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H-Atom Energy Spectrum in Open RW Space Time via Approximated "Curved" Coulomb Potential

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Abstract

The Dirac equation with approximated "curved" Coulomb potential is studied in open Robertson Walker space-time by the Newman-Penrose formalism. Under suitable approximations on both the cosmological background and short range of the action of the potential, the confined solutions of the Dirac equation are exactly determined. By a Schrôdinger like quantization assumption they are interpreted to describe the Hydrogen atom. The corresponding discrete energy spectrum is determined.

Keywords: RW metric - " Curved" Coulomb potential - Dirac equation - Discrete energy spectrum

1 Introduction

The study of the spectrum of the Hydrogen atom is usually performed by the Dirac equation with the conventional Coulomb potential. In the context of general relativity the Coulomb potential results to be have a modified expression due to both curvature and expansion of the space time. Due to the physical interest the H-atom has been studied not only by the Dirac equation (see e. g., [6, 7, 10, 11, 12, 17] and References therein) but also by other point if view such as Path integral methods (e. g., [1] and References therein).

In the present paper the H-atom is studied by the Dirac equation method by using the curved expression of Coulomb potential recently proposed [17].

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The solution of the equation is obtained under the assumption that the H-atom exists in a microscopic confined form so that the curved Coulomb potential is approximated to the first order in r. Other approximations are done that essentially correspond to consider a static cosmological background far from the moment of the big bang. The angular dependence of the Dirac spinor equation is separated by using the two components Newmann Penrose formalism [13] as in [16]. One is then left with two coupled equations that generalize those of the Minkowski space time case [4]. They are then disentangled into two second order ODE's that are reported, by making use of the approximation assumptions, to confluent hypergeometric equations. The series expansion of the hypergeometric equations is then truncated in order to have an integral convergence required by a Schrodinger like interpretation assumption. Accordingly one obtains the discrete energy spectrum of the H-atom that reduces to the conventional one if the "curved" potential is reduced to the standard Coulomb potential.

2 Dirac equation with central potential

In curved space time the Dirac field can be descried by the pair $(\phi_A, \chi_{B'})$ of two component spinor field. Following [3], by setting $P_A \equiv \phi_A$, $\bar{Q}^{B'} \equiv -\chi^{B'}$ the Dirac equation for a particle of mass μ and subjected to the potential $V_{AA'}$ can be consistently written [5]:

$$(\nabla_{AX'} + iV_{AX'})P^A + i\mu_\star \bar{Q}_{X'} = 0 \tag{1}$$

$$(\nabla_{AX'} - i\bar{V}_{AX'})Q^A + i\mu_\star \bar{P}_{X'} = 0 \quad (\mu_\star = \mu/\sqrt{2}) \tag{2}$$

In the following the Dirac equation will be studied in the Robertson Walker space time of metric tensor $g_{\mu\nu}$ given by [14]

$$g_{\mu\nu} = diag\Big\{1;\, -\frac{R(t)^2}{1-\kappa r^2};\, -r^2 R(t)^2;\, -r^2 R(t)^2 (\sin\theta)^2\Big\} \qquad \kappa = 0, \pm 1 \tag{3}$$

The Dirac equation is studied by applying the Newman Penrose formalism [13], based on the null tetrad frame defined in [15], namely by expanding the spinorial derivatives in terms of directional derivatives and spin coefficients. The Dirac equation can be separated under the condition s $V_{AX'} = V_{AX'}(t,r)$ and $V_{01} = V_{10}$ by setting

$$P^A \equiv \frac{1}{rR(t)}\Big(H_1(t,r)S_1(\theta),\, H_2(t,r)S_2(\theta)\Big)e^{im\varphi} \tag{4}$$

$$\bar{Q}^A \equiv \frac{1}{rR(t)}\Big(-H_1(t,r)S_2(\theta),\, H_2(t,r)S_1(\theta)\Big)e^{im\varphi} \tag{5}$$

$$V_{AA'} = \sigma^\alpha_{AA'} V_\alpha = \frac{V}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Big(V_\alpha \equiv (V,0,0,0)\Big) \tag{6}$$

