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Considering Measurement Dynamics in Classical Mechanics

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Abstract

In this paper, we consider modeling measurement as part of the overall system dynamics in a way that allows a dynamic interaction between the target system and the measuring system/device. For example, the interaction between the target system and the measuring system can be modeled as a form of collision, scattering process, or some coupling between the two systems depending on how they interact with each other. We show that following this line of thinking about the two systems as intertwined/coupled, in the classical picture, naturally leads to the general dynamics analogous to Schrodinger's wave dynamics.

Keywords: Interaction, Collision, Measurement, Wave function, Phase, Trajectory, Schrodinger, Hamilton-Jacobi Equation, Potential, Lagrangian, Action

1 Introduction

Measurement is central to all of Physics because it is the standard by which theories are evaluated and verified. In formulating theories and deriving laws of nature in Physics, most of the effort is focused on the dynamics of the target system, while very little is said about the role played by measurement dynamics. This is not as big an issue in classical mechanics as it is in quantum mechanics, because most classical systems of interest can interact with a measurement device without suffering significant disruption in their dynamics caused by the interaction process. A large number of problems (philosophical or interpretational) in quantum mechanics can be associated with the so-called measurement problem [1, 2, 3, 4, 5]. It is widely accepted that measurement plays a crucial role in quantum mechanics and this issue has given rise to several different interpretations of the theory, which include among others, the Copenhagen [6], the Everettian [6, 7], the de Broglie-Bohm [6, 7, 8, 9, 4, 10], and the Quantum Bayesian interpretations [6, 11, 12, 13, 14, 15]. An important reason for the differences between approaches lies in how the wave function is interpreted; whether it is thought as just a helpful Mathematical construct with no physical correspondence or if it is viewed as a representation of some physical quantity in nature [16]. Moreover, it is also debated whether the wave function constitutes a full description of the whole system or whether there are some hidden variables that are crucial in the description of the system [17, 18, 19, 20, 21, 22].

In this paper, we attempt to show that it is possible to start from classical mechanics and arrive at something analogous to the Schrodinger's equation by simply including the measurement dynamics as part of the overall system dynamics. In section 2 we discuss an interaction problem by considering a two-particle interaction model with one particle acting as the measuring device while the other particle acts as the target system. Section 3 presents the two-particle interaction problem in the Hamilton-Jacobi formalism by first proposing a complex action from which two coupled Hamilton-Jacobi equations are derived. We explain how these two coupled equations relate to the interaction or coupling between the measuring device and target system. Section 4 presents some obvious symmetries (and their physical meaning) that follow from the proposal of a complex action. Section 5 applies the dynamics derived in the previous section to some well-known classical problems of two particles interacting under an external uniform field. In section 6 we use the Madelung transformation to show that Schrodinger's equation results into two coupled equations which are similar in form with our coupled Hamilton-Jacobi equations derived from a classical complex action. It is shown in this section that there is a probability density function which is conserved and its physical meaning is inferred from the phase-trajectory relationship inherent in the Hamilton-Jacobi formalism. In section 7, through an example, we show how our coupled Hamilton-Jacobi equations reduce to Newtonian dynamics. There we show how the Newtonian picture follows from one of the two equations and by ignoring the second equation associated with the measuring device. Section 8 concludes this paper and proposes a possible path by which gravity and quantum mechanics could be unified through the consideration of a complex Einstein-Hamilton action.

2 An Interaction Problem

2.1 Motivation From Drude Model

Drude model is a model of electrical conduction which treats electrons as a gas or cloud and applies the kinetic theory of gases on this cloud of electrons [23]. In this section we adapt some of the basics of the Drude model with the aim of introducing a general interaction between two particles. Although Drude model considers the interactions that are primarily due to collisions between two particles, we will later consider the general interaction between two particles.

Consider a particle (among many) with momentum $\mathbf{p}_1(t)$ at some time t under the influence of some force $\mathbf{F}_{g1}$, which we will call the guiding force. At time $t+dt$ it will, with probability dt/τ_1 (with τ_1 as the mean time between collisions), collide with some other particle thus scattering to $\mathbf{p}_{c1}(t+dt)$. At the same time, the probability of no collision will be $1 - dt/\tau_1$. If it does not collide then it can accelerate based on the applied guiding force $\mathbf{F}_{g1}$, with the usual equation of motion $\dot{\mathbf{p}}_1(t) = \mathbf{F}_{g1}$ or $\mathbf{p}_1(t+dt) = \mathbf{p}_1(t) + \mathbf{F}_{g1}dt$. Now we can combine these two terms and weigh them with their probabilities to give an expectation value for $\mathbf{p}_1(t+dt)$ as,

$$\begin{aligned}
\mathbf{p}_1(t+dt) &= (\mathbf{p}_1(t) + \mathbf{F}_{g1}dt)(1 - \frac{dt}{\tau_1}) + \mathbf{p}_{c1}(t+dt)\frac{dt}{\tau_1} && \text{(1a)}\\
&\Downarrow \\
d\mathbf{p}_1(t) &= \mathbf{F}_{g1}dt(1 - \frac{dt}{\tau_1}) + (\mathbf{p}_{c1}(t+dt) - \mathbf{p}_1(t))\frac{dt}{\tau_1} && \text{(1b)}\\
&\Downarrow \\
\frac{d\mathbf{p}_1}{dt} &= \mathbf{F}_{g1}(1 - \frac{dt}{\tau_1}) + \mathbf{F}_{c1}\frac{dt}{\tau_1} && \text{(1c)}
\end{aligned}$$

This is the basis for the derivation of the Drude theory [23, 24]. From equation (1c) we have two terms on the right-hand side; the first being the guiding force term $\mathbf{F}_{g1}$ while the second one is what we can loosely call the coupling or interaction force $\mathbf{F}_{c1} = (\mathbf{p}_{c1}(t+dt) - \mathbf{p}_1(t))/dt = d\mathbf{p}_{c1}/dt$. We can now turn our attention to the second particle which collided with the first one above.

