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Construction of Hall States by Means of a Model Quantum Matrix Model

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Abstract

The mathematical structure of quantum matrix oscillator models is studied. There is an equivalence between these models and the Calogero model in one dimension. By transforming to the Bargmann representation using a type of similarity transformation, the quantum matrix oscillator is found to represent the quantum mechanics of electrons in the lowest Landau level with ground state accounted for by a Laughlin-type wavefunction.

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1 Introduction to the Matrix Model

A class of models will be defined which turn out to have important applications to the quantum Hall effect [**1-5**]. The quantum Hall effect has many important implications for physics, such as the nature of fractional charge and applications of topolgy to physics [**6-8**]. It may be that further mathematical applications might result as the effect is further studied [**9-10**].

Consider the $N \times N$ matrices $\mathbf{X}$ and $\mathbf{P}$ with operator valued matrix elements and a matrix $\mathcal{M}$ referring to particle masses. The matrix Hamiltonian, generally non-Hermitian, $\hbar = 1$, is given by

$$\mathbf{H} = \frac{1}{2}\big(\mathbf{P}\mathbf{M}^{-1}\mathbf{P} + \omega^2\,\mathbf{X}\mathbf{M}\mathbf{X}\big). \tag{1.1}$$

This represents a matrix generalization of a single harmonic oscillator. Suppose the following matrix commutation relations are postulated

$$[\mathbf{X}, \mathbf{P}] = i\mathcal{V}, \tag{1.2}$$

where $\mathcal{V}$ is a Hermitian $N \times N$ matrix with constant real, symmetric matrix elements $\nu_{ij} = \nu_{ji}$, with $i, j = 1, \ldots, N$, and the diagonal elements are set to equal to unity. Suppose also that the matrix $\mathbf{X}$ is Hermitian and can be represented by a diagonal matrix with real matrix elements x_i, with $i = 1, 2, \ldots, N$ and $m_i > 0$, so that the matrix elements of $\mathbf{X}$ and $\mathbf{M}$ are

$$\mathbf{X}_{ij} = x_i \delta_{ij}, \qquad \mathbf{M}_{ij} = m_i \, \delta_{ij}. \tag{1.3}$$

Introduce a matrix operator $\mathcal{D}$ such that $\mathbf{P} = -i\mathcal{D}$ holds [**1**]. The result (1.2) can be calculated in the following manner for all i, j,

$$\mathcal{D}_{ij} x_j - x_i \mathcal{D}_{ij} = \nu_{ij}, \quad i \neq j, \qquad \mathcal{D}_{ii} x_i - x_i \mathcal{D}_{ii} = 1. \tag{1.4}$$

The addition of a diagonal contribution to $\mathcal{D}_{ij}$ which depends only on coordinates doesn't influence these equations, as there are many solutions to (1.4). This can be expressed more explicitly by the similarity transformation $\mathcal{D}^f = f^{-1}\mathcal{D}\, f$ of the operator $\mathcal{D}$ with respect to f, an arbitrary function of the coordinates.

The corresponding Hamiltonians are connected by non-unitary gauge transformations, or similarity transformations, of the form $\mathcal{H}^f = f^{-1}\mathcal{H}\, f$. The presentation shall be restricted to gauge transformations defined by $f = \prod_{i<j} (x_i - x_j)^{\lambda_{ij}}$ where the λ_{ij} are symmetric $\lambda_{ij} = \lambda_{ji}$ gauge parameters. The operators $\mathcal{D}_{ij}$ will be shown to constitute a solution to (1.4)

$$\mathcal{D}_{ij} = \delta_{ij}\Big(\frac{\partial}{\partial x_i} + \sum_{k \neq i} \frac{\lambda_{ik}}{x_i - x_k}\Big) - \nu_{ij} \frac{1 - \delta_{ij}}{x_i - x_j}. \tag{1.5}$$