$m = 0, \pm 1, \pm 2, \ldots$, σ^0 the σ-matrix relative to the mentioned null tetrad frame. This leads to an eigenvalue problem for S_1, S_2 that under a suitable continuity condition gives the angular functions that are the same of those in absence of potential and result to be essentially given by Jacobi polynomials (see, e. g., [8, 15, 16]). One is then left with the separated equation in the t, r variables:

$$(D + \epsilon + iV_{00})H_1 = H_2\Big(i\mu_\star + \frac{\lambda}{rR}\Big) \tag{7}$$

$$(\Delta + \epsilon + iV_{00})H_2 = H_1\Big(i\mu_\star - \frac{\lambda}{rR}\Big) \tag{8}$$

where $\lambda^2 = l(l+1),\ l = 0, 1, 2, ..,$ is the eigenvalue of the separated angular equations [16], the directional derivatives and spin coefficient being given by

$$\sqrt{2}\, D = \partial_t + R^{-1}\sqrt{1 - \kappa r^2}\partial_r, \quad \sqrt{2}\,\Delta = \partial_t - R^{-1}\sqrt{1 - \kappa r^2}\,\partial_r \tag{9}$$

and $\epsilon = 2^{-2/3}\dot{R}/R$.

3 Approximated H-atom description

The solution of equations (7), (8) seems very difficult also for for simple analytical form of $V(r)$ and some approximations assumptions seems to be unavoidable. Here it is assumed that possible solutions representing the H-atom in the open space time case must be spatially confined into atomic dimensions. Moreover the radius $R(t)$ of the Universe is taken constant (in the following also large with respect to atomic dimensions) during time intervals the atomic interactions take place. It is then assumed that:

$$|\kappa r^2| \ << 1, \quad R(t) = R(t_0) = R_0 \quad (\kappa = -1) \tag{10}$$

$$V(r) = -\frac{e_0}{R_0^3}\Big[\frac{\sqrt{1+r^2}}{r} - \log(r + \sqrt{1+r^2})\Big] \tag{11}$$

$$\cong \frac{\chi}{R_0\sqrt{2}}\Big(\frac{1}{r} - \frac{r}{2}\Big), \quad r << 1, \quad (\kappa = -1) \tag{12}$$

where $V(r)$ is the curved Coulomb potential previously obtained that makes sense only in the open space time case [17]. The coefficient of V in (14) has given that form for convenience in the following development.

By the position

$$H_1 = r\frac{f(r) + g(r)}{2}e^{ikt}, \quad H_2 = r\frac{f(r) - g(r)}{2}e^{ikt} \tag{13}$$

the equations (7), (8) with (9) and the approximation (10) give :

$$f' + \frac{1-\lambda}{r}\, f + i(k_0 - m_0 + VR_0)\, g = 0 \tag{14}$$

$$g' + \frac{1+\lambda}{r}\, g + i(k_0 + m_0 + VR_0)\, f = 0 \tag{15}$$

where $k_0 = kR_0,\ m_0 = \mu R_0$. By further setting [4], :

$$f = \alpha\, e^{-\rho/2} \rho^{\gamma-1} (q_1 - q_2)(\rho), \quad g = \beta\, e^{-\rho/2} \rho^{\gamma-1} (q_1 + q_2)(\rho) \tag{16}$$

$$\rho = 2\delta r, \quad \delta = R_0\sqrt{\mu^2 - k^2}, \quad \alpha/\beta = \sqrt{(k-\mu)/(k+\mu)} \tag{17}$$

and by taking into account the approximation (12), one finally obtains the equations for q_1, q_2:

$$\rho q_1' + q_1[\gamma + iA\, F(\rho)] + q_2[-\lambda - iB\, F(\rho)] = 0 \tag{18}$$

$$\rho q_2' + q_2[\gamma - \rho - iA\, F(\rho)] + q_1[-\lambda + iB\, F(\rho)] = 0 \tag{19}$$

$$A = \frac{\chi k}{\sqrt{k^2 - \mu^2}}, \quad B = \frac{\chi \mu}{\sqrt{k^2 - \mu^2}}, \quad F(\rho) = 1 - \rho^2/(8\delta^2). \tag{20}$$

