Massless Field Equations in Comoving
Spherically Symmetric Metric and
Solution in LTB Cosmology

Antonio Zecca

Dipartimento di Fisica dell' Università degli Studi di Milano (Retired)
GNFM, Gruppo Nazionale per la Fisica Matematica - Milano - Italy

Abstract

The massless field equation are studied in LTB space time. General properties of the equations are first recalled in type D space-time. This requires to consider an algebraic constraint equation on the field components that involves the Weyl spinor, that reduces the number of the non zero field components and that distinguishes between fermions and bosons. The equations are studied by the Newman Penrose formalism based on a previously introduced null tetrad frame. The equation of the a priori non vanishing field components are exactly integrated. The solution depends on two arbitrary functions relative to the metric tensor that depend both on time and radial coordinate. The study is specialized to LTB cosmological model whose solution is represented in parametric form. All field equations are then exactly integrated. The solution depends on two arbitrary radial functions relative to the integration of the cosmological model. Some comments on the Weyl spinor are discussed.

Keywords: Massless field equation - LTB space time - LTB Cosmology - Solution of equations

1Address: Dipartimento di Fisica, Via Celoria, 16 Milano - Italy
1 Introduction

On account of the results obtained by [3, 6, 7] the formulation of spin field equations in general relativity has to take into account consistency conditions. They arise from space time curvature. The effect works both for massive and massless field equations (See, e.g., [4]).

In the language of two components spinors, the massless field is described by a field equation subjected to algebraic constraint:

\begin{align}
\nabla_{AA'}\phi^A_{A_1A_2...A_n} &= 0, \\
\phi_{AA_1A_2...A_n} &= \phi_{(AA_1A_2...A_n)} \\
(n-1)\phi_{AA_1M(A_2A_3...A_{n-1}} \Psi^{A_1M}_{A_n)} &= 0, \quad n \geq 1
\end{align}

The scheme describes fields of massless particles of spin \( s = (n+1)/2 \), \( s \geq 1 \) and it is assumed to be suitable for both bosons and fermions. The algebraic constraint equation strongly reduces the number of non zero independent components of \( \phi \). The equation have been widely discussed on general grounds (See [7] and References therein and in particular Ref.[2]). Solution in Kerr metric has been given in [1] for \( s = 3/2 \) and in [11] for arbitrary spin value with generalization to type D metric. In Lemaitre-Tolman-Bondi (LTB) cosmology they have been studied for \( s = 3/2, 2 \) in [9].

Here the eqs. (1), (2) are studied in the type D LTB metric. The choice is based on the fact that such metric is time dependent and on the fact that it is the base for the well known associated cosmological model [5]. The study is first performed in the LTB space time for arbitrary spin value. In that context some equation remain however suitable for further investigation. By assuming the space time background to be that of the LTB cosmological model, all equations, both for fermions and bosons, are exactly integrated by separation procedure and the final results depend only on the two integration functions of the cosmological model.

2 Assumptions and preliminary results

The equation (1) can be expanded in a general space-time in terms of the directional derivatives \( D = l^i \partial_i, \Delta = n^i \partial_i, \delta = m^i \partial_i, \delta^* = (m^i)^* \partial_i \) with \( \{l^i, n^i, m^i, m^*\} \) the null tetrad frame, and in terms of the spin coefficients [7]. The notation \( \phi_h \equiv \phi_{AA_1A_2...A_n} \) iff \( A + A_1 + A_1A_1... + A_1 = h, \ h = 0, 1, 2, ..., (n+1) \), is useful for symmetric spinors.

In a type D space time the spin coefficients \( \kappa, \sigma, \lambda, \nu \) vanish as well as it happens for the components of the Weyl spinor except \( \Psi_2 \). Accordingly the expanded equation (1) can be compactly written [11]

\[
\left[ D - (n - 1 - 2j)e - (n + 1 - j)\rho \right] \phi_{j+1} -
\]
Massless field equations in comoving spherically symmetric metric and ...

\[
-\left[\delta^* + (j + 1)\pi - (n + 1 - 2j)\alpha\right]\phi_j = 0
\]

\[
[\Delta + (j + 1)\mu - (n + 1 - 2j)\gamma]\phi_j - \left[\delta - (n + 1 - j)\tau + (2j - n + 1)\beta\right]\phi_{j+1} = 0, \quad j = 0, 1, 2, \ldots, n
\]  

Moreover, by making explicit the summation, from eq. (2) one has \([11, 9]\):

i) \(n\) is an even number (fermion) then \(\phi_h = 0\), \(h \neq 0, n + 1\)

ii) \(n\) is an odd number (boson) then \(\phi_h = 0\), \(h \neq 0, \frac{n + 1}{2}, n + 1\).

