it is compatible with an inflationary phase in the evolution of the Universe [4]. A theoretical extension of general relativity to include torsion can also be done simply by re-defining the signature parameter in the metric of the standard cosmology [11].
In the present paper a cosmological model with torsion is proposed that is based on the Robertson-Walker metric and a homogeneous isotropic torsion. The Einstein-like field equations are obtained from the Einstein-Hilbert action by varying the metric, but not the torsion, and equating the result to the energy momentum tensor. Consistency in the metric divergence of the field equations is explicitly required. Some special situations are then studied by noting that the conservation equation is implied by the other equations. The empty space case implies, contrarily to the Standard Model, a non static (open) Universe whose time evolution and torsion are explicitly obtained.
In the present model the equations do not completely determine the functions involved. It is then possible to assume, a priori, particular physical configurations. For zero pressure and constant torsion the universe may expand with constant density in the flat, open and closed case. A situation is finally identified that allows an inflationary growing of the universe. This is compatible with different state equations. In the simplest case of proportionality between energy density and pressure the solution is completely determined. In particular the vacuum energy case, that produces constant pressure and energy density in the standard cosmological model [9, 13] is characterized here by an exponential inflation to which corresponds an exponential decreasing of the torsion.

2 Assumptions and general formulation.

The following discussion is developed in a four dimensional Lorentz manifold with affine connection $\widetilde{\nabla}$ that is required to be compatible with the metric $g_{\mu\nu}$ (For definitions and mathematical assumptions we refer to [12]). The affine coefficients Γ can be expressed in terms of the contorsion tensor $K^{\kappa}_{.\mu\nu}$, the Christoffel symbols $\{^{\kappa}_{\mu\nu}\}$ and the torsion tensor $T^{\kappa}_{.\mu\nu}$ through the relations

$$\begin{aligned} \Gamma^{\kappa}_{\ \mu\nu} &= \{^{\kappa}_{\mu\nu}\} + K^{\kappa}_{.\mu\nu} \qquad (K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}) \\ K^{\kappa}_{.\mu\nu} &= \tfrac{1}{2}(T^{\kappa}_{.\mu\nu} + T_{\mu.\nu}^{\ \,k} + T_{\nu.\mu}^{\ \,\kappa}) \\ T^{\kappa}_{.\mu\nu} &= \Gamma^{\kappa}_{\ \mu\nu} - \Gamma^{\kappa}_{\ \nu\mu} \end{aligned} \tag{1}$$

The contorsion tensor can be decomposed into independent parts as [1, 5, 7]

$$\begin{aligned} 6K_{\alpha\mu\nu} &= 2(g_{\alpha\mu}\tau_\nu - g_{\nu\mu}\tau_\alpha) + 3\mathcal{A}^\sigma\epsilon_{\sigma\alpha\mu\nu} + 6U_{\alpha\mu\nu} \\ \tau_\mu &= g^{\alpha\beta}K_{\alpha\beta\mu} \\ 3\mathcal{A}^\sigma &= \epsilon^{\sigma\alpha\beta\mu}K_{\alpha\beta\mu} \end{aligned} \tag{2}$$

a relation that indeed defines also $U_{\alpha\mu\nu}$ ($\epsilon_{\alpha\beta\mu\nu}$ is the completely antisymmetric tensor such that $\epsilon_{0123} = 1$). Accordingly, the affine curvature $\widetilde{R}$ can be decomposed, up to a divergence term, as [1, 5, 17, 20]

$$\widetilde{R} = R - \frac{1}{3}\tau^2 + \frac{3}{2}\mathcal{A}^2 + U_{\alpha\mu\nu}U^{\mu\alpha\nu} \tag{3}$$

where R is the metric curvature obtained from the Ricci (metric) tensor $R^{\mu\nu}$. The gravitational action can then be assumed to be $S_g = -\int d^4x\sqrt{g}\widetilde{R}$ where $g = |det\ g_{\mu\nu}|$. The volume element in the action is required to be metric compatible but not connection compatible as sometimes alternatively required (See, e.g. [15, 16]). By varying with respect to the metric $g_{\mu\nu}$ but not with respect to the torsion, and by no variations of the boundaries, one obtains (e.g. [21]):