The equations (18), (19) can be disentangled by a substitution procedure. For what concerns q_1, one obtains the equation

$$q_1'' \rho^2 (\lambda + iBF) + \rho Q_1 q_1' + P_1 q_1 = 0 \tag{21}$$

$$\frac{Q_1}{\lambda + iBF} = 1 + 2\gamma - \rho + \frac{B}{4\delta^2}\, \frac{1}{\lambda + iBF} \tag{22}$$

$$\cong 1 + 2\gamma - \rho, \quad R_0 >> 1 \tag{23}$$

$$\frac{P_1}{\lambda + iBF} = iF'\rho\Big(A - B\frac{\gamma + iAF}{\lambda + iBF}\Big) - \rho\Big(\gamma + iA - \frac{iA}{8\delta^2}\rho^2\Big) + \\ + \frac{\rho^2 (A^2 - B^2)}{4\delta^2}\Big(1 + \frac{\rho^2}{16\delta^2}\Big) \tag{24}$$

$$\cong \frac{\chi^2 \rho^2}{4(\mu^2 - k^2)} - \Big(\gamma + \frac{k\chi}{\sqrt{\mu^2 - k^2}}\Big)\rho, \quad R_0 >> 1 \tag{25}$$

The approximation done are based on the dependence of δ on R_0 in (17). It has also been set

$$\gamma = \sqrt{\lambda^2 - \chi^2} \tag{26}$$

a condition that makes to disappear some terms in P_1. By a completely similar procedure, one obtains, from (18), (19), the approximated equation for $q_2(\rho)$. The final approximated equations are then recast into the form

$$q_1'' + \frac{a_0 + a_1\rho}{\rho} q_1' + \frac{b_0 + b_1 \rho + b_2 \rho^2}{\rho^2} q_1 = 0 \tag{27}$$

$$q_2'' + \frac{a_0 + a_1\rho}{\rho} q_2' + \frac{b_0 + (b_1 - 1)\rho + b_2 \rho^2}{\rho^2} q_2 = 0 \tag{28}$$

$$a_0 = 1 + 2\gamma, \quad a_1 = -1, \quad b_0 = 0 \tag{29}$$

$$b_1 = -\gamma - \frac{\chi}{\sqrt{\mu^2 - k^2}}, \quad b_2 = \frac{\chi^2}{4(\mu^2 - k^2)} \tag{30}$$

By further setting $q_i(\rho) = e^{x\rho} p_i(\rho),\ i = 1, 2$, and then $\xi = -(2x + a_1)\rho$, with $x^2 + a_1 x + b_2 = 0$ or $2x_\pm = 1 \pm \sqrt{1 - 4b_2}$, the equations (27), (28) can be reported to confluent hypergeometric equations:

$$\xi p_1'' + (a_0 - \xi) p_1' - \frac{a_0 x + b_1}{2x + +a_1} p_1 = 0 \tag{31}$$

$$\xi p_2'' + (a_0 - \xi) p_2' - \frac{a_0 x + b_1 - 1}{2x + +a_1} p_2 = 0 \tag{32}$$

whose solutions are respectively

$$q_1 = C e^{x\rho} \phi\Big(\frac{a_0 x + b_1}{2x + +a_1}; 1 + 2\gamma; -(2x + a_1)\rho\Big) \tag{33}$$

$$q_2 = D e^{x\rho} \phi\Big(\frac{a_0 x + b_1 - 1}{2x + +a_1}; 1 + 2\gamma; -(2x + a_1)\rho\Big) \tag{34}$$

The constants C, D are related by

$$C = \frac{\lambda - \chi\mu/\sqrt{\mu^2 - k^2}}{\gamma - \chi\mu/\sqrt{\mu^2 - k^2}} D \tag{35}$$

that follows from the q_1, q_2 equations.

In the following, the range of the energy k is supposed to take value lower then the mass μ of the particle. It is also assumed that $\mu^2 - k^2 > \chi^2$.

