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Coulomb Field and Potential in LTB Metric via NP Formalism

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Abstract

The electromagnetic field equation for a fixed charged point is considered in LTB space time. The electromagnetic spin 1 spinor field equation is studied by assuming a spherically symmetric current source. A solution is obtained by using the Newman Penrose formalism together with a suitable angular factorization assumption of the spinor field. The result is translated into the the coordinate formalism. Accordingly one obtains qualitatively expected results for both electromagnetic field and potential that reduce to the standard ones in Minkowski space time. Consistency of the integration procedure requires a constraint on one of the metric tensor coefficients. The condition is satisfied for a still wide subclass of LTB cosmological models. In the Robertson-Walker metric the constraint condition is automatically satisfied and the results essentially agree with existing ones.

Keywords: Coulomb field and potentilal, LTB space time, LTB Cosmology, RW space time

1 Introduction

The cosmological observation of the spectrum of Hydrogen atom is of interest. It furnishes information about Hydrogen distribution in the Universe. Deviations of the observed spectrum from the canonical one also give information

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about gravity in the region of the emitting source and its velocity and finally on the Universe expansion.

From an elementary theoretical point of view, previsions can be done by studying the Dirac equation with Coulomb potential in a general curved space-time. It is therefore of interest the expression of the Coulomb potential entering the Dirac equation. Such expression can be derived by a perturbative study in general space time (e. g., [8, 9]) as well as by a direct evaluation from Maxwell equations in explicit space time models. (For solutions in RW space one can see e. g.,[2, 7]; in Anti deSitter space time see e. g., [5]).

In the present paper attention is towards the calculation of Coulomb field and potential in a general spherically symmetric co-moving space time, a context where, as far as the author knows, the problem has not explicitly considered in the literature. To that end the preliminary assumption is the definition of a current vector associated to a fixed charged point. To determine the Coulomb potential it seems better to deal with first order equation and to determine the electromagnetic field. Accordingly one has to solve a spin 1 spinor filed equation with the electromagnetic current as a source. The equation is formulated by the Newman Penrose formalism based on a previously considered null tetrad frame. The solution, that is determined by a suitable coordinate dependence assumption of the spinor field, is translated into the coordinate formalism. The electromagnetic tensor and the four potential vector are then determined.

The procedure applied to obtain the results requires a consistency condition that is not fulfilled by a general LTB metric tensor. Moreover the vector potential does not in general verify the Lorentz condition. Both conditions are however satisfied in a general RW space time and in a still wide class of LTB cosmological models of physical interest. Once specialized to the Minkowski space time the results agree with well known physical laws. When specialized to the RW space time they are similar, but not completely identical to existing results (e. g., [7, 2, 3]) that, however, are not all identical among themselves.

2 Definitions and Assumptions

The study is developed within a general spherically symmetric co-moving space time whose metric tensor $g_{\mu\nu}$ (here called LTB metric) is defined by [4]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - e^{\Gamma(r,t)}dr^2 - Y^2(r,t)\Big(d\theta^2 + \sin\theta^2 d\varphi^2\Big) \tag{1}$$

In the calculation, the Newmn Penrose [6] formalism is adopted based on the previously defined null tetrad frame $\{l^i, n^i, m^i, m^{*i}\}$ [12]:

$$l^i = \frac{1}{\sqrt{2}}(1, e^{-\Gamma/2}, 0, 0), \qquad n^i = \frac{1}{\sqrt{2}}(1, -e^{-\Gamma/2}, 0, 0), \tag{2}$$

$$m^i = \frac{1}{Y\sqrt{2}}(0,0,1,i\csc\theta), \qquad m^{*i} = \frac{1}{Y\sqrt{2}}(0,0,1,-i\csc\theta) \tag{3}$$

whose corresponding non null spin coefficients are:

$$\tag{4}$$

$$\rho = -\frac{1}{Y\sqrt{2}}(\dot{Y} + Y'e^{-\Gamma/2}), \qquad \mu = \frac{1}{Y\sqrt{2}}(\dot{Y} - Y'e^{-\Gamma/2}), \tag{5}$$

$$\beta = -\alpha = \frac{1}{2Y\sqrt{2}}\cot\theta, \qquad \epsilon = -\gamma = \frac{1}{4\sqrt{2}}\dot{\Gamma} \tag{6}$$

As usual the directional derivatives will be denoted by $D = l^i\partial_i,\ \Delta = n^i\partial_i,\ \delta = m^i\partial_i,\ (\delta^*)^i = (m^*)^i\partial_i$.