Therefore, we find that

$$\mathcal{D}_{ij} x_j - x_i \mathcal{D}_{ij} = \delta_{ij} + x_j \delta_{ij} \frac{\partial}{\partial x_i} + \sum_{i \neq k} \frac{\lambda_{ik}}{x_i - x_k} x_j \delta_{ij} - \frac{\nu_{ij}(1 - \delta_{ij})}{x_i - x_j}\, x_j$$

$$-\delta_{ij} x_i \frac{\partial}{\partial x_i} - x_i \delta_{ij} \sum_{k \neq i} \frac{\lambda_{ik}}{x_i - x_k} + \nu_{ij} \frac{(1 - \delta_{ij})}{x_i - x_j} x_i = \nu_{ij} \frac{1 - \delta_{ij}}{x_i - x_j}\,(x_i - x_j) = \nu_{ij}, \tag{1.6}$$

for $i \neq j$, and since $1 - \delta_{ii} = 0$, so when $i = j$,

$$\mathcal{D}_{ii}\, x_i - x_i\, \mathcal{D}_{ii} = x_i \frac{\partial}{\partial x_i} - x_i\, \frac{\partial}{\partial x_i} + \delta_{ii} = 1. \tag{1.7}$$

The dependence of the Hamiltonian $\mathcal{H}$ on the gauge parameters λ_{ij} enters the model through the gauge parameters.

The matrix Schrödinger equation has the following form,

$$\mathcal{H}\mathcal{I}\,\Psi(\mathbf{X}) = E\,\Psi(\mathbf{X}), \tag{1.8}$$

where $\Psi(\mathbf{X})$ is a column wave function such that $\big(\psi_i(\mathbf{X})\big)$, $i = 1, \ldots, N$ and $\mathcal{I}$ is an $N \times N$ matrix with a one in all the entries. The ground state in the gauge $\lambda_{ij} = \nu_{ij}$, for all i and j is described by

$$\Psi_g(\mathbf{X}) = \frac{1}{\sqrt{N}}\, e^{-\frac{\omega}{2}\sum_{i=1}^{N} m_i x_i^2} \cdot C. \tag{1.9}$$

In (1.9), C is a column matrix all of whose elements are one.

One can introduce the left action of $\mathcal{H}$ on the row wave functions. When $\lambda_{ij} = \nu_{ij}$, the ground state can be represented as

$$\Psi_g^{\dagger}(\mathbf{X}) = \frac{1}{\sqrt{N}}\, R\, e^{-\frac{\omega}{2}\sum_{i=1}^{N} m_i x_i^2}, \tag{1.10}$$

It is the case that $R = C^T$ and $C^T C = N$, $C^T C = N$, $CC^T = \mathcal{I}$ with $\mathcal{I}\Psi_g = N\Psi_g$ and $\Psi_g^{\dagger}\Psi_g = 1$. It should be said that $\Psi_g(\mathbf{X})$ is not an eigenstate of $\mathcal{H}$ in the usual sense. It does satisfy a type of eigenvalue equation in which the eigenvalue E is a matrix. A well-defined meaning can be given to the equation after multiplying it on both sides from the left by $R = C^T$. Doing so reduces (1.8) to a simpler more standard eigenvalue equation $H\,\Psi = E\Psi$, where H is an operator which corresponds to the matrix operator $\mathcal{H}$ in the following way

$$H = C^{\dagger}\,\mathcal{H}C = \mathrm{tr}(\mathcal{H}\mathcal{I}). \tag{1.11}$$

When the particular gauge in which $\lambda_{ij} = \nu_{ij}$ is used, the more familiar eigenvalue equation $\hat{H}\psi = E\,\psi$ where $\hat{H}$ is a Hamiltonian for a multispecies Calogero model. In this event, $\hat{H}$ is obtained from $\mathcal{H}$ by means of (1.11)

$$\hat{H} = \mathrm{tr}(\mathcal{H}\mathcal{I}) =$$

$$-\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2}\sum_{i=1}^{N} m_i x_i^2 - \frac{1}{2}\sum_{i\neq j}\frac{\nu_{ij}}{x_i - x_j}\Big(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}\Big) = -T_- + \omega^2\, T_+. \tag{1.12}$$

The $T_{\pm}$ are $SU(1,1)$ generators. After performing the similarity transformation with the factor $\prod_{i<j}(x_i - x_j)^{-\nu_{ij}}$, a Hamiltonian for the multispecies Calogero model with inverse-square two and three-body interactions is obtained.