Consider that this second particle was also under the influence of some guiding force $\mathbf{F}_{g2}$ and had momentum $\mathbf{p}_2(t)$ at time t just before the collision. We can derive an equation, similar to equation (1c) above, for the dynamics of this second particle,

$$\frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{g2}(1 - \frac{dt}{\tau_2}) + \mathbf{F}_{c2}\frac{dt}{\tau_2} \tag{2}$$

Similar to equation (1c), we can associate two potentials with the two forces on the right-hand side of the equation (2). The interaction between two colliding particles is associated with the two momentum transfer terms, $\mathbf{F}_{c1}$ and $\mathbf{F}_{c2}$ which should be equal in magnitude and have opposite signs if there is no other particle involved in the collision. The limitations of the two equations (1c, 2) are that the guiding force is assumed to be off or insignificant during the collision which is not necessarily the case in general. The equations are also based on probabilities but they serve as a good motivation for the two-particle dynamics we wish to model in this paper.

The collision-based interaction (or coupling) forces $\mathbf{F}_{c1}$ and $\mathbf{F}_{c2}$ are non-conservative and as such we cannot associate with each force a position-dependent potential. It is worth noting however, that some non-conservative forces such as those with velocity dependence, can be associated with velocity-dependent potentials using Rayleigh dissipation function [25, 26]. In general, not all interaction forces are non-conservative as there are also conservative interaction forces such as gravitational force and Coulomb force which can couple two particles. In the rest of this paper we will limit ourselves to the case in which the guiding force and the interaction force can each be associated with some kind of potential.

2.2 Two-Particle Interaction Dynamics

In a general setting, the guiding force and the interaction force should be added as vectors (not as a statistically weighted vector sum) to give us the following set of more general equations,

$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{g1} + \mathbf{F}_{c1} \tag{3a}$$

$$\frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{g2} + \mathbf{F}_{c2} \tag{3b}$$

The coupling forces $\mathbf{F}_{c1}$ and $\mathbf{F}_{c2}$ are responsible for momentum transfer between the two particles. In a closed system of two particles momentum is conserved between the two particles thus we have $\mathbf{F}_{c1} + \mathbf{F}_{c2} = \mathbf{0}$. In an open system momentum is conserved among the two particles and their environment thus we have $\mathbf{F}_{c1} + \mathbf{F}_{c2} + \mathbf{F}_{ce} = \mathbf{0}$, with $\mathbf{F}_{ce}$ as the coupling force between the particles and their environment or a third particle [27, 28].

2.3 Relation To Measurement Model

Let us consider that we don't have direct access to the dynamics of the first particle (i.e. the target system) but we can interact with the first particle indirectly through the second particle (i.e. the measuring system), whose dynamics are known to us. The information about the first particle reported to us by the second particle is only available as the effects of a coupling or collision with the second particle. This is the model of measurement we are considering in this paper. That is, we are using the second particle as a means/device to measure some physical quantity associated with the first particle. In a generalized measurement model the interaction between the two particles/systems isn't just restricted to collision but can be any interaction such as electrical, magnetic, gravitational, etc. This is in the same spirit as the thought experiment proposed by Einstein, Podolsky and Rosen, although in a different context [29]. In the next section, we will adapt the Hamilton-Jacobi formalism to represent this set of coupled dynamics.

3 Hamilton-Jacobi Formalism

3.1 A Complex Action Proposal

Associated with each force term on the right-hand side of each of the two equations in the equation set (3a - 3b) is some potential. This gives us a total of four potentials denoted as, V_{g1}, V_{g2}, V_{c1} and V_{c2} respectively with the coupling potentials V_{c1} and V_{c2} being related to one another. The Lagrangians associated with the two equations will then be,

$$\mathcal{L}_1 = \frac{\mathbf{p}_1^2}{2m_1} - V_{g1} - V_{c1} \tag{4a}$$

$$\mathcal{L}_2 = \frac{\mathbf{p}_2^2}{2m_2} - V_{g2} - V_{c2} \tag{4b}$$

The Lagrangian is a function of position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and time t. In classical mechanics we would add equation (4a) and equation (4b) to get an overall Lagrangian. This way of arriving at an overall Lagrangian assumes that the particles are not necessarily coupled with one another. In an attempt to describe dissipative systems (i.e. systems coupled to their environment like a resistor thermally coupling to its environment), several forms of what can be thought of as non-standard Lagrangians have been proposed. In [30, 31] an $n-$dimensional dissipative system is modeled by introducing additional $n-$dimensional space (as a dual space adding more degrees of freedom) and the Lagrangian spans both spaces.