In the following we are concerned with the determination of the electromagnetic field of a point of charge e_o located in the origin of coordinates. Correspondingly, the associated conserved current vector is assumed to be [11]:

$$j^\mu = (e_o g^{-\frac{1}{2}}\delta_3(x), 0, 0, 0) \tag{7}$$

g the determinant of the metric tensor $g_{\mu\nu}$. On account of spherical space time symmetry, it is possible to define

$$J^\mu = \int d\Omega\, j^\mu \equiv \Big(e_o\frac{e^{-\Gamma/2}}{Y^2}\delta(r), 0, 0, 0\Big) \tag{8}$$

that is conserved, $\nabla_\mu J^\mu = 0$. J^μ will be assumed as the electromagnetic current vector relative to the charge e_o. The current spinor $J_{AA'}$ corresponding to J^μ is then

$$J_{AA'} = \sigma^\alpha_{AA'}J_\alpha = \frac{J_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_0 = e_o\frac{e^{-\Gamma/2}}{Y^2}\,\delta(r) \tag{9}$$

where the spin matrices $\sigma^\alpha_{AA'}$ readily follows from eqs. (2), (3). (e. g., [10, 1]).

3 Coulomb electromagnetic spinor field

It is better to first solve the equation for the electromagnetic tensor field. To that end the Newman Penrose formalism is adopted. The equation to solve is then (e. g., [10]):

$$\nabla_{BA'}\phi_A{}^B = 2\pi J_{AA'} \tag{10}$$

$\phi_{AB} = \phi_{BA}$ the electromagnetic spinor field, $J_{AA'}$ the spinor current (9). By making explicit the covariant spinor derivatives in terms of directional derivatives and spin coefficients according to the assumptions of the previous Section, the equation (10) gives:

$$(D-2\rho)\phi_1-(\delta^*-2\alpha)\phi_0=-e_o\frac{e^{-\Gamma/2}}{Y^2}\,\delta(r) \tag{11}$$

$$(D-\rho+2\epsilon)\phi_2=\delta^*\phi_1 \tag{12}$$

$$(\Delta+\mu-2\gamma)\phi_0=\delta\phi_1 \tag{13}$$

$$(\delta+2\beta)\phi_2-(\Delta+2\mu)\phi_1=-e_o\frac{e^{-\Gamma/2}}{Y^2}\,\delta(r) \tag{14}$$

It has been set $\phi_{00}=\phi_0,\ \phi_{11}=\phi_2,\ \phi_{01}=\phi_{10}=\phi_1$.

One can try to separate the φ dependence in the form $\phi_k(t,r,\theta,\varphi)=\phi_k(t,r,\theta)\exp(im_k\varphi)$, $m_k\in\mathbf{R}$, $k=0,1,2$. However, from (9)-(12), this is possible only for $m_k=0$. Therefore we look for solution φ independent and of the form

$$\phi_k=\psi_k(t,r)S_k(\theta),\quad k=0,1,2 \tag{15}$$

$$S_1(\theta)=const.=1 \tag{16}$$

$$\phi_0=\phi_2 \tag{17}$$

Accordingly one has that $\delta^*\phi_1=\psi_1\delta S_1=0$ while $\phi_0=\phi_2$ satisfy the equations:

$$(D-\rho+2\epsilon)\psi_0=0,\qquad(\Delta+\mu-2\gamma)\psi_0=0 \tag{18}$$

From the very definition of D, Δ, ρ, μ, ϵ, by summing and subtracting the equations (18) one if left, respectively, with the equations ($\dot X=\partial_tX$, $X'=\partial_rX$)

$$\dot\psi_0+\frac{\dot Y}{Y}\psi_0+\frac{1}{2}\dot\Gamma\psi_0=0,\qquad\psi_0'+\frac{Y'}{Y}\psi_0=0 \tag{19}$$

Solutions of the first (19) should be also solution of the second (19). Hence

$$\psi_0=\frac{e^{-\Gamma/2}F(r)}{Y}\quad\wedge\quad e^{-\Gamma/2}F(r)=G(t) \tag{20}$$