In a way similar to the introduction of the matrix Hamiltonian $\mathcal{H}$, a set of matrix generators with operator-valued matrix elements can be introduced as follows

$$\begin{aligned}\mathcal{T}_+ &= \frac{1}{2}\,\mathbf{X}\mathcal{M}\mathbf{X},\\ \mathcal{T}_- &= \frac{1}{2}\,\mathcal{D}\mathcal{M}^{-1}\mathcal{D},\\ \mathcal{T}_0 &= \frac{1}{4}\,(\mathbf{X}\mathcal{D}+\mathcal{D}\mathbf{X}) = \frac{1}{4}\,(\mathbf{X}\mathcal{D}+\mathbf{X}\mathcal{D}+\mathcal{V}) = \frac{1}{2}\mathcal{X}\mathcal{D}+\frac{1}{4}\mathcal{V}.\end{aligned} \tag{1.13}$$

Here we have $[\mathbf{X},\mathcal{D}] = -\mathcal{V}$ or $\mathcal{D}\mathbf{X} = \mathbf{X}\mathcal{D}+\mathcal{V}$.

The operators (1.13) satisfy the $\mathcal{I}$-commutation relations

$$R[\mathcal{T}_-,\mathcal{T}_+]_{\mathcal{I}}\mathbf{C} = 2R\,\mathcal{T}_0\,\mathbf{C}, \qquad R[\mathcal{T}_0,\mathcal{T}_\pm]_{\mathcal{I}}\mathbf{C} = \pm R\mathcal{T}_\pm\mathbf{C}. \tag{1.14}$$

The commutator $[,]_{\mathcal{I}}$ in (1.14) is defined as

$$[A,B]_{\mathcal{I}} = A\mathcal{I}B - B\mathcal{I}\,A. \tag{1.15}$$

The set of generators $\mathcal{T}_\pm$, $\mathcal{T}_0$ are related to the generators $T_\pm, T_0$ of a $SU(1,1)$ algebra in the following way

$$T_\pm = R\mathcal{T}_\pm C = \mathrm{tr}(\mathcal{T}_\pm\mathcal{I}), \qquad T_0 = R\mathcal{T}_0 C = \mathrm{tr}(\mathcal{T}_0\mathcal{I}). \tag{1.16}$$

The wavefunctions of the $\hat{H}$ Hamiltonian are related to the column wave functions $\Psi(\mathbf{X})$ which appear in (1.8) and given by

$$\psi(x_1,\ldots,x_N) = R\Psi(\mathbf{X}), \qquad \psi^*(x_1,\ldots,x_N) = \psi^\dagger(\mathbf{X})C. \tag{1.17}$$

The model described by Hamiltonian in (1.12) has solutions in the matrix formulation obtained in terms of the following pairs of creation and annihilation operators given as

$$\mathcal{A}_1^\pm = \frac{1}{\sqrt{2\mathrm{tr}\mathcal{M}}}\big(\sqrt{\omega}\mathbf{X}\mathcal{M} \mp \frac{1}{\sqrt{\omega}}\,D\big), \qquad \mathcal{A}_2^\pm = \frac{1}{2}\big(\omega\mathcal{T}_+ + \frac{1}{\omega}\,\mathcal{T}_-\big) \mp \mathcal{I}_0. \tag{1.18}$$

Note that the case in which all masses m_i are equal there is a simple relation between these two sets of operators

$$NR\mathcal{A}_1^{\pm\,2}C = R\mathcal{A}_2^\pm C. \tag{1.19}$$

The generators $\mathcal{T}_\pm$ are defined in (1.13) and play the role of collective radial variables corresponding to dilatation modes. The first pair of operators in (1.18) describes a center of mass modes, while the second pair describes collective radial modes. These operators satisfy the following commutation relations

$$R[\mathcal{A}_1^-,\mathcal{A}_1^+]_{\mathcal{I}}C = \mathbf{1}, \qquad R[\mathcal{A}_2^-,\mathcal{A}_2^+]_{\mathcal{I}}C = \frac{1}{\omega}\,R\mathcal{H}C,$$

$$R[\mathcal{A}_1^{\mp},\mathcal{A}_2^{\mp}]_{\mathcal{I}}C=0, \qquad R[\mathcal{A}_1^{\mp},\mathcal{A}_2^{\pm}]_{\mathcal{I}}=\pm R\mathcal{A}_1^{\pm}C, \tag{1.20}$$
$$R[\mathcal{H},\mathcal{A}_1^{\pm}]_{\mathcal{I}}C=\pm\omega\, R\mathcal{A}_1^{\pm}C, \qquad R[\mathcal{H},\mathcal{A}_2^{\pm}]_{\mathcal{I}}\,C=\pm 2\omega\, R\mathcal{A}_2^{\pm}\,C.$$