4 Discrete spectrum

By the conserved current $J^{AA'} = \bar{P}^A P^{A'} + \bar{Q}^A Q^{A'}$ one can define an inner product of solutions of the free Dirac equation [9]. The same holds for the Dirac equation with the potential adopted here, because, as it can be checked, the current $J^{AA'}$ is still conserved . Accordingly, by a Schrôdinger like interpretation the H atom corresponds to solutions for which [9]

$$\int d_3x \sqrt{-g}(|P^0|^2 + |P^1|^2 + |Q^0|^2 + |Q^1|^2) < \infty \tag{36}$$

or, by factoring out the (bounded) angular integral, such that

$$\int_0^\infty d\rho\, \rho^{2\gamma} e^{-\rho}(|q_1|^2 + |q_2|^2) < \infty \tag{37}$$

By taking into account the asymptotic behavior of the confluent hypergeometric function $\phi(a; c; z)$ for large z, the integrating function in (37) behaves like

$$e^{(2x_\pm - 1)\rho}\Big[e^{-(2x_\pm - 1)\rho} + v(\rho)\Big], \quad \rho >> 1 \tag{38}$$

where $u(\rho), v(\rho)$ are polynomials in ρ, $2x_\pm - 1 = \pm\sqrt{(\mu^2 - k^2 - \chi^2)/(\mu^2 - k^2)}$ and the expression in squared brackets represents the asymptotic behavior of ϕ. Hence for both x_+ and x_- the integral in (37) is not bounded. Convergence can then be obtained by truncating the hypergeometric series and by choosing $x = x_-$. In case of q_1 in (33) we set then $(a_0 x_- + b_1)/(2x_- + a_1) = -n,\ n = 1, 2, ...$ that is

$$-\frac{1}{2}\left[(1+2\gamma)\Big(1-\sqrt{\frac{\mu^2 - k^2 - \chi^2}{\mu^2 - k^2}}\Big)\right] - \gamma - \frac{\chi k}{\sqrt{\mu^2 - k^2}} = -n\sqrt{\frac{\mu^2 - k^2 - \chi^2}{\mu^2 - k^2}} \quad (39)$$

By putting $y^2 = \mu^2 - k^2$, the equation (39) can be solved algebraically with respect to y. Hence $k/\mu = [(y/k)^2 + 1]^{-\frac{1}{2}}$ or

$$\frac{k}{\mu} = \left[1 + \left(\frac{2\chi + \sqrt{4\chi^2 + [(1+2n+2\gamma)^2 - 1][4\chi^2 + (1+2n+2\gamma)^2]}}{(1+2n+2\gamma)^2 - 1}\right)^2\right]^{-\frac{1}{2}} \quad (40)$$

In case of (34) one reproduces the spectrum (40) by noting that, as far as the condition (39) is concerned, $b_1 \to b_1 - 1 \Leftrightarrow \gamma \to \gamma + 1 \wedge n \to n - 1$.

Correspondingly, the solutions of (33), (34) are

$$q_1 = C\, e^{-\rho\sqrt{\frac{\mu^2-k^2-\chi^2}{\mu^2-k^2}}}\ \phi\Big(-n; 2\gamma+1; \sqrt{\frac{\mu^2 - k^2 - \chi^2}{\mu^2 - k^2}}\rho\Big) \quad (41)$$

$$q_2 = D\, e^{-\rho\sqrt{\frac{\mu^2-k^2-\chi^2}{\mu^2-k^2}}}\ \phi\Big(-n+1; 2\gamma+3; \sqrt{\frac{\mu^2 - k^2 - \chi^2}{\mu^2 - k^2}}\rho\Big) \quad (42)$$

The spectrum of the conventional $1/r$ Coulomb potential corresponds to the choice $b_2 = 0$ in the previous development. The discrete spectrum is then:

$$\frac{k}{\mu} = \left[1 + \frac{\chi^2}{(1 + n + \sqrt{\lambda^2 - \chi^2})^2}\right]^{-\frac{1}{2}}, \quad n = 1, 2, ... \quad (43)$$

that is the conventional relativistic spectrum of the H-atom [2].

5 Remarks and comments

In previous calculations the H-atom has been approximately described by the Dirac equation with curved Coulomb potential in open RW space time. The approximations done concern both the cosmological background and the Dirac equation. It has been assumed the H-atom to be still of atomic dimension. Accordingly the curved Coulomb potential has been approximated to the first order in r. The radius of the universe has been taken sufficiently large with respect to atomic dimensions and constant during the time intervals the interactions take place. Accordingly the exact solution of the approximated Dirac equation has been given and a "discrete" energy spectrum of the H-atom determined.

Under the approximations done, the scheme does not seem immediately suitable for a treatment of the scattering states of the Dirac equation on account of the asymptotic behavior of the potential (11) for large r. Instead the discrete energy H-spectrum determined could possibly have some cosmological interest, far from the big bang time.

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