On the other hand, in [30] Bateman proposed that the Lagrangian for non-dissipative system be multiplied by a time-dependent factor $\exp(\Gamma t)$ that counters the decay rate $\exp(-\Gamma t)$ in order to get the corresponding Lagrangian for a dissipative system. In [32] Dekker proposed the use of complex variables to describe the Lagrangian for a dissipative system. All these three approaches lead to the same Euler-Lagrange dynamics. In this paper we adapt the approach of a complex Lagrangian to couple two systems. That is, rather than aggregate the Lagrangians in equations (4a - 4b) above as a simple sum (as done for decoupled particles), we propose to represent them as component of a complex Lagrangian $\mathcal{L}$ as follows,

$$\mathcal{L} = \mathcal{L}_1 + i\mathcal{L}_2 \tag{5}$$

The effect of a complex Lagrangian of this kind is that the resulting phase will have two components; one oscillatory (i.e. imaginary part) and another dissipative (i.e. real part). It is the dissipation (i.e. from one particle to another) that couples the two systems. The principle of stationary action can then be applied to this complex Lagrangian in equation (5) to obtain the full dynamics for both the target system (first particle) and the measuring system (second particle) which also include their coupling.

3.2 Coupled Hamilton-Jacobi Dynamics

The Hamilton-Jacobi equation of classical mechanics can be recast in terms of a density and a phase [33], in effect relating the particle trajectory with a wave. It follows from having a complex Lagrangian in equation (5) that the corresponding action will also be complex and given by [19],

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}dt = \int_{t_1}^{t_2} (\mathcal{L}_1 + i\mathcal{L}_2)dt = \mathcal{S}_1 + i\mathcal{S}_2 \tag{6}$$

Note that the action $\mathcal{S}$ is related to the momentum $\mathbf{p}$ as follows,

$$\mathbf{p} = \nabla\mathcal{S} = \mathbf{p}_1 + i\mathbf{p}_2 = \nabla\mathcal{S}_1 + i\nabla\mathcal{S}_2 \tag{7}$$

whereby the momentum $\mathbf{p}_n = m_n d\mathbf{r}_n/dt$ is for particle n of mass m_n and displacement $\mathbf{r}_n$. It follows from equation (6) that the Hamilton-Jacobi equation will be as follows,

$$\begin{aligned} \tfrac{\partial\mathcal{S}_1}{\partial t} + i\tfrac{\partial\mathcal{S}_2}{\partial t} + \tfrac{1}{2}\left(\tfrac{\nabla\mathcal{S}_1}{m_1} + i\tfrac{\nabla\mathcal{S}_2}{m_2}\right)\cdot(\nabla\mathcal{S}_1 + i\nabla\mathcal{S}_2) &= \\ -(V_{g1} + V_{c1}) - i(V_{g2} + V_{c2}) & \end{aligned} \tag{8}$$

Equation (8) above can be decomposed into two coupled Hamilton-Jacobi equations as shown below,

$$\frac{\partial\mathcal{S}_1}{\partial t} + \frac{\nabla\mathcal{S}_1\cdot\nabla\mathcal{S}_1}{2m_1} - \frac{\nabla\mathcal{S}_2\cdot\nabla\mathcal{S}_2}{2m_2} + V_{g1} + V_{c1} = 0 \tag{9a}$$

$$\frac{\partial\mathcal{S}_2}{\partial t} + \frac{\nabla\mathcal{S}_1\cdot\nabla\mathcal{S}_2}{2m_1} + \frac{\nabla\mathcal{S}_2\cdot\nabla\mathcal{S}_1}{2m_2} + V_{g2} + V_{c2} = 0 \tag{9b}$$

These two equations in equations (9a - 9b) describe the dynamics of the target system and the observation/measuring system collectively. In a way, one could compare this set of two equations to a state-space model in control theory with the first equation being the state equation while the second equation is taken as the measurement or observation equation. One clear difference from the usual state-space is that in this case, the act of observing the state of the target system seems to be modifying the dynamics of the target system through the coupling terms (i.e. kinetic energy term $-\nabla\mathcal{S}_2\cdot\nabla\mathcal{S}_2/(2m_2)$ and potential energy term V_{c1}).

In the case whereby the momentum of the observation system is zero (i.e. $\nabla\mathcal{S}_2 = \mathbf{0}$) there is no coupling between the two particles and each system evolves in the usual Newtonian way. This corresponds to switching off the measurement device. It is worth noting that the interaction potentials V_{c1} and V_{c2} may be either position-dependent or velocity-dependent or even both, depending on whether the underlying interaction forces were conservative or not. A conservative force is generally associated with a position-dependent potential. On the other hand some dissipative (hence non-conservative) forces can be associated with velocity-dependent potentials, presented in terms of the Rayleigh dissipation function [25, 26].

The coupling potential V_{c1} can be associated with momentum $\nabla\mathcal{S}_2$ of the second particle such that in the absence of the second particle, both the kinetic energy term $\nabla\mathcal{S}_2 \cdot \nabla\mathcal{S}_2/(2m_2)$ and the associated coupling potential term V_{c1} due to this particle vanish, leaving behind the usual Hamilton Jacobi equation for a single particle of mass m_1 and momentum $\nabla\mathcal{S}_1$. This conjecture will be more evident later when making a term-by-term comparison between this set of equations and the Schrodinger's equation. The second equation (i.e. measurement equation) in equation set (9a - 9b) has a coupled kinetic energy term that cannot be attributed to either one of the particles but rather to their reduced mass $m_* = m_1 m_2/(m_1 + m_2)$ as in the case of the two-body problem. That is, we can write this equation more compactly using the reduced mass as follows,

$$\frac{\partial\mathcal{S}_2}{\partial t} + \frac{\nabla\mathcal{S}_1 \cdot \nabla\mathcal{S}_2}{2m_*} + V_{g2} + V_{c2} = 0 \tag{10}$$

This seems to be intertwining these two particle such that the second particle can't be described without the other particle and this feature is indicative of the inherent coupling between the two interacting particles. We will later rewrite this same equation in probability density formulation for purposes of comparing it to the conservation of probability inherent in the Schrodinger's equation.