The partial matrix Fock space corresponding to CM modes and collective radial modes is spanned by states of the form,

$$||n_1,n_2\rangle=(\mathcal{A}_1^{+}\mathcal{I})^{n_1}(\mathcal{A}_2^{+}\mathcal{I})^{n_2}\,||0\rangle.$$

These are evolved by the matrix Schrödinger equation

$$\mathcal{H}\mathcal{I}||n_1,n_2\rangle=\mathcal{E}_{n_1n_2}||n_1,n_2\rangle.$$

Here $\mathcal{E}_{n_1,n_2}$ is the matrix that satisfies the relation

$$\frac{1}{\sqrt{N}}R\mathcal{E}_{n_1,n_2}||n_1,n_2\rangle=\frac{1}{\sqrt{N}}E_{n_1,n_2}R||n_1,n_2\rangle=E_{n_1,n_2}|n_1,n_2\rangle=(E_0+\omega(n_1+2n_2))\,|n_1,n_2\rangle.$$

where $E_0=\omega(N/2+1/2\sum_{i\neq j}\nu_{ij})$ is the energy of the ground state. The state $||n_1,n_2\rangle$ is the column state Fock and $||0\rangle$ is the vacuum state defined by (1.9), while E_{n_1,n_2} and $|n_1,n_2\rangle$ are eigenvalues and eigenvectors of the partial Fock space of the corresponding multispecies Calogero problem. The ground state is well defined if $E_0>1/2$.

Note that the correspondence between the matrix ladder operators (1.18) and the analogous operators that define the partial Fock space in the multispecies Calogero model is simply given by

$$A_1^{\pm}=R\mathcal{A}_1^{\pm}C=\mathrm{tr}(\mathcal{A}_1^{\pm}\mathcal{I}), \qquad A_2^{\pm}=R\mathcal{A}_1^{\pm}C=\mathrm{tr}(\mathcal{A}_2^{\pm}\mathcal{I}),$$

2 Quantum Hall Effect

Quantum levels of nonrelativistic electrons in a uniform magnetic field plays a role in the study of the quantum Hall effect. Electrons in the lowest Landau level exhibits interesting properties, such as the existence of incompressible fluid like states of condensed electrons whose excitations have fractional charge and obey fractional statistics. Such states appear only when the electron densities are certain rational fractions of the density corresponding to a fully filled Landau lowest Landau level and the gap in their excitation spectrum gives rise to the fractional Hall effect observed experimentally. Here a matrix oscillator model is introduced and a connection is made to the finite matrix model. From the matrix model, an equation of motion arising from the matrix model action is identified as the quatization condition imposed on the matrix coordinates of the electron. The matrices that solve the quantization condition can be found.

An action for the matrix oscillator described by the $N\times N$ matrices $\mathbf{X}$ and $\mathbf{P}$ is constructed which have operator valued matrix elements $(\mathbf{X_{ij}})^{\dagger}=\mathbf{X}_{ji}$,

$(\mathbf{P}_{ij})^{\dagger} = \mathbf{P}_{ji}$, $i, j = 1, \ldots, N$. The matrix $\mathbf{X}$ can be real and diagonal. The Hamiltonian is

$$\mathcal{H} = R\big(\frac{1}{2m}\mathbf{P}^2 + \frac{1}{2}m\omega^2\mathbf{X}^2\big)C,$$
$$[\mathbf{X}, \mathbf{P}] = i\mathcal{V}, \tag{2.1}$$
$$\mathcal{V} = (1-\nu)\mathbf{I} + \nu\mathcal{I}.$$

In (2.1) $R = (1, \ldots, 1)$ is a row vector with a one in each position, $C = R^T$ is the transpose of R and $RC = N$ and $CR = \mathcal{I}$ is the $N \times N$ matrix with one in all its positions. The matrix $\mathcal{V}$ is symmetric so $\mathcal{V}^T = \mathcal{V}$ and $\nu > -1/N$ is a real parameter, m the mass. Here $\mathcal{V}$ is a Hermitian matrix such that $\nu_{ij}^* = \nu_{ji}$ and $\nu_{ii} = 1$ for all i, j.