$$\delta S_g = \int d^4x\sqrt{g}(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R + \tau^{\mu\nu} + \mathcal{A}^{\mu\nu} + U^{\mu\nu}) \tag{4}$$

$$\tau^{\mu\nu} = \tfrac{1}{3}(\tfrac{1}{2}g^{\mu\nu}\tau^2 - \tau^\mu\tau^\nu) \tag{5}$$

$$\mathcal{A}^{\mu\nu} = -\tfrac{3}{2}(\tfrac{1}{2}g^{\mu\nu}\mathcal{A}^2 - \mathcal{A}^\mu\mathcal{A}^\nu) \tag{6}$$

$$U^{\mu\nu} = -\tfrac{1}{2}g^{\mu\nu}U_{\alpha\beta\gamma}U^{\beta\alpha\gamma} + 2U^{\nu}_{.\alpha\beta}U^{\alpha\mu\beta} + U_{\beta\alpha.}^{\ \ \nu}U^{\alpha\beta\mu} \tag{7}$$

The field equations can then be assumed to be

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \tau^{\mu\nu} + \mathcal{A}^{\mu\nu} + U^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{8}$$

where $T^{\mu\nu}$ is the energy momentum tensor whose form depends on the particular physical situation one is considering. It can be noted that, by taking the metric divergence ∇ of eq. (8) and using the corresponding Bianchi identity, one has

$$\nabla_\mu(\tau^{\mu\nu} + \mathcal{A}^{\mu\nu} + U^{\mu\nu}) = 8\pi G\nabla_\mu T^{\mu\nu} \tag{9}$$

This equation is assumed to generalize, in presence of torsion, the covariant conservation equation $\nabla_\mu T^{\mu\nu} = 0$. In the following, the equations (8), (9) will be used as the basic scheme for the formulation of a cosmological model.

3 Cosmology with torsion.

The cosmological principle that the Universe is spatially homogeneous and isotropic can be translated by requiring that the 4-dimensional space-time manifold admits maximally symmetric three dimensional subspaces whose metric has positive eigenvalues and arbitrary curvature. This leads to the Robertson-Walker form of the metric tensor such that [19]:

$$ds^2 = dt^2 - R^2(t)\Big[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big] \quad (a = 0, \pm 1). \qquad (10)$$

We denote $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$. The same principle requires the invariance of cosmic fields under all the isometries under which the Robertson-Walker metric is, and therefore it implies restrictions on the tensor fields. If the principle is extended to space-time with torsion, there results that the non zero components of the torsion tensor are of the form [18] (See also [4, 6])

$$T_{ijk} = \epsilon_{ijk} f(t), \qquad T_{ij0} = -T_{i0j} = g(t)\delta_{ij} \qquad (11)$$

with $f(t)$, $g(t)$ a priori general function of the time and ϵ_{ijk} is the completely antisymmetric tensor such that $\epsilon_{123} = 1$ (space-time indexes are denoted by greek letters, spatial indexes by latin letters). As a consequence the contortion tensor and its components can be expressed in terms of f, g. One obtains

$$\begin{aligned} \tau_\alpha &= g(t)M\delta_{\alpha 0}, \qquad \mathcal{A}_\sigma = f(t)\delta_{\sigma 0} \\ U^{\mu\nu} &= g^2(t)M_1(\delta^{\mu 0}\delta^{\nu 0} - \frac{1}{2}g^{\mu\nu}) + 2g^2(t)M_{kk}\delta^{\mu k}\delta^{\nu k} \qquad (12) \\ M_1 &= \sum_{i=1}^{3}(g^{ii} - \frac{1}{3}M)^2; \quad M_{kk} = g^{kk}(g^{kk} - \frac{1}{3}M)^2 \end{aligned}$$