Therefore, independently of the spin value, the equations to be solved concern two field components in case of fermion field and three in case of boson field.

The equations relative to \(\phi_0\) and \(\phi_{n+1}\) are, for both fermions and bosons,

\[
[(\delta^* + \pi - (n + 1)\alpha)\phi_0 = 0, \quad [\Delta + \mu - (n + 1)\gamma]\phi_0 = 0
\]

\[
[D + (n + 1)\epsilon - \rho]\phi_{n+1} = 0, \quad [\delta + (2 - n)\tau + (n + 1)\beta]\phi_{n+1} = 0
\]  

In boson case, \(n\) odd, \(n + 1 = 2s\) there are also the equations relative to \(\phi_s\):

\[
[\Delta + (s + 1)\mu]\phi_s = 0, \quad [D - (s + 1)\rho]\phi_s = 0
\]

\[
[\delta - (1 + s)\tau]\phi_s = 0, \quad [\delta^* + (s + 1)\pi]\phi_s = 0
\]

All equations are of first order and in one only \(\phi\)-component.

3 LTB space-time

The equations (5-8) are now studied in the LTB space-time whose metric tensor \(g_{\mu\nu}\) is given by [5]

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - \exp(\Gamma(t,r)) dr^2 - Y^2(r,t) \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)
\]  

The following calculations are based on the null tetrad frame \(\{l^i, n^i, m^i, m^i\}^*\) introduced in [8]. Correspondingly the directional derivatives and the non zero spin coefficients are \((\dot{X} = \partial_t X, X' = \partial_t X)\):

\[
D = \frac{1}{\sqrt{2}}(\partial_t + e^{-\Gamma/2}\partial_r), \quad \Delta = \frac{1}{\sqrt{2}}(\partial_t - e^{-\Gamma/2}\partial_r)
\]

\[
\delta = \frac{1}{Y\sqrt{2}}(\partial_\theta + \frac{i}{\sin \theta}\partial_\varphi), \quad \delta^* = \frac{1}{Y\sqrt{2}}(\partial_\theta - \frac{i}{\sin \theta}\partial_\varphi)
\]

\[
\rho = -\frac{1}{Y\sqrt{2}}(\dot{Y} + Y' e^{-\Gamma/2}), \quad \mu = \frac{1}{Y\sqrt{2}}(\dot{Y} - Y' e^{-\Gamma/2})
\]

\[
\beta = -\alpha = \frac{1}{2Y\sqrt{2}} \cot \theta, \quad \epsilon = -\gamma = \frac{1}{4\sqrt{2}} \dot{\Gamma}
\]

With such assumptions and by also setting \(\phi_0 = e^{im\varphi} \phi_0(t, r, \theta), \quad m = 0, \pm 1, \pm 2, \ldots\)

the first equation in (5) read:

\[
(\partial_\theta + \frac{m}{\sin \theta})\phi_0 + \frac{n + 1}{2} \cot \phi_0 = 0
\]
with solution
\[ \phi_0(t, r, \theta) = |\sin \theta|^{-\frac{n+1}{2}} \left( \tan \frac{\theta}{2} \right)^{-m} G_0(t, r) \] (15)

\( G_0(t, r) \) the arbitrary integration function to be determined by substitution in the second equation (5).

Similarly by setting \( \phi_{n+1}(t, r, \theta, \varphi) = e^{im'\varphi} \phi_{n+1}(t, r, \theta), \) \( m' = 0, \pm 1, \pm 2, \ldots \), from the second equation (6) one obtains
\[ \phi_{n+1}(t, r, \theta) = e^{im'\varphi} \left( \tan \frac{\theta}{2} \right)^{m'} G_{n+1}(t, r)|\sin \theta|^{-\frac{n+1}{2}} G_{n+1}(t, r), \] (16)

\( G_{n+1}(t, r) \) the integration function that must be determined by substitution in the first equation in (6). Therefore \( G_0(t, r), G_{n+1}(t, r) \) must be solutions of the equations
\[ \Delta(\log G_0 Y) = -\frac{n+1}{4\sqrt{2}} \dot{\Gamma}, \quad D(\log G_{n+1} Y) = -\frac{n+1}{4\sqrt{2}} \dot{\Gamma} \] (17)