To describe two-dimensional systems of N charged particles with charge e in a magnetic field B, we introduce matrix $\mathbf{X}_1 = \mathbf{X}$ and a second matrix $\mathbf{X}_2$ defined in terms of $\mathbf{P}$ as

$$\mathbf{X}_2 = -\frac{1}{eB}\,\mathbf{P} = -\frac{1}{m\omega}\,\mathbf{P}, \tag{2.2}$$

where $\omega = eB/m$ and in accord with (2.1),

$$\mathrm{tr}\,[\mathbf{X}_1, \mathbf{X}_2] = -\frac{1}{eB}\mathrm{tr}\,[\mathbf{X}, \mathbf{P}] = -\frac{i}{eB}\,\mathrm{tr}\,\mathcal{V} = -\frac{N}{eB}.$$

The coordinates of the electrons can be globally parametrized by introducing $N \times N$ Hermitian matrices $\mathbf{X}_a$, $a = 1, 2$. The action leading to the quantum matrix oscillator is then given by the regularized finite matrix Chern-Simons model introduced by Polychronakos

$$S = \frac{1}{2}\,eB\int dt\,\mathrm{tr}\,\epsilon_{ab}\mathbf{X}_a(\dot{\mathbf{X}}_b - i[A_0, \mathbf{X}_b]) + 2\vartheta\,A_0\mathbf{1}) - \frac{e\omega BN}{2\bar{\psi}\psi}\int dt\,\bar{\psi}\mathbf{X}_a\mathbf{X}_a\psi$$
$$-\int dt\,\bar{\psi}(t\partial_t + A_0)\psi, \tag{2.3}$$

where $eB\vartheta = k$ and $\mathbf{A}_0$ is a matrix which enters only linearly into (2.3) with $\psi, \bar{\psi} = \psi^{*T}$ is a boundary field. Action (2.3) is invariant under the transformation $\mathbf{X}_a \to \mathbf{U}\mathbf{X}_a\mathbf{U}^{-1}$, $\psi \to \mathbf{U}\psi$, $\bar{\psi} \to \bar{\psi}\mathbf{U}^{-1}$ and $\mathbf{A}_0 \to \mathbf{U}\mathbf{A}_0\mathbf{U}^{-1} + i\,\mathbf{U}\partial_t\mathbf{U}^{-1}$, such that $\mathbf{U}$ is a unitary matrix $\mathbf{U} \in U(N)$. The term with ω is a potential term which acts to localize states and it provides a Hamiltonian with a unique ground state while the last term can be thought of as a boundary term.

The variation of the action S in the field variable $\mathbf{A}_0$ gives the equation of motion for the time component $\mathbf{A}_0$ of the gauge field which has the form

$$ieB[\mathbf{X}_0, \mathbf{X}_2] + k\mathbf{1} - \psi\bar{\psi} = 0. \tag{2.4}$$

This can be interpreted as a Gauss's law constraint. The constraint (2.4) can be seen as a quantization condition imposed on the matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ after which their matrix elements become operators. It can also be required that one of the $\mathbf{X}_a$ matrices , $\mathbf{X}_1$ for example, can be diagonalized and moreover,

$$ieB\mathrm{tr}\,[\mathbf{X}_1, \mathbf{X}_2] = -k\,\mathrm{tr}\mathbf{1} + \mathrm{tr}\,\bar{\psi}\psi = \mathrm{tr}\,\bar{\psi}\psi - N\,k,$$

which can be written,

$$\mathrm{tr}[\mathbf{X}_1, \mathbf{X}_2] = \frac{1}{ieB}\left(\mathrm{tr}\,\bar{\psi}\psi - N\,k\right) = \frac{N}{ieB}.$$

This implies that $\mathrm{tr}\,\psi\,\bar{\psi} = N(k+1)$.