3.3 Extension To N-Particles Interaction

In the case of three particles we have an action as $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2 + j\mathcal{S}_3$ and it follows then that the Hamilton-Jacobi equation will be of the form,

$$\frac{\partial\mathcal{S}_1}{\partial t} + \frac{\nabla\mathcal{S}_1 \cdot \nabla\mathcal{S}_1}{2m_1} - \sum_{n=2}^{3} \frac{\nabla\mathcal{S}_n \cdot \nabla\mathcal{S}_n}{2m_n} + V_{g1} + V_{c1} = 0 \tag{11a}$$

$$\frac{\partial\mathcal{S}_2}{\partial t} + \frac{\nabla\mathcal{S}_1 \cdot \nabla\mathcal{S}_2}{2m_1} + \frac{\nabla\mathcal{S}_2 \cdot \nabla\mathcal{S}_1}{2m_2} + V_{g2} + V_{c2} = 0 \tag{11b}$$

$$\frac{\partial\mathcal{S}_3}{\partial t} + \frac{\nabla\mathcal{S}_1 \cdot \nabla\mathcal{S}_3}{2m_1} + \frac{\nabla\mathcal{S}_3 \cdot \nabla\mathcal{S}_1}{2m_3} + V_{g3} + V_{c3} = 0 \tag{11c}$$

In the case of three particles we have an action as $\mathcal{S} = \mathcal{S}_1 + i\mathcal{S}_2 + j\mathcal{S}_3 + k\mathcal{S}_4$ and it follows then that the Hamilton-Jacobi equation will be of the form,

$$\frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m_1} - \sum_{n=2}^{4} \frac{\nabla \mathcal{S}_n \cdot \nabla \mathcal{S}_n}{2m_n} + V_{g1} + V_{c1} = 0 \tag{12a}$$

$$\frac{\partial \mathcal{S}_2}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_2}{2m_1} + \frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_1}{2m_2} + V_{g2} + V_{c2} = 0 \tag{12b}$$

$$\frac{\partial \mathcal{S}_3}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_3}{2m_1} + \frac{\nabla \mathcal{S}_3 \cdot \nabla \mathcal{S}_1}{2m_3} + V_{g3} + V_{c3} = 0 \tag{12c}$$

$$\frac{\partial \mathcal{S}_4}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_4}{2m_1} + \frac{\nabla \mathcal{S}_4 \cdot \nabla \mathcal{S}_1}{2m_4} + V_{g4} + V_{c4} = 0 \tag{12d}$$

Note that the action $\mathcal{S}$ is still related to the momentum $\mathbf{p}$ by $\mathbf{p} = \nabla \mathcal{S}$. Extrapolating from the pattern in equation sets (9a - 9b), (11a - 11c) and (12a - 12d) we generalize the coupled Hamilton-Jacobi equations to N interacting particles as follows,

$$\frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m_1} - \sum_{n=2}^{N} \frac{\nabla \mathcal{S}_n \cdot \nabla \mathcal{S}_n}{2m_n} + V_{g1} + V_{c1} = 0 \tag{13a}$$

$$\frac{\partial \mathcal{S}_2}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_2}{2m_1} + \frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_1}{2m_2} + V_{g2} + V_{c2} = 0 \tag{13b}$$

$$\frac{\partial \mathcal{S}_3}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_3}{2m_1} + \frac{\nabla \mathcal{S}_3 \cdot \nabla \mathcal{S}_1}{2m_3} + V_{g3} + V_{c3} = 0 \tag{13c}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{\partial \mathcal{S}_N}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_N}{2m_1} + \frac{\nabla \mathcal{S}_N \cdot \nabla \mathcal{S}_1}{2m_N} + V_{gN} + V_{cN} = 0 \tag{13d}$$

The equations of motion, similar in form to Newton's second law of motion, can be obtained by applying a gradient operator ∇ on equation set (13a - 13d) above. This approach towards mechanics could be useful when looking at problems like the many-body problem. In the next sections we will once again restrict ourselves to the case of just two-particle interactions. Continuing our exploration, we next look at some trivial symmetries associated with the two-particle dynamics.

4 Some Trivial Symmetries

4.1 Invariance Under Phase Conjugation

If we take the complex conjugate of the Lagrangian in equation (5), the action is transformed into $\mathcal{S} = \mathcal{S}_1 - i\mathcal{S}_2$ and the resulting dynamics are,

$$\frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m_1} - \frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_2}{2m_2} + V_{g1} + V_{c1} = 0 \tag{14a}$$

$$\frac{\partial \mathcal{S}_2}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_2}{2m_1} + \frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_1}{2m_2} + V_{g2} + V_{c2} = 0 \tag{14b}$$

Equation set (14a - 14b) is exactly the same as equation set (9a - 9b) and this shows that the coupled Hamilton-Jacobi dynamics are invariant under complex conjugation of phase $\mathcal{S}/\hbar$ or Lagrangian $\mathcal{L}$. This is expected since the principle of stationary action accepts all critical points (minima, maxima, and saddle points) and the act of complex-conjugating the phase, only swaps minima with maxima of one component of the action (i.e. $\mathcal{S}_2$) and that does not affect the stationarity of action at those swapped points as they remain critical points.