where $M = \sum_{i=1}^{3} g^{ii}$, no sum over k in the expression of M_{kk}. By using these expressions and the fact that $R_{\mu\nu} = 0$ for $\mu \neq \nu$ in the Robertson-Walker metric, the equation (8) implies that T must be diagonal. On account of the cosmological principle we are induced to assume, for the energy momentum tensor, the form adopted in the Standard Cosmology:

$$T^\mu_\nu = diag\ \{\rho(t), -p(t), -p(t), -p(t)\} \qquad (13)$$

where the energy density ρ and the pressure p depend only on the time. It is a fact that the conditions so far assumed imply

$$g(t) \;=\; 0 \tag{14}$$

Indeed for $\mu = \nu = 0$ all terms of eq. (8) depend only on time, except the term $U^{00} + \tau^{00}$ that, according to eqs. (5), (7) is of the form $U^{00} + \tau^{00} = g^2(t)F(r,\theta)/R^4(t)$. Therefore eq. (8) can be separated for $\mu = \nu = 0$. This implies, unless $g(t) = 0$, that $F(r,\theta) = \textit{constant}$ which is not possible because F is a well defined non constant function of its arguments. As a consequence $\tau_\alpha = 0$, $U^{\mu\nu} = 0$ and the axial torsion is the only part of the torsion that is of cosmological relevance. Under the results (11)-(14) the field equations and the conservation equation become respectively ($\dot{R} = dR/dr$):

$$3\Big(\frac{\dot{R}^2}{R^2} + \frac{a}{R^2}\Big) + \frac{3}{4}f^2(t) \;=\; 8\pi G\rho \qquad (\mu = \nu = 0) \tag{15}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{a}{R^2} - \frac{3}{4}f^2(t) \;=\; -8\pi G p \quad (\mu = \nu = r, \theta, \varphi) \tag{16}$$

$$\frac{3}{2}f\dot{f} + \frac{9}{2}\frac{\dot{R}}{R}f^2 \;=\; 8\pi G[\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p)] \tag{17}$$

The eq. (17) is eq. (9) detailed for $\nu = 0$, the other ones being automatically satisfied. An useful equation that follows combining eqs. (15), (16), is

$$6\frac{\ddot{R}}{R} \;=\; 3f^2 - 8\pi G(\rho + 3p) \tag{18}$$

It can be noted that, inserting the expression of ρ obtained from (15) and that of $\rho{+}p$ obtained from eqs. (15), (16) into eq. (17), one finds that the eq. (17) is automatically satisfied. Therefore the present cosmological model with torsion is essentially described by eqs. (15), (16) in the functions of time R, f, ρ, p. These equations are slightly different from the final equations obtained in Ref. [4] where however the spin tensor of matter and the cosmological constant have been taken into consideration. The present scheme differs also from Ref. [11] because the opposite sign f^2 has in eqs. (15), (16) does not allow here to possibly re-define the signature parameter a. It should also be noted that the eqs. (15), (16), (17) are formally those of a cosmological model without torsion but with pressure and energy density defined by $p - 3f^2/(32\pi)$, $\rho - 3f^2/(32\pi)$.

4 Discussion.

The present model does not completely define the functions involved. One can then consider different physical situations.

Empty space. If one sets $\rho = p = 0$ into eqs. (15), (16), one is left with the equations

$$f = \alpha R^{-3}, \qquad R^4(\dot{R}^2 + a) = -\frac{1}{4}\alpha^2 \tag{19}$$

α a real integration constant if one looks for f and R real functions. There follows that the cases $a = 0, 1$ are impossible. If $a = -1$ the second equation in (19) can be integrated by separation of variables by setting $2R^2/\alpha = \cosh x$, and then $y = \exp x$. One finds

$$\frac{\alpha^{3/2}}{8}\left[\log C\Big(y + \sqrt{1+y^2}\Big) - \frac{\sqrt{1+y^2}}{y}\right] = \pm t + \beta \tag{20}$$

β, C integration constants. The eq. (20) gives in principle $y = y(t)$ (at least in some time regions) so that $R^2(t) = \frac{\alpha}{4}[y(t) + \frac{1}{y(t)}]$ that toghether with the first equation (19) gives also the expression of $f(t)$. Therefore if the torsion is given a physical interpretation, torsion has a role also in case of empty space-time.