Clearly certain quantization conditions ($k \in \mathbb{Z}$) can be imposed on the parameter k which may be justified by group theoretic considerations. The redundent number of degrees of freedom is reduced to effectively $2N$ phase space variables by means of the Gauss law constraint and $U(N)$ gauge symmetry. After diagonalizing $\mathbf{X}_1$ and solving the Gauss law constraint, from initially $2N^2$ degrees of freedom, there remain $2N$ degrees of freedom corresponding to N electrons.

It is remarkable to note that after the diagonalization of matrix $\mathbf{X}_1$, the quadratic potential becomes equal to the quantum matrix oscillator Hamiltonian (2.1). Explicitly, $\mathbf{U}\mathbf{X}_1\mathbf{U}^\dagger = \mathrm{diag}(x_1, \ldots, x_N)$ will tranform vector ψ into a vector $\varphi = \mathbf{U}\psi$ and $\mathbf{X}_2$ into $\mathbf{X}_2'$. After solving the quantization condition (2.1), using operator-valued matrix elements, X_2' is represented as

$$-ie\,B(\mathbf{X}_2')_{ij} = \big(\frac{\partial}{\partial x_j} + \sum_{k\neq j}\frac{\lambda_{jk}}{x_j - x_k}\big)\delta_{ij} = \frac{1-\delta_{ij}}{x_i - x_j}\varphi_i\bar{\varphi}_j, \tag{2.5}$$

The eigenvalues of $\mathbf{X}_1$ can be interpreted as the particle coordinates in the x_1 direction. The parameters λ_{jk}, $j, k = 1, \ldots, N$ are gauge parameters and φ is the vector which results after ψ is rotated by transformation $\mathbf{U}$. It is possible to work in the gauge where all gauge parameters λ_{ik} are equal to zero. Gauss's law becomes a deformed quantization condition the second equation of (2.1) that can be put in the form,

$$-eB[\mathbf{X}_1', \mathbf{X}_2'] = i\mathcal{V}', \qquad \mathcal{V}' = -k\mathbf{1} + \varphi\bar{\varphi}. \tag{2.6}$$

If one of the matrices $\mathbf{X}_1'$, $\mathbf{X}_2'$ in (2.6) is diagonal, as is the case here, then the consistency of the solution of the commutation requires the matrix $\mathcal{V}$ to have the form $\mathcal{V}' = -k\mathbf{1} + (k+1)\,\mathcal{I}$. The residual $U(1)$ gauge freedom can be used to choose the phase factors of φ_i such that $\varphi_i = \sqrt{k+1}$. This means matrix $\mathcal{V}$ is equal to the matrix $\mathcal{V} = (1-\nu)\mathbf{1} + \nu\mathcal{I}$ where $\nu = k+1$. In the classical limit where $\nu \to \infty$, $\mathrm{tr}[\mathbf{X}_1, \mathbf{X}_2] = 0$ and the diagonal elements in $\mathcal{V}$ are equal

to zero. Two interesting cases for k are $k=-1$ ($\nu=0$) corresponding to a Bose system and $k=0$ ($\nu=1$) a Fermi system.

Matrix operators are now introduced by setting

$$Q^{\pm}=\sqrt{\frac{m\omega}{2}}(X_1'\pm iX_2') \tag{2.7}$$

such that the following commutation relation holds

$$[Q^-,Q^+]=-k\mathbf{1}+\varphi\bar{\varphi}=(1-\nu)\mathbf{1}+\nu\mathcal{I}. \tag{2.8}$$

As the φ and $\bar{\varphi}$ fields are proportional to the R,C matrices

$$\varphi=\sqrt{k+1}\,C,\qquad \bar{\varphi}=\sqrt{k+1}\,R,$$

the Hamiltonian can now be expressed as in (1.1)

$$H=\frac{\omega}{2(k+1)}\bar{\varphi}[Q^-,Q^+]\varphi=\frac{1}{2}\,\omega\,R\,[Q^-,Q^+]\,C. \tag{2.9}$$