4.2 Phase And Particle Exchange Symmetry

We wish to show that there is nothing special about the choice we made earlier in equation (5). In other words, we could have made a different choice $\mathcal{L} = \mathcal{L}_2 + i\mathcal{L}_1$ (instead of $\mathcal{L} = \mathcal{L}_1 + i\mathcal{L}_2$) and under this choice we get the action as $\mathcal{S} = \mathcal{S}_2 + i\mathcal{S}_1$ (which trivially leads to exchanging phase components). The resulting dynamics are,

$$\frac{\partial \mathcal{S}_2}{\partial t} + \frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_2}{2m_2} - \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m_1} + V_{g2} + V_{c2} = 0 \tag{15a}$$

$$\frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_2}{2m_1} + \frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_1}{2m_2} + V_{g1} + V_{c1} = 0 \tag{15b}$$

Equation set (15a - 15b) is similar to equation set (9a - 9b) in structure with the exception that the particles seem to be swapped or exchanged. This suggests that exchanging the real and imaginary components of the phase (or action and consequently the Lagrangian) has the effect of exchanging particles in our overall system of two particles.

5 Some Classical Examples

In this section we apply the coupled Hamilton-Jacobi equations in equation set (9a - 9b) to demonstrate, with examples, the role played by measurement dynamics in the overall system dynamics. The coupling mechanism is different in each of the chosen examples and that is meant to demonstrate contextuality in measurement. That is, for the same target system, coupling could be done differently and the choice of coupling will influence the overall dynamics and hence the information gained through measurement.

5.1 Charged Particles In Uniform Electric Field

Consider a particle of mass m_1 and charge q_1 moving in uniform electric field $\mathbf{E}$. The Lagrangian for this one-particle system can be written as,

$$\mathcal{L}_1 = \frac{\mathbf{p}_1^2}{2m_1} - q_1 V \tag{16}$$

for some potential V such that $\mathbf{E} = -\nabla V$. The corresponding Hamilton-Jacobi equation becomes,

$$\frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m_1} = -q_1 V \tag{17}$$

We can obtain the equation of motion by applying the gradient operator ∇ on equation (17) as shown below,

$$\begin{aligned}
\frac{\partial (\nabla \mathcal{S}_1)}{\partial t} + \nabla\left(\frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m_1}\right) &= -q_1 \nabla V &\text{(18a)}\\
\left(\frac{\partial}{\partial t} + \frac{\nabla \mathcal{S}_1}{m_1} \cdot \nabla\right)\nabla \mathcal{S}_1 &= q_1 \mathbf{E} &\text{(18b)}\\
\frac{d}{dt}(\nabla \mathcal{S}_1) &= \mathbf{F}_1 &\text{(18c)}\\
\frac{d}{dt}(m_1 \dot{\mathbf{r}}_1) &= \mathbf{F}_1 &\text{(18d)}\\
m_1 \ddot{\mathbf{r}}_1 &= \mathbf{F}_1 &\text{(18e)}
\end{aligned}$$

where $d/dt = \partial/\partial t + m_1^{-1} \nabla \mathcal{S}_1 \cdot \nabla$ is the convective/material derivatives definition and $\nabla \mathcal{S}_1 = m_1 \dot{\mathbf{r}}_1$ the momentum relation. The derivation in equation set

(18a - 18e) shows that the resulting equation of motion is actually Newton's second law. So far measurement dynamics have not yet been integrated into the system. To include measurement into the system we need to think about a measuring system that will be able to couple with the target system and get some information from it. One possible measuring system could be a second particle of mass m_2 and charge $q_2$moving the same electric $\mathbf{E}$. On its own, this particle will have the same general form of the Lagrangian as the first particle. However, in the presence of the first particle, the second particle will experience the coupling force due to the first particle and by Newton's third law, the first particle will experience the same magnitude of force due to the second particle. The overall Lagrangian will therefore be as follows,

$$\mathcal{L} = \mathcal{L}_1 + i\mathcal{L}_2 = (\frac{p_1^2}{2m_1} - q_1 V - q_1 V_{c1}) + i(\frac{p_2^2}{2m_2} - q_2 V - q_2 V_{c2}) \tag{19}$$

with $V_{cj} = kq_k / |\mathbf{r}_j - \mathbf{r}_k|$ as the Coulomb coupling potential. The resulting Hamilton-Jacobi equations are,

$$\begin{aligned} \frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m_1} - \frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_2}{2m_2} + V_{g1} + V_{c1} &= 0 & \text{(20a)} \\ \frac{\partial \mathcal{S}_2}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_2}{2m_*} + V_{g2} + V_{c2} &= 0 & \text{(20b)} \end{aligned}$$

Following the same procedure as in equation set (18a - 18e) we obtain,

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_1 + \nabla(\frac{\nabla \mathcal{S}_2 \cdot \nabla \mathcal{S}_2}{2m_2}) + \frac{kq_1 q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{\|\mathbf{r}_2 - \mathbf{r}_1\|^3} \tag{21}$$

which is different from the equation $m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_1$ obtained without interaction effects of the second particle. The two additional terms on the right hand-side of equation (21) above result from the coupling between the two particles and demonstrate how the coupling changes the trajectory of the first particle from an interaction-free case.

5.2 Neutral Particles In Gravitational Field

The Coulomb interaction force example above can be adapted for gravitational interaction force case in which the same particles (but now neutral) are now under a fairly uniform gravitational field while also interacting with each other under their own gravitational force. The resulting equation of motion would take the same form as equation (21) with the difference that the guiding force $\mathbf{F}_1$ is now due to the external uniform gravitational force on two particles while the Coulomb interaction force term is replaced by the gravitation force term between the two particles.