$F(r),G(t)$ arbitrary integration functions. This restricts the class of LTB metrics to which the present integration method applies because it requires $\exp\Gamma$ to be a function of t times a function of r. As to $S_0(\theta)$ one can require (see (11), (14) and assumption (17)):

$$(\delta^*-2\alpha)\phi_0=(\delta+2\beta)\phi_0\equiv\frac{\psi_0}{Y\sqrt{2}}(\partial_\theta+\cot\theta)S_0=0 \tag{21}$$

After integration one finally obtains:

$$\phi_0=\psi_0S_0(\theta)=\frac{c_0}{\sin\theta}\frac{e^{\Gamma/2}F(r)}{Y} \tag{22}$$

where c_0 is constant of integration.
By taking into account the last results and the assumptions one can now determine the ψ_1 solution. By subtracting and summing (11) and (14), and one is left, respectively, with the equations:

$$[D + \Delta + 2(\mu - \rho)]\psi_1 = 0 \tag{23}$$

$$[D - \Delta - 2(\mu + \rho)]\psi_1 = -2\pi e_o \frac{e^{-\Gamma/2}}{Y^2}\delta(r) \tag{24}$$

Then, by using the expressions of the directional derivatives and spin coefficients:

$$\dot{\psi}_1 + 2\frac{\dot{Y}}{Y}\psi_1 = 0 \tag{25}$$

$$\psi_1' + 2\frac{Y'}{Y}\psi_1 = \sqrt{2}\pi e_o \frac{e^{-\Gamma/2}}{Y^2}\delta(r) \tag{26}$$

whose solutions have to have both the forms:

$$\psi_1 = -\frac{F(r)}{Y^2}, \qquad \psi_1 = \frac{1}{Y^2}(-\sqrt{2}\pi e_o + c_1) \tag{27}$$

c_1, $F(r)$ integration constants. By a suitable choice of the constants one has finally:

$$\phi_1 = \psi_1 = -\frac{e_o}{Y^2}. \tag{28}$$

4 Coulomb field and potential

By the standard correspondence between complex tensor of rank n and spinor of type (n, n) one can obtain the expression of the electromagnetic tensor field F_{ab} from the e. m. spinor ϕ_{AB} [1, 10]:

$$F_{ab} = \sigma_a^{AX'}\sigma_b^{BY'}(\phi_{AB}\epsilon_{X'Y'} + \epsilon_{AB}\phi_{X'Y'}) \tag{29}$$

$$= \Big\{\frac{1}{2}g_{aa}g_{bb}[\phi_0(n^b m^{*a} - n^a m^{*b}) + \phi_2(l^a m^b - m^a l^b) + +\phi_1(n^a l^b - m^{*a}m^b + m^{*b}m^a - l^a n^b)]\Big\} + \Big\{C.C.\Big\} \tag{30}$$

$\sigma_a^{AX'}$ the spin matrices relative to the null tetrad frame (2), (3). An explicit calculation by the expressions of ϕ_1, $\phi_0 = \phi_2$ obtained in the previous Section and by the null tetrad frame (2), (3) give then:

$$F^{ab} = e_o \frac{e^{-\Gamma/2}}{Y^2}(\delta^{at}\delta^{br} - \delta^{bt}\delta^{ar}) \tag{31}$$

therefore, as expected, the electric field E_i has only the radial component $E_r = e_o e^{-\Gamma/2}/Y^2$, $E_\theta = E_\varphi = 0$, while for the magnetic field $H_i = 0$, $i = r, \theta, \varphi$. As to the 4-potential vector A_μ, it satisfies $\nabla_a A_b - \nabla_b A_a = F_{ba}$. A solution is

$$A^\mu = \delta^{\mu t} \int F^{01} dr \, = e_o \delta^{\mu t} \int \frac{e^{-\Gamma/2}}{Y^2} dr \tag{32}$$

The results can be specialized to the general Robertson Walker metric. The constraint (20) is satisfied because the coefficient $e^{\Gamma/2}$ has a factorized dependence on t and r . Therefore in RW space time, by suitably choosing the integration functions and constants, one has:

$$e^\Gamma = R^2(t)(1-\kappa r^2)^{-1}, \quad \kappa = 0, \pm 1 \tag{33}$$
$$Y = rR(t) \tag{34}$$
$$\phi_0 \equiv \phi_2 = \frac{1}{rR^2(t)} \, \frac{1}{\sin\theta} \tag{35}$$
$$\phi_1 = -\frac{e_0}{r^2 R^2(t)} \tag{36}$$
$$F^{ab} = e_0 \frac{\sqrt{1-\kappa r^2}}{r^2 R^3} \left(\delta^{at}\delta^{br} - \delta^{bt}\delta^{ar}\right) \tag{37}$$
$$A^\mu = e_o \delta^{\mu t} \int \frac{\sqrt{1-\kappa r^2}}{r^2 R^3} dr \tag{38}$$
$$= -e_o \delta^{\mu t} \, \frac{1}{rR^3}, \quad \kappa = 0 \tag{39}$$
$$= -e_o \delta^{\mu t} \frac{1}{R^3} \left[\frac{\sqrt{1+r^2}}{r} - \log(r + \sqrt{1+r^2}) \right], \quad \kappa = -1 \tag{40}$$
$$= -e_o \delta^{\mu t} \, \frac{1}{R^3} \left[\frac{\sqrt{1-r^2}}{r} + \sin^{-1} r \right], \quad \kappa = 1 \tag{41}$$
$$\nabla_\mu A^\mu = g^{-1/2} \partial_\mu (g^{1/2} A^\mu) = 0, \qquad \kappa = 0, \pm 1 \tag{42}$$

The validity of the Lorentz gauge condition (42), does not hold for a general LTB metric. (For further comments see the last Section).

The results reduce to the standard ones in Minkowski space time. One can note here the new special behavior of the potential for large r and $\kappa = -1$. The form of both electric and potential Coulomb field in RW space time obtained here are in a qualitative agreement with other existing results (see, e.g., [7, 2, 3]). One can see, however, that the radial component of the electric field (37) has similar radial coordinate dependence expression, but slightly different $R(t)$ dependence, with respect to those of [7, 2, 3], that in turns differ among themselves.

5 Remarks and comments

The results of the previous Sections are essentially obtained under a generalized spherical symmetry assumption for the e. m. current generated by a fixed charged point. The solutions of the equations are looked for under the assumption (15-17) in the context of general LTB space time. In this regards it remains open the problem of the existence of solutions with not so a special φ-dependence. More generally to knowledge of explicit non factorized solutions would be of interest.

It seems useful to note that the results obtained above hold in LTB space times more general than the RW ones. More precisely one can see that both the consistency condition in (20) as well as the validity of the Lorentz condition (42) do hold for a still large class of LTB cosmological models. To show this, one can recalls that the LTB cosmological model is solution of the Einstein field equationin the metric (1) for a Universe filled with dust matter without pressure. The model can be equivalently described by the equations [4]:

$$e^{\Gamma} = \frac{Y'^2}{1+2E}, \qquad \frac{\dot{Y}^2}{2} - \frac{M}{Y} = E, \qquad M = 4\pi G \int_0^r d(t,r)\, Y^2 Y' dr \tag{43}$$

$E(r), M(r)$ are arbitrary integration functions of the cosmological equations; $d(t,r)$ the density of the dust matter.The scheme (43) can be solved in general in parametric form [4]. If one assumes

$$(GM)^2 = (2|E|)^3, \tag{44}$$

the form of the general parametric solution of (39) specializes to:

$$Y = (9/2)^{1/3}\, M)^{1/3}\, t)^{2/3}, \qquad E = 0, \tag{45}$$

$$Y = \sqrt{2|E|}\; (1-\cos\eta), \quad t = \eta - \sin\eta, \quad 0 \le \eta \le 2\pi, \quad E < 0 \tag{46}$$

$$Y = \sqrt{2E}\,(\cosh\eta - 1), \quad t = \sinh\eta - \eta, \quad \eta > 0, \quad E > 0 \tag{47}$$

Since for both $E > 0$ and $E < 0$ the function $t = t(\eta)$ is an invertible function, Y is of the form $Y = y(r) f(t)$ in all E cases. Therefore, from the first (43) and (44), $\exp\Gamma$ results to be a function of t times a function of r as it is required in (20). As to the Lorentz gauge condition, its validity follows as in (42) because the expression $g^{1/2} A^{\mu}$ results to be time independent for $Y = y(r) f(t)$ and taking into account (32).

Finally one can note that condition (44) is also a sufficient condition for the separability of the Dirac equation in LTB cosmology [14]. By that condition the LTB cosmological model depends on one only arbitrary integration function. The model is therefore still suitable to describe a wide class of gravitational situations. It seems therefore of interest the study of the spectrum of the Hydrogen atom in such LTB cosmological models as well as in Standard Cosmology in analogy to what done in [13].

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