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Cosmological Exact Solutions of Petrov Type D of a Nonlinear Asymptotic Fluid to a Dark Energy Fluid in Real and Complex Geometries with Double Singularity. Second Case.

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Abstract

In this paper, exact solutions to the Einstein's equations are obtained for an anisotropic and homogeneous symmetry of Petrov Type D from a nonlinear fluid that responds to the equation of state $Q_+Q_- = 0$ where $Q_\pm = \sqrt{\mu_1}\,(-2\,P_1 + \mu_1)^{3/2} \pm 2\,\frac{BP_1}{a}$ wherein $\mu_1 = \mu - \Lambda$, $P_1 = P + \Lambda$, and μ, P and Λ are the volumetric energy density, the pressure, and a constant linked to the concept of dark energy. That equation of state and what it represents in certain limits of time (when $t \to 0$ and when $t \to \infty$) are also analyzed. Two general solutions which are different because of the degree of initial expansion that a coordinate can have in relation to a perpendicular plane are obtained. For each solution, two cases are present: one represents a space-time with real geometry ($\mathbb{R}$) for all the values of t, and asymptotically in time, this case becomes an isotropic space-time of FLRW of a dark energy fluid; and the other one presents a double singularity, so that since the first singularity, space-time is complex ($\mathbb{C}$) until a certain time $t = a$ (when the second singularity is present) from which space-time is real ($\mathbb{R}$) and with the increase of time, it tends to an isotropic space-time of FLRW from a dark energy fluid. Then, temperature behavior in relation to time is obtained.

Keywords: cosmology, exact solution, Einstein, temperature, nonlinear, Kretschmann, singularity, complex space-time geometry

1 Introduction

Cosmological space-times of Petrov Type D of nonlinear fluids and which can present double singularities have been studied previously in [1] and [2] where there have been discussions about the great interest that these solutions and their characteristics close to singularities generate without mentioning other relevant aspects and their respective literature related to solutions with non-linear fluids within this symmetry and others. In [2], some of the possible solutions obtained were complex and had double singularity. Thus, the cosmological space-time starts from a first singularity from which it is a real space-time ($\mathbb{R}$) until a time when a second singularity is present, and space-time becomes complex ($\mathbb{C}$). The complex part of the interval represents a space-time that tends to contract meanwhile the real part represents a space-time that tends to expand asymptotically when $t \to \infty$ transforms into an isotropic space-time of FLRW that corresponds to the one obtained for the dark energy model. In this article, solutions are obtained from a nonlinear fluid equation of which the equation of state has the form $Q_+Q_- = 0$ where

$Q_\pm = \sqrt{\mu - \Lambda}\,(-2\,P - 3\,\Lambda + \mu)^{3/2} \pm 2\,\frac{B(P+\Lambda)}{a}$

μ, P and Λ are the volumetric energy density, the pressure, and a constant linked to the concept of dark energy respectively. That equation of state represents a fluid that can be initially interpreted as a diverse-fluid "cocktail" and that transforms over time. Solutions could be complex or non-complex ($\mathbb{C}$) in a time lag between two singularities when the constant of the equation of state $a < 0$.

2 Symmetry, Einstein's Equations, Solutions and Kretschmann's Invariant

In this work, the anisotropic and homogeneous symmetry of Petrov Type D will be used and whose form is [3]

$$ds^2 = F dt^2 - t^{2/3} K (dx^2 + dy^2) - \frac{t^{2/3}}{K^2} dz^2, \tag{1}$$

where in F and K are functions of t.
An equivalent analysis was conducted in [2] considering that the equation of state complies with the following equality:

$$Q_+Q_- = 0,\ where\ Q_\pm = \sqrt{\mu_1}\,(-2\,P_1 + \mu_1)^{3/2} \pm 2\,\frac{BP_1}{a}, \tag{2}$$

wherein $\mu_1 = \mu - \Lambda$, $P_1 = P + \Lambda$ and Λ is the constant linked to the concept of dark energy and that leads to the next equalities

$$K = K_0 e^{C_1 \int \frac{F^{1/2}}{t} dt}, \tag{3}$$

where the constant K_0 without lossing generalities in (3) is considered equal to 1 and $C_1 = \pm 2/3$ (for each possible value of C_1 a different model is obtained).