5.3 Charged Particles In Gravitational Field

In the case of charged particles, both Coulomb and gravitational force terms can be considered at the same time and equation (21) would be modified to include both gravitational force terms and Coulomb force terms. For any interaction, the general form of equation (21) holds but the interaction potential will differ.

5.4 Neutral Particles In Collision

The general form of equation (21) still hold in this of a non-conservative interaction force with the exception that the interaction potential will be replaced by a Rayleigh potential for particle collision. In section 7, a similar case of a photon colliding and scattering off a massive object is considered and it is shown how in the limit when the interaction potential and the second particle's momentum are sufficiently weaker than the guiding potential and the first particle's momentum respectively, the Newton's second law is recovered.

6 Relation To Quantum Mechanics

6.1 From Schrodinger's Equation To The Phase Equation

The Schrodinger's time-dependent equation for a particle of mass m under the influence of some guiding potential V_g is presented as shown below [8, 19],

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V_g\psi(\mathbf{r},t) \tag{22}$$

To rewrite Schrodinger's time-dependent equation in terms of the phase of the wave, we adapt the transformation [33, 19, 8] as shown below,

$$\psi(\mathbf{r},t) = \exp(i\mathcal{S}(\mathbf{r},t)/\hbar) = \exp(i\mathcal{S}_1(\mathbf{r},t)/\hbar - \mathcal{S}_2(\mathbf{r},t)/\hbar) \tag{23}$$

Substituting this relation (23) in Schrodinger's time-dependent equation (22) and rearranging terms we obtain,

$$\frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla\mathcal{S}_1 \cdot \nabla\mathcal{S}_1}{2m} - \frac{\nabla\mathcal{S}_2 \cdot \nabla\mathcal{S}_2}{2m} + V_g + \frac{\hbar\nabla^2\mathcal{S}_2}{2m} = 0 \tag{24a}$$

$$\frac{\partial \mathcal{S}_2}{\partial t} + \frac{\nabla\mathcal{S}_1 \cdot \nabla\mathcal{S}_2}{m} - \frac{\hbar\nabla^2\mathcal{S}_1}{2m} = 0 \tag{24b}$$

This set of equations (24a - 24b) has the same general form as the Hamilton-Jacobi equations derived in the section 3 for two coupled particles. Below we turn our attention to some of the similarities between the two sets of equations.

6.2 The Phase Equation And Hamilton-Jacobi Equations

We can draw some analogs with the Hamilton-Jacobi equations derived in the previous section. Specifically, we can recover the Schrodinger equation set (24a - 24b) by making the following settings on the equation set (9a - 9b) of coupled Hamilton-Jacobi equations in the previous section.

$$m_1 = m_2 \tag{25a}$$

$$V_{g2} = 0 \tag{25b}$$

$$V_{c1} = \frac{\hbar}{2m}\nabla^2\mathcal{S}_2 \tag{25c}$$

$$V_{c2} = -\frac{\hbar}{2m}\nabla^2\mathcal{S}_1 \tag{25d}$$

It is worth noting that under the Schrodinger's dynamics the second particle (i.e. measuring system) would be moving freely in the absence of interaction since its guiding potential is zero ($V_{g2} = 0$). Notice that in the Schrodinger's dynamics the second particle has the same mass as the first particle and one wonders if that is just a special case of a more general setting as indicated by our coupled Hamilton-Jacobi equations. It can seem as if the second particle is an image (or conjoined twin) of the first particle or perhaps as if the particle is interacting with itself. Looking at the potential V_{c1} we see that it does depend on the momentum $\nabla\mathcal{S}_2$ of the second particle as we indicated in the previous section since it is proportional to the divergence of the second particle's momentum. The same is true about the dependence of the potential V_{c2} on the first particle's momentum $\nabla\mathcal{S}_1$. Below we present a set of coupling potentials consistent with those in equation set (25a - 25d) above,

$$V_{c1} = \frac{\hbar}{2m_*}\nabla^2\mathcal{S}_2 = \frac{\hbar(m_1+m_2)}{2m_1m_2}\nabla\cdot(\nabla\mathcal{S}_2) \tag{26a}$$

$$V_{c2} = -\frac{\hbar}{2m_*}\nabla^2\mathcal{S}_1 = -\frac{\hbar(m_1+m_2)}{2m_1m_2}\nabla\cdot(\nabla\mathcal{S}_1) \tag{26b}$$

We can update our interaction potentials V_{c1} and V_{c2} in equation set (9a - 9b) using this particular set of coupling potentials in equation set (26a - 26b) to obtain,

$$\frac{\partial\mathcal{S}_1}{\partial t} + \frac{\nabla\mathcal{S}_1\cdot\nabla\mathcal{S}_1}{2m_1} - \frac{\nabla\mathcal{S}_2\cdot\nabla\mathcal{S}_2}{2m_2} + \frac{\hbar\nabla^2\mathcal{S}_2}{2m_*} = -V_{g1} \tag{27a}$$

$$\frac{\partial\mathcal{S}_2}{\partial t} + \frac{\nabla\mathcal{S}_1\cdot\nabla\mathcal{S}_2}{2m_*} - \frac{\hbar\nabla^2\mathcal{S}_1}{2m_*} = -V_{g2} \tag{27b}$$

written in terms of the reduced mass m_*. We can then show that, for this selection of coupling potentials, there exist a real quantity $\mathcal{P} = \exp(-2\mathcal{S}_2/\hbar)$ associated with the second particle and this quantity is conserved. Substituting this quantity in the second equation in equation set (27a - 27b) we obtain the following relation,