For boson fields, \( \phi_s \) is determined from (7), (8) and by the fact that the spin coefficients \( \tau, \pi \) vanish. From (8) one has first:
\[ (\delta + \delta^*) \phi_s = (\delta - \delta^*) \phi_s = 0 \Rightarrow \partial_\theta \phi_s = \partial_\varphi \phi_s = 0 \Rightarrow \phi_s = \phi(t, r) \] (18)

Instead the equations (7) become
\[ \dot{\phi_s} + (s+1) \frac{\dot{Y}}{Y} \phi_s = 0, \quad \phi'_s + (s+1) \frac{Y'}{Y} \phi_s = 0 \] (19)

Therefore \( \phi_s = Y^{-(s+1)} G_s(r) \equiv Y^{-(s+1)} H_s(t) \) for every \( t, r \), or
\[ \phi_s = \frac{H_{s0}}{[Y(t, r)]^{(s+1)}} \] (20)

\( H_{s0} \) a numerical constant.

4 The LTB cosmological model

It is possible to integrate further the equations (17) by assuming the underlying space-time to be that of the LTB cosmological model. Such model is a solution of the Einstein equation in the metric (9) for a Universe filled with dust matter without pressure. The model is equivalently described by the equations
\[ e^\Gamma = \frac{Y'^2}{1+2E}, \quad \frac{\dot{Y}^2}{2} - \frac{M(r)}{Y} = E \] (21)

\[ M(r) = 4\pi G \int_0^r d(t, r) Y^2 Y' dr + M_0 \] (22)
$E(r), M(r)$ are arbitrary integration functions of the cosmological equations; $d(t, r)$ the density of the dust matter, and $M_0$ a constant. [The form (22) is possible because the structure of the cosmological equations imply $\partial_t (YY''d) = 0$ (e.g. [10]).] As it follows from the expression of the spin coefficients (12), (13), the LTB metric (9) is of type D. Therefore all components of the Weyl spinor vanishes except $Ψ_2$. In the LTB cosmological model one has [9]:

$$Ψ_2 = \frac{1}{4} \left[ \frac{M(r)}{Y^3} - \frac{4}{3} \pi G d(t, r) \right]$$

(23)

The solution of the cosmological LTB model can be represented in the parametric form [5]:

$$Y = \alpha y(r) \xi(t), \quad \alpha = \left(\frac{9}{2}\right)^{\frac{1}{3}}, \quad y = M^{\frac{1}{3}}, \quad \xi = t^{\frac{3}{2}} \quad (E = 0)$$

(24)

$$Y = G \frac{M}{2E} (\cosh η - 1), \quad η > 0$$

$$t - t_0(r) = G \frac{M}{(2E)^{\frac{1}{2}}} (\sinh η - η) \quad (E > 0)$$

(25)

$$Y = G \frac{M}{2|E|} (1 - \cos η), \quad 0 \leq η \leq 2\pi$$

$$t - t_0(r) = G \frac{M}{(2|E|)^{\frac{1}{2}}} (\eta - \sin η) \quad (E < 0)$$

(26)

(In case $E = 0$ it has been chosen $t_0(r) = 0$). By the cosmological solution it is possible to integrate the equations (17).

**Case** $E = 0$. By denoting $G_- = G_0(t, r), G_+ = G_{n+1}(t, r)$, the eqs. (17) become:

$$\frac{\dot{G}_\pm}{G_\pm} + \frac{\dot{Y}}{Y} \pm \frac{1}{2} \left(\frac{G_\pm'}{G_\pm} + \frac{Y'}{Y}\right) = -\frac{n + 1}{2} \frac{\dot{Y}'}{Y'}$$

(27)

The equations can be separated by further setting $G_\pm = g_\pm(r) T_\pm(t)$. One gets

$$\frac{\dot{T}_\pm}{T_\pm} \xi + \frac{n + 3}{3} \frac{\xi}{t} = K_\pm$$

(28)

$$\frac{g_\pm'}{g_\pm} \frac{1}{\alpha y} + \frac{1}{\alpha y} = \mp K_\pm$$

(29)

The final solutions are then:

$$G_\pm(t, r) = D'_{\pm} \frac{M^{\frac{1}{3}}}{t^{\frac{3}{2} + \alpha}} \exp \left[ K_\pm \left(3t^{\frac{3}{2}} \mp \alpha M^{\frac{1}{3}}\right)\right]$$

(30)
$K_\pm$ the separation constant, $D_\pm$ the integration constant.