From Einstein's equations $G_\alpha^\beta = \kappa T_\alpha^\beta$, the equality $T^{\mu\nu}{}_{;\mu} = 0$ (see [2]) of (3) and (2) gets for any C_1 that the solution of F is

$$F = \frac{1}{3\left(B\sqrt{t^2+at} + \Lambda t^2 + 1\right)}, \tag{4}$$

wherein a is a constant whose sense will be determined later.
When considering Einstein's equations, from the solution (4), (3), G_0^0 and G_1^1 (see [2]) the pressure is

$$P = \frac{Ba}{2t\sqrt{t^2+at}} - \Lambda \tag{5}$$

and the density μ is

$$\mu = \frac{B\sqrt{t^2+at}}{t^2} + \Lambda. \tag{6}$$

The K function in (3) can be written as

$$K = e^{\pm 2\sigma/3}, \tag{7}$$

where σ has two possible solutions when $a > 0$,

$$\sigma_1 = \frac{-4\left(e_1 F\left(T\xi, R\right) + \left(e_2 - e_1\right)\Pi\left(T\xi, \frac{e_2}{e_1}T^{-2}, R\right) - S_1\right)}{\sqrt{-e_2+e_4}\sqrt{e_3-e_1}\sqrt{1+3\,\Lambda a^2 e_2 e_1}} \tag{8}$$

and when $a = -b < 0$,

$$\sigma_2 = \frac{4\left(e_2 F\left(T\eta, R\right) + \left(e_1 - e_2\right)\Pi\left(T\eta, T^{-2}, R\right) - S_2\right)}{\sqrt{-e_2+e_4}\sqrt{e_3-e_1}} \tag{9}$$

in (8) and (9) functions $F(\nu, m)$ and $\Pi(\nu, n, m)$ are the incomplete elliptic integral of the first kind and the incomplete elliptic integral of the third kind respectively. ν is the sine of amplitude, n is the characteristic and m the parameter, variable $\xi = \sqrt{\frac{\left(t+\sqrt{t(t+a)}\right)(e_1-1)+e_1 a}{\left(t+\sqrt{t(t+a)}\right)(e_2-1)+e_2 a}}$, constants $T = \sqrt{\frac{-e_2+e_4}{e_4-e_1}}$, $R = \sqrt{\frac{e_2-e_3}{-e_3+e_1}}T^{-1}$, variable $\eta = \sqrt{\frac{\left(t+\sqrt{t(t-b)}\right)(e_1-1)+b}{\left(t+\sqrt{t(t-b)}\right)(e_2-1)+b}}$, constants of integration $S_1 = e_1 F\left(T\beta, R\right) + \left(e_2 - e_1\right)\Pi\left(T\beta, \frac{e_2}{e_1}T^{-2}, R\right)$ and $S_2 = e_2 F\left(T\beta, R\right) + \left(e_1 - e_2\right)\Pi\left(T\beta, T^{-2}, R\right)$ wherein $\beta = \sqrt{\frac{1-e_1}{1-e_2}}$ and constants e_k in solution (8) are the roots of the equation

$(1-x^2)\,(x^2-3\,Bax-1)=3\,\Lambda\,a^2x^4$, and in solution (9),

$(1-x^2)\,(x^2-3\,Bbx-1)=3\,\Lambda\,b^2$, so the roots (usually complex) are taken counterclockwise increasing with the index $k=1..4$ and when graphed in a rectangular, coordinate systems x and y where a root e_k has the form $e_k = x_k + y_k i$. For roots that are multiples of others, ($e_k = Ae_l$), the first one to be taken is the one with the lowest absolute value.

3 Singularties and Kretschmann's Invariant

In the study of possible singularities of a given space-time, Kretschmann's invariant is used ($Krets = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$), and its relevance is discussed in [3]. For the solutions found (with $C_1 = \pm 2/3$), the invariant has the form

$$Krets_{\pm} = W_{\pm} + U, \tag{10}$$

where

$$U = \frac{9\,B^2t\,(20\,t^2+52\,ta+41\,a^2)-24\,\sqrt{t\,(t+a)}\,(-6\,\Lambda\,t^3+3\,\Lambda\,t^2a)\,B}{108t^4\,(t+a)}$$

and

$$W_{\pm} = \frac{8}{27}\,\frac{2\left(\sqrt{3\,B\sqrt{t\,(t+a)}+3\,\Lambda\,t^2+1}\pm 1\right)^2+9\,t^4\Lambda^2}{t^4}$$

and the positive sign is taken when $C_1 = 2/3$ and the negative one if $C_1 = -2/3$. From Kretschmann's invariant (10), it is known that a singularity exists in $t=0$ for any value of C_1 and of a. When $C_1 = 2/3$, Kretschmann's invariant presents a singularity of $t=0$ equal to Kasner's E_{D_1} (with a depth order of t^{-4}) discussed in [3]. When $C_1 = -2/3$, the singularity has a depth order of t^{-3} when $a \neq 0$. If $a < 0$, Kretschmann's invariant shows one more singularity when $t = |a|$.