$$\frac{\partial\mathcal{P}}{\partial t} + \nabla\cdot(\frac{\mathcal{P}\nabla\mathcal{S}_1}{2m_*} - \frac{2}{\hbar}\mathbf{W}_{g2}) = 0 \tag{28}$$

with $V_{g2} = \exp(2\mathcal{S}_2/\hbar)\nabla\cdot\mathbf{W}_{g2}$. Equation (28) has a structure of a continuity equation, thus indicating that $\mathcal{P}$ is a conserved quantity. In general any coupling potential of the following form leads to a continuity equation,

$$V_{c2} = -\frac{\hbar}{2m_*}\nabla^2\mathcal{S}_1 + \exp(2\mathcal{S}_2/\hbar)\nabla\cdot\mathbf{W}_{c2} \tag{29}$$

for some vector field $\mathbf{W}_{c2}$. The phase term $\mathcal{S}_2/\hbar$ holds information about the allowed trajectories (for the second particle). According to the principle of stationary action, the allowed trajectories are those which are, at some time t, all normal to the wavefront corresponding to a stationary or constant phase term [19, 22]. In a closed system, a particle would follow one particular trajectory, however, if some interaction is brought into the picture the particle can deviate from one allowed trajectory to another. We believe that the conserved quantity $\mathcal{P}$ can be thought of as a function assigning weights to each of the allowed trajectories and as such, it takes the form of the probability density function associated with the allowed trajectories.

6.3 Bohmian Mechanics And Hamilton-Jacobi Equations

Bohmian mechanics can be derived from Schrodinger's equation by writing the wave function in its polar form as $\psi(\mathbf{r},t) = \sqrt{\mathcal{P}(\mathbf{r},t)}\exp(i\mathcal{S}_1(\mathbf{r},t)/\hbar)$, using Madelung transformation [33, 19, 8], and this leads to the following two equations [8],

$$\frac{\partial \mathcal{S}_1}{\partial t} + \frac{\nabla \mathcal{S}_1 \cdot \nabla \mathcal{S}_1}{2m} = -(\mathcal{Q} + V_{g1}) \tag{30a}$$

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla \cdot (\mathcal{P}\frac{\nabla \mathcal{S}_1}{m}) = 0 \tag{30b}$$

where the term $\mathcal{Q}$, referred to as the quantum potential is given by,

$$\mathcal{Q} = \frac{\hbar^2}{4m}(\frac{\nabla \mathcal{P} \cdot \nabla \mathcal{P}}{2\mathcal{P}^2} - \frac{\nabla^2 \mathcal{P}}{\mathcal{P}}) \tag{31}$$

Equation (30b) is the statement of the conservation of the probability density $\mathcal{P}(\mathbf{r},t)$. This continuity equation is similar to the one in equation (28) with the difference being that in equation (28) $\mathcal{P}$ is a direct property of the measuring system and not that of target system, as Bohmian view seems to indicate. This confusion about which part belongs to the measuring system and which belongs to the target system is inherent in Schrodinger's equation since it restricts itself to a special case (i.e. of $m_1 = m_2 = m$) from which one can no longer trace which terms belong to m_1(measured system or first particle) and which belong to m_2 (the measuring system or the second particle). By making comparison with our coupled Hamilton-Jacobi equations one can trace back and resolve the confusion. Another restriction in the Schrodinger's view (and hence Bohmian's view too) is that the guiding potential term is zero (i.e. $\nabla \cdot \mathbf{W}_{g2} = 0$). The quantum potential term in the Bohmian view can be decomposed further and shown to actually have the kinetic energy term belonging to the measuring system. Making the substitution $\mathcal{P} = \exp(-2\mathcal{S}_2/\hbar)$ in the quantum potential term and simplifying we obtain,

$$\mathcal{Q} = \frac{\hbar^2}{4m}\left(\frac{\nabla\mathcal{P}\cdot\nabla\mathcal{P}}{2\mathcal{P}^2} - \frac{\nabla^2\mathcal{P}}{\mathcal{P}}\right) = \frac{\nabla\mathcal{S}_2\cdot\nabla\mathcal{S}_2}{2m} + \frac{\hbar}{2m}\nabla^2\mathcal{S}_2 \tag{32}$$

The term $\nabla\mathcal{S}_2 \cdot \nabla\mathcal{S}_2/(2m)$ in equation set (32) is clearly the kinetic energy of the measuring system with the same mass m as the target system. Writing the quantum potential term in this explicit way recovers the first particle's equation in our coupled Hamilton-Jacobi dynamics. By making reference to our coupled Hamilton-Jacobi equations, we can show that the hidden variables in Bohmian mechanics refer to the position coordinates of the first particle. We start by substitution the Madelung transformation in equation (23) into the guiding equation in Bohm's view [34] as shown below,

$$m\frac{d\mathbf{r}(t)}{dt} = \hbar\mathrm{Im}\left(\frac{\nabla\psi(\mathbf{r}(t),t)}{\psi(\mathbf{r}(t),t)}\right) = \frac{\hbar}{\psi}\mathrm{Im}\left(i\frac{\psi\nabla\mathcal{S}_1}{\hbar} - \frac{\psi\nabla\mathcal{S}_2}{\hbar}\right) = \nabla\mathcal{S}_1 \tag{33}$$

Equation (33) shows that the momentum (i.e $md\mathbf{r}(t)/dt$) of the hidden variable $\mathbf{r}(t)$ in the Bohmian picture is actually the momentum $\nabla\mathcal{S}_1$ of the first particle (i.e target system). Bohmian mechanics goes further indicating that in the limit of reduced Planck constant approaching zero (i.e. $\mathcal{Q} \to 0$ as $\hbar \to 0$), Bohmian dynamics recover classical mechanics or Newtonian picture [8]. However, if it was made clear in the first place that the quantum potential $\mathcal{Q}$ is made up of terms belonging to the second particle (measuring system) rather than the first particle then it wouldn't be necessary to recover Newtonian picture by setting $\hbar \to 0$. In the next section we show that Newtonian picture is recovered naturally from having the momentum and coupling potential of the second particle sufficiently dominated by the momentum and guiding potential of the first particle respectively.