The ground state is a column vector $||0\rangle$ which is annihilated by the operator Q^-

$$A^-\mathcal{I}||0\rangle_\nu=Q^-||0\rangle_\nu=0,$$

and is given by

$$||0\rangle_\nu=K\prod_{i<j}(x_i-x_j)^\nu e^{-m\omega/2\sum_{i=1}^n x_i^2}. \tag{2.10}$$

The full Fock space is given by the following states

$$\prod_n(\mathrm{tr}(Q^{\dagger\,n}\mathcal{I}))^{m_n}||0\rangle_\nu=\prod_n\Big(\sum_i(a_i^\dagger)^n\Big)^{m_n}|0\rangle_\nu. \tag{2.11}$$

where,

$$(a_i^+)^n=(R\,Q^{\dagger\,n})_i,\qquad (a_i^-)^n=(Q^{\dagger\,n}C)_i,\quad i=1,2,\ldots,N. \tag{2.12}$$

The a_i^+, a_i^- are one-particle creation and annihilation operators for the Hamiltonian H. The corresponding energies are calculated by means of

$$E_{(m)}=E_0+\omega\sum_n n\,m_n, \tag{2.13}$$

such that

$$E_0=\frac{1}{2}\,\omega\big(N+\nu\,N(N+1)\big),\quad \nu\geq 0. \tag{2.14}$$

Note that this spectrum is the same as that following from the term $\mathrm{tr}(\mathbf{X}_a^2)$, because the corresponding Hamiltonian acts as a number operator up to the constant E_0 on the singlet part of the Fock space of states.

The structure of energy eigenstates can be studied in tahe Bargmann representation. Starting from the matrices $\mathbf{X}_1$ and $\mathbf{X}_2$, define the $Q^{\pm}_{\mathcal{B}}$ as the linear combinations $Q^{\pm}_{\mathcal{B}} = \sqrt{m\omega/2}(\mathbf{X}_1 \pm i\mathbf{X}_2)$, where $\mathcal{B}$ denotes the Bargmann representation. A transformation to this representation is given by

$$Q^{+}_{\mathcal{B}} = S^{-1}Q^{+}\,S, \qquad Q^{-}_{\mathcal{B}} = S^{-1}Q^{-}S, \qquad S = e^{-\omega T_+}\cdot e^{-\frac{i}{2\omega}T_-}, \tag{2.15}$$

The operators $T_{\pm}$ together with T_0 are the generators for the $SU(1,1)$ algebra and are given by

$$T_+ = \frac{m}{2}\,R\mathbf{X}'_1\,C, \quad T_- = -\frac{1}{2}m\omega^2 R\mathbf{X}'_2 C, \quad T_0 = -\frac{i}{4}\,m\omega\,R(\mathbf{X}'_1\mathbf{X}'_2 + \mathbf{X}'_2\mathbf{X}'_1)\,C. \tag{2.16}$$

The frequency ω is assumed to be different from zero. The same transformation connects the Hamiltonian $H = (\omega/2)\,R\{\mathbf{Q}^{-}\mathbf{Q}^{+}\}\,C$ and $H_{\mathcal{B}} = (\omega/2)\,R\{\mathbf{Q}^{-}_{\mathcal{B}}, \mathbf{Q}^{+}_{\mathcal{B}}\}\,C$ with their corresponding ground states

$$H = SH_{\mathcal{B}}S^{-1} = 2\omega\,ST_0S^{-1}, \quad |0\rangle_\nu = S|0\rangle^{\mathcal{B}}_\nu. \tag{2.17}$$

The commutation relations satisfied by operators $\mathbf{Q}^{-}_{\mathcal{B}}$ and $\mathbf{Q}^{+}_{\mathcal{B}}$ are unchanged

$$[\mathbf{Q}^{-}_{\mathcal{B}}, \mathbf{Q}^{+}_{\mathcal{B}}] = -k\mathbf{1} + \varphi\bar{\varphi} = (1-\nu)\mathbf{1} + \nu\mathcal{I}.$$

Up to a factor of 2ω, relation (2.17) is satisfied by T_0, which when written explicitly in the form $(1/2)(\sum_i x_i\partial/\partial x_i + N/2)$, we infer the Hamiltonian in the Bargmann representation is exactly operator $2\omega\mathbf{T}_0$.