4 Analysis of Solutions

4.1 Case $a > 0$

When $a > 0$ and $t \geqslant 0$, solutions present only one singularity in $t = 0$ as it can be seen in (10). Constant S_1 in (8) has a form similar to σ_1, but it requires of the change of $\xi \to L$. In this case, the solution for values of $t \longrightarrow \infty$ becomes isotropic and of the form of dark energy analyzed in [3]. The density and pressure, μ and P, in (6) and (5) when $t \to 0$ tends to $\mu \longrightarrow \infty$ and $P \longrightarrow \infty$.

In close proximity to $t = 0$, pressure and density, P and μ, can be written as an addition form

$$P = -\Lambda - \frac{2B}{3}\sum_{n=0}^{\infty}\left(\prod_{k=0}^{n}(\frac{3}{2} - k)\right)\frac{(n-\frac{1}{2})t^{n-3/2}}{a^{n-1/2}n!} \tag{11}$$

and

$$\mu = \Lambda + \frac{2B}{3}\sum_{n=0}^{\infty}\left(\prod_{k=0}^{n}(\frac{3}{2} - k)\right)\frac{t^{n-3/2}}{a^{n-1/2}n!}. \tag{12}$$

Models with linear equations of state of type $P_\lambda = \lambda\mu_\lambda$ in anisotropic space-times of Petro type D comply with (see [3])

$$\mu_\lambda = \frac{\alpha_\lambda}{t^{1+\lambda}},\ \textit{and}\ P_\lambda = \frac{\lambda\alpha_\lambda}{t^{1+\lambda}},$$

thus, when comparing terms from (11) and (12), it is possible to come to a conclusion that the fluid behaves as a Hard Universe mixture (cocktail) ($\lambda = 1/2$), quintessence ($\lambda = -1/2$), dark energy ($\lambda = -1$), Phantom ($\lambda = -(2n + 1)/2$, $n \in \mathbb{N}$), so that Phantom energy values, like n, are odd. That energy is extracted from the energy of a fluid (negative energy); on the contrary, even n adds energy to a fluid (possitive energy).
As t increases, the fluid changes and for long-term times, it shows that

$$\mu = \Lambda + \frac{B}{t}\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{\frac{3}{2} - k}{k}\right)\frac{a^n}{t^n}\ \textit{and} \tag{13}$$

$$P = -\Lambda + \frac{Ba}{2t^2}\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{\frac{1}{2} - k}{k}\right)\frac{a^n}{t^n}, \tag{14}$$

when comparing terms (13) and (14), the fluid acts like a dark energy mixture (cocktail), dust ($\lambda = 0$), Zeldovich ($\lambda = 1$) and ekpyrotics fluids ($\lambda = n + 1$, $n \in \mathbb{N}$), and for values when n is odd, that energy is extracted from the energy of a fluid (negative energy), but when n is even, it adds energy to a fluid (possitive energy).

4.2 Case $a < 0$

When $a < 0$ and $t \geqslant 0$ of (10), solutions present double singularity in $t = 0$ and in $t = |a| = b$. They are complex ($\mathbb{C}$) for values of $t \in]0, b[$ so $\mu = \Lambda + i\mu_I$ and $P = -\Lambda + iP_I$ where μ and P represent a fluid mixture similar to the one in (12) and (11) but with energies from Hard Universe and dark energy which are the possitive ones. When considering (4), (6) and (5) likewise (3), the interval takes the form $ds^2 = ds_R^2 - i ds_I^2$ where

$$ds_R{}^2 = FR\,dt^2 - e^{B_c}\cos(B_s)\,t^{2/3}\left(dx^2 + dy^2\right) - e^{-2\,B_c}\cos(2\,B_s)\,t^{2/3}dz^2, \quad (15)$$

$$ds_I{}^2 = -FI\,dt^2 + e^{B_c}\sin(B_s)\,t^{2/3}\left(dx^2 + dy^2\right) - e^{-2\,B_c}\sin(2\,B_s)\,t^{2/3}dz^2, \quad (16)$$

and

$$B_c = C_1 \int \sqrt[4]{FR^2 + FI^2}\cos\left(\frac{1}{2}\arctan\left(\frac{FI}{FR}\right)\right)t^{-1}dt, \quad (17)$$

$$B_s = C_1 \int \sqrt[4]{FR^2 + FI^2}\sin\left(\frac{1}{2}\arctan\left(\frac{FI}{FR}\right)\right)t^{-1}dt, \quad (18)$$

$$\begin{aligned} FR &= \frac{3\,t^2\Lambda + 1}{\left(3\,t^2\Lambda + 1\right)^2 + 9\,B^2\left(-t^2 + tb\right)}, \\ FI &= -3\,\frac{B\sqrt{-t^2 + tb}}{\left(3\,t^2\Lambda + 1\right)^2 + 9\,B^2\left(-t^2 + tb\right)}. \end{aligned} \quad (19)$$