7 Recovery of Newtonian Dynamics

Imagine observing a macroscopic object (i.e. first particle) of mass m_1 moving with momentum $\nabla\mathcal{S}_1$ under the influence of some potential V_{g1}, which be gravitational, electrical, etc. To be able to observe this object requires that a photon (i.e. second particle) scatters off this object (after their interaction) and move at velocity c (or momentum $\nabla\mathcal{S}_2 = \hbar\omega/c$) to a detector. In this case, the photon momentum is much smaller than the object momentum (i.e.

$\nabla\mathcal{S}_1 \gg \nabla\mathcal{S}_2$). Also the scattering potential V_{c1} much smaller than the object's guiding potential (i.e. $V_{g1} \gg V_{c1}$) and as a result the recoil of the object is insignificant. The consequence of these conditions is that the first equation in equation set (9a - 9b) transforms in the following way,

$$\begin{aligned} \frac{\partial\mathcal{S}_1}{\partial t} + \frac{\nabla\mathcal{S}_1\cdot\nabla\mathcal{S}_1}{2m_1} - \frac{\nabla\mathcal{S}_2\cdot\nabla\mathcal{S}_2}{2m_2} &= -(V_{g1}+V_{c1}) \qquad (34a)\\ &\Downarrow \\ \frac{\partial\mathcal{S}_1}{\partial t} + \frac{\nabla\mathcal{S}_1\cdot\nabla\mathcal{S}_1}{2m_1} &= -V_{g1} \qquad (34b)\end{aligned}$$

Next, we apply the gradient operator ∇ on the second equation in equation set (34a - 34b) to get,

$$\begin{aligned} \frac{\partial(\nabla\mathcal{S}_1)}{\partial t} + \nabla(\frac{\nabla\mathcal{S}_1\cdot\nabla\mathcal{S}_1}{2m_1}) &= -\nabla V_{g1} \qquad (35a)\\ (\frac{\partial}{\partial t} + \frac{\nabla\mathcal{S}_1}{m_1}\cdot\nabla)\nabla\mathcal{S}_1 &= -\nabla V_{g1} \qquad (35b)\end{aligned}$$

Lastly we employ the convective/material derivatives definition $d/dt = \partial/\partial t + m_1^{-1}\nabla\mathcal{S}_1\cdot\nabla$, the conservative force definition $\mathbf{F}_1 = -\nabla V_{g1}$ as well as the momentum relation $\nabla\mathcal{S}_1 = m_1\dot{\mathbf{r}}_1$ to obtain the following set of equations,

$$\begin{aligned} (\frac{\partial}{\partial t} + \frac{\nabla\mathcal{S}_1}{m_1}\cdot\nabla)\nabla\mathcal{S}_1 &= -\nabla V_{g1} \qquad (36a)\\ m_1\ddot{\mathbf{r}}_1 &= \mathbf{F}_1 \qquad (36b)\end{aligned}$$

which is Newton's second law of motion for our object of interest. The derivation above clearly demonstrates that our laws in classical mechanics do not take into account the dynamics of the observer (or measuring system) and this can lead to their failure whenever the object of interest has a momentum that is comparable with that of the observer, as it is the case in quantum mechanics. It would therefore be wise to formulate all our theories in a way that includes the measurement model as part of the system dynamics even if the interaction potential is not known a priori until the experiment is set up. This suggestion is in line with one of the No-Go theorems in quantum mechanics called the Kochen-Specker theorem [35]. This theorem, about contextuality, states that the observables of a system cannot both have definite values and at the same time those values be independent of the device used to measure them.

8 Conclusion

In this paper, we modeled the measurement process as part of the dynamics describing the interaction between the target system and the measuring system. This approach led to dynamics that seem to be consistent with both quantum mechanics and classical mechanics. We also learnt that hidden within Schrodinger's equation are two equations (similar in structure to our coupled Hamilton-Jacobi equations) one of which describes the dynamics of our system and the other equation describes the dynamics of the observing device and the interaction between the two. The crucial step in our treatment of the measurement problem in the classical Hamilton-Jacobi picture was to generalize the domain and range of action such that it maps the space of complex functions (instead of only real functions) to a space of complex numbers (rather than the limited real number set) and force the Lagrangian of the two interacting systems to be the real and imaginary components leading to this complex action.

The relativistic Hamilton-Jacobi equation has been addressed in the literature [10] and it would be interesting to do the same for the coupled Hamilton-Jacobi equations presented above. Also one wonders if general relativity theory reformulated in terms of coupled Hamilton-Jacobi equations would yield a theory of gravity that is compatible with quantum mechanics. A good place to start might be with the Einstein-Hilbert action [36] which perhaps could be modified to include the imaginary component in the Lagrangian as a way to augment the observer dynamics in the total action.

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