The Fock space for H_0 can be constructed by applying the creation operators $(a^{+}_i)^n_{\mathcal{B}}(R\mathbf{Q}^{+n}_{\mathcal{B}})_i$ to the vacuum state

$$|0\rangle^{\mathcal{B}}_\nu = \prod_{i<j}(x_i - x_j)^\nu,$$

which is annihilated by

$$\partial_i|0\rangle^{\mathcal{B}}_\nu = \Big(\frac{\partial}{\partial x_i} - \nu\sum_{l\neq i}\big(\frac{1}{x_i - x_l}\big)\Big)\prod_{j<k}(x_j - x_k)^\nu = 0.$$

The operators x_i, ∂_i satisfy the commutation relations $[\partial_i, x_j] = \delta_{ij}$, $[\partial_i, \partial_j] = 0$ $i, j = 1, \ldots, N$. Hamiltonian $H_{\mathcal{B}}$ takes the form

$$H_{\mathcal{B}} = E_0 + \omega\sum_i x_i\partial_i, \qquad [H_0, x_i] = x_i, \quad [H_B, \partial_i] = -\partial_i, \tag{2.18}$$

which allows x_i, ∂_i to be thought of as a pair of creation and annihilation operators for $H_{\mathcal{B}}$. Only totally symmetric combinations of these operators

have physical import. The true Fock space for $H_{\mathcal{B}}$ is constructed by applying the operators

$$B_n^+ = \sum_i x_i^n, \qquad B_n^- = \sum_i \partial_i^n \tag{2.19}$$

to the vacuum $|0\rangle_\nu^{\mathcal{B}}$.

The $SU(N)$ invariant ground state vacuum in the Bargmann picture for a fixed ν is

$$|0\rangle_\nu^{\mathcal{B}} \equiv (\epsilon_{i_0\cdots i_{N-1}} \prod_{k=0}^{N-1} (a_{i_n}^+)_{\mathcal{B}}^k)^\nu |0\rangle_0^{\mathcal{B}},$$

where $(a_{i_k}^+)_{\mathcal{B}}^k = (R\mathbf{Q}_{\mathcal{B}}^{+k})_{i_k}$, $(a_{i_k}^-)_{\mathcal{B}}^k = (\mathbf{Q}_{\mathcal{B}}^{+k}\, C)_{i_k}$ $i_k = 1, \ldots, N$ where $(a_{i_k}^+)_{\mathcal{B}}$, $(a_{i_k}^-)_{\mathcal{B}}$ the one-particle creation and annihilation operators for $H_{\mathcal{B}}$. Knowing the transformation from $H_{\mathcal{B}}$ to H implies knowing the transformation between the corresponding ladder operators

$$S \sum_i (a_i^\pm)_{\mathcal{B}}^n \, S^{-1} = \sum_i (a_i^\pm)^n.$$

In this representaion, the expression for the ground state takes the form

$$||0\rangle_\nu^{\mathcal{B}} = S^{-1} \, ||0\rangle_\nu = K \prod_{i<j} (x_i - x_j)^\nu \tag{2.20}$$

with $\mathbf{Q}_{\mathcal{B}}^- ||0\rangle_\nu^{\mathcal{B}} = 0$.

The lowest state in a given tower with fixed ν is just a Laughlin wave function. For $\nu = 0$ ($k = -1$), both operators $\mathbf{Q}^\pm$ are diagonal and the system is equivalent to N ordinary one-dimensional harmonic oscillators. The Laughlin wavefunction exponent $\nu = k+1$ is an integer and if ν is even behaves as a Bose system and for ν odd it behaves as a Fermi. This transformation to the Bargmann picture allows us to eliminate the gauge degrees of freedom reducing the finite Chern-Simons matrix model to an N variable quantum model with wavefunction of the Laughlin form. This kind of picture is quite interesting and may be useful as well. In spite of the localization of states there may exist extended states composed of Landau level states which are correlated over lengths characterized by the sample size. This may result in a contribution to the effect that resembles or is like a standing wave anchored at the boundary. These could play a significant role in giving the observations seen, such as when the magnetic field through the sample is varied.

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