Real and imaginary parts of the metric for very small values of t have the form

$$\begin{aligned} ds_R{}^2 = \left(1 - 9\,B^2 tb\right)dt^2 - \left(1 \pm \frac{1}{4}B^2 bt\,(9 \pm 8)\right)t^{2/3\pm 2/3}\left(dx^2 + dy^2\right) + \\ - \left(1 + \frac{1}{2}B^2 bt\,(\mp 9 + 16)\right)t^{2/3\mp 4/3}dz^2, \end{aligned} \quad (20)$$

and

$$ds_i{}^2 = 3\,B\sqrt{tb}\left(dt^2 \mp \frac{2}{3}t^{2/3\pm 2/3}\left(dx^2 + dy^2\right) \pm \frac{4}{3}t^{2/3\mp 4/3}dz^2\right) \quad (21)$$

wherein the sign for $\pm$ is possitive when $C_1 = 2/3$ and negative when $C_1 = -2/3$, and in $\mp$ case, it is the opposite. From (21), the imaginary part of space-time strictly for short-term times is conformal of Petrov type O if $C_1 = -2/3$ (see [3]). In this case, the imaginary space-time emerges when time starts as a spot, and if $C_1 = 2/3$, it surfaces as a line in z of Petrov type D. It is obtained from (16) that in the expansion process when $C_1 = 2/3$, the plane x, y is expanded to a maximum given value in a time that complies with the equation $3C_1\sin(\theta + \alpha) + 2G\sin(\alpha) = 0$ wherein $G = \sqrt[4]{\left(3\,t^2\Lambda + 1\right)^2 - 9\,B^2 t\,(t - b)}$,

$$\theta = \frac{1}{2}\arctan\left(3\,\frac{B\sqrt{t\,(-t + b)}}{3\,t^2\Lambda + 1}\right)\ y\ \alpha = \int \frac{C_1\sin(\theta)\,dt}{tG}$$

later, it disappears in $t = b$. Then, axis z contracts since the beginning until it vanishes in $t = b$. When $C_1 = -2/3$, the imaginary space-time emerges as a spot that expands itself so axis z passes through a maximum expansion of time t that meets the equation criteria $3C_1 \sin(\theta + \alpha) - G \sin(\alpha) = 0$. The imaginary space-time ends disappearing in $t = b$.

5 Temperature

In an equivalent way as the one stated on [2], temperature depends on the time that forms it

$$T = \frac{T_0 \left(t + \frac{3a}{2}\right)}{\sqrt{t^2 + at}}, \tag{22}$$

where $T_0 > 0$ is a constant of integration. From (22), when $t \to \infty$ temperature tends to T_0 for any value of a (currently, it should be $\approx 2.7K$). For values of $a = -b < 0$, temperature is a magnitude that shows at the beginning of space-time and in the imaginary space-time only. By being singular in $t = b$, going through negative values in the interval $t \in]b, 3b/2[$, being null in $t = 3b/2$, it finally tends to $T \longrightarrow T_0$ when t$\longrightarrow \infty$.

6 Conclusions

Cosmological exact solutions were obtained from Einstein's equations (two general solutions that differ from one another by the degree of initial expansion). For the nonlinear fluid case (nonlinear equation of state), if a constant $a < 0$, they have double singularity, and if $a > 0$, they have only one singularity. If $a < 0$, the first singularity in $t = 0$ presents a volumetric energy density μ and an infinite pressure, and when $t = |a|$, the second singularity has a tendency for $|P| \to \infty$ but finite to μ. If constant $a > 0$, the interval, P and μ are real ($\mathbb{R}$), and as time increases, it becomes into an interval of the usual type for the dark energy model of FLRW. However, if $a < 0$, the interval is complex ($\mathbb{C}$) in a lapse where singularities are present ($ds^2 = ds_R^2 - ids_i^2 \in \mathbb{C}$). The imaginary space-time ds_i^2 can emerge as a line in axis z or as a spot depending on the sign of a constant, but in any case, the imaginary interval disappears in $t > |a|$. The logic behind the nonlinear equation of state $Q_+Q_- = 0$ used in this paper can be interpreted as a linear fluid mixture (cocktail) that changes in given time lags. Once temperature was analyzed, it was determined that when time is complex, this one is imaginary and tends to be constant when time increases, in any case.

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