"Matter dominated" Universe. Suppose $p = 0$, $f = constant = f_0$. The equation (17) gives, by choosing zero the integration constant,

$$16\pi G\,\rho \;=\; 3\, f_0^2 \tag{21}$$

The system (15), (16) can then be easily integrated to obtain

$$\begin{aligned} R \;&=\; \alpha \exp(\pm\sqrt{\rho_0}\; t), & a = 0 \\ R \;&=\; \tfrac{1}{\sqrt{\rho_0}} \cosh(\pm\sqrt{\rho_0}\, t + \gamma) & a = 1 \\ R \;&=\; \tfrac{1}{\sqrt{\rho_0}} \sinh(\pm\sqrt{\rho_0}\, t + \gamma) & a = -1 \end{aligned} \tag{22}$$

where α, γ are integration constants and $\rho_0 = \frac{4}{3}\pi G\rho$. If the mentioned integration constant is not taken to be zero, the equation are difficult to be solved, but ρ is allowed to depend on t. Therefore the scheme differs from the matter dominated case of the Standard Cosmology where the behaviour of the energy density is only of the form $\rho \propto R^{-3}$ (e.g. [9, 13].

Inflation. Possible exponential expansion of the Universe happens, for instance, when $\ddot{R}/R \cong constant = \alpha^2$. Then

$$R \;=\; \beta \exp(\pm\alpha t) \tag{23}$$

Therefore, under the present assumptions, eqs. (16), (18) become respectively

$$f^2 = 2\alpha^2 + \frac{8}{3}(\rho + 3p)\pi G \tag{24}$$

$$\frac{3}{2}\alpha^2 + \frac{a}{\beta^2}\exp(-2\alpha t) = 2\pi G(\rho - p), \quad a = 0, \pm 1 \tag{25}$$

One can then check that all the other equations are satisfied identically. An inflationary expansion is therefore compatible with different state equations. Suppose $p = w\rho$. Then

$$\rho = \frac{3\alpha^2/2 + a\exp(-2\alpha t)/\beta^2}{2\pi G(1-w)} \tag{26}$$

$$f^2 = 4\alpha^2\frac{1+w}{1-w} + \frac{4}{3}\frac{a}{\beta^2}\frac{1+3w}{1-w}\exp(-2\alpha t) \tag{27}$$

The situation $w = -1$ (vacuum energy of the Standard Cosmology) is possible only for the closed case $a = 1$ and implies $f^2 \approx 1/R^2$. Instead, for $w > 1,\ \alpha \ll 1,\ \alpha^2 \approx 0$, the eqs. (27) are possible only in the open Universe case $a = -1$ and again one has $f^2 \approx 1/R^2$. Finally if $w = -1/3$ or $a = 0$ the torsion remains constant.

The field equations of the present cosmological scheme do not include the spin angular momentum equation as done, e. g., in Ref. [3,4] in the line of the formulation of Ref. [8]. (The cosmological constant term has not been considered here to better put into evidence the role the torsion plays). The surviving function that characterizes torsion is therefore treated, since the beginning, as a further parameter of the theory whose interpretation has not yet been given. To partially overcome that problem note that from the above discussion of possible different configurations of the cosmological model, it appears that torsion adds an energy like term to the equations. In particular this is true in the empty space in which case also an expansion of the universe is possible. Therefore torsion could be interpreted as a parameter characterizing an a priori energy content of the empty "state" of the universe. This interpretation however does not justify the presence of torsion. Its origin (like that of the origin of spin of matter in the conventional cosmologies with torsion) seems to remain an open problem.

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Received: March 31, 2008