**Case** $E \neq 0$. The equation (17) can be solved by variable separation in a unified manner for both $E > 0$ and $E < 0$. This is possible by the change of variables $(t, r) \rightarrow (\eta, r)$ where:

$$
\tau = (2|E|)^{\frac{1}{2}} \int \frac{dt}{Y} = \eta \quad (\dot{\tau} = \partial \tau / \partial t = (2|E|)^{\frac{1}{2}} / Y)
$$

(31)
a result that follows from (25) (26). Indeed by noting that the solutions (25), (26) are of the form $Y(t, r) = y(r)\xi(\eta)$, by setting again $G_+ = G_0(t, r), G_- = G_{n+1}(t, r)$, $G_\pm = g_\pm(r)T_\pm(\eta)$, the eqs. (17) give successively ($\dot{\cdot} = \partial / \partial \eta$):

$$
\dot{G}_\pm + \frac{Y'}{Y} \pm \frac{\sqrt{1+2E}}{G_\pm} \left( \frac{G'_\pm}{G_\pm} + \frac{Y'}{Y} \right) = -\frac{n+1}{2} \frac{(2|E|)^{\frac{1}{2}} \partial \xi / \partial r Y}{Y}
$$

(32)

$$
\frac{\dot{T}_\pm}{T_\pm} + \frac{n+3}{2} \frac{\dot{\xi}}{\xi} \pm \frac{\sqrt{1+2E}}{2|E|} \left( \frac{g'_\pm y'}{g_\pm y} + 1 \right) = 0
$$

(33)
The last equation can be separated to get:

$$
\frac{\dot{T}_\pm}{T_\pm} + \frac{n+3}{2} \frac{\dot{\xi}}{\xi} = H_\pm
$$

(34)

$$
\sqrt{\frac{1+2E}{2|E|}} \left( \frac{g'_\pm y'}{g_\pm y} + 1 \right) = \mp H_\pm
$$

(35)

whose integration gives

$$
T_\pm(\eta) = C_\pm \frac{e^{H_\pm \eta}}{[\xi(\eta)]^{\frac{n+3}{2}}}
$$

(36)

$$
g_\pm = \frac{D_\pm}{y(r)} \exp \left[ \mp \int \frac{y'}{y} \sqrt{\frac{2|E|}{1+2E}} dr \right]
$$

(37)

$K_\pm$ the separation constant, $C_\pm, D_\pm$ the integration constants.

**Case** $E > 0$. From (25) one has: $y(r) = G \frac{M}{2E}, \xi(\eta) = \cosh \eta - 1$. Therefore

$$
g_\pm(r) = 2 \frac{D'_\pm E(r)}{GM(r)} \exp \left[ \mp H_\pm \int \sqrt{\frac{2|E|}{1+2E}} \left( \frac{E'}{E} - \frac{M'}{M} \right) dr \right]
$$

(38)

$$
T_\pm = C_\pm \frac{e^{H_\pm \eta}}{[\cosh \eta - 1]^{\frac{n+3}{2}}}
$$

(39)

**Case** $E < 0$. From (26) $y(r) = G \frac{M}{2E}, \xi(\eta) = 1 - \cosh \eta$. Therefore:

$$
g_\pm(r) = 2 \frac{D'_\pm |E(r)|}{GM(r)} \exp \left[ \mp H_\pm \int \sqrt{\frac{2|E|}{1+2E}} \left( \frac{M'}{M} - \frac{E'}{E} \right) dr \right]
$$

(40)
\[ T_\pm = C_\pm \frac{e^{\pm H_\pm \eta}}{[1 - \cos \eta]^{n+1/2}} \]  

(41)

The solution for \( \phi_s(r, \eta) \) relative to the LTB cosmological model follows by using in (20) the expression \( t = t(r, \eta) \) of (25), (26).

One can note that by choosing \( M \) to be the function of \( E \) given by:

\[ \sqrt{\frac{2|E|}{1 + 2E}} \left( \frac{M'}{M} - \frac{E'}{E} \right) = \frac{1}{v_0}, \]

(42)

and by further setting \( D_- = D_+ \), \( C_- = C_+ \), the solutions \( G_\pm \) become of the form

\[ G_\pm = F(\eta, r) \exp \left[ \frac{H_\pm}{v_0} (v_0 \eta \mp r) \right] \]

(43)

By combining eqs. (43) and (15), (16) one finds then that \( \phi_0, \phi_n+1 \) propagate radially (in the \( \eta \) parameter) in the opposite direction. Such an aspect is also possible in Kerr metric with the usual time parameter \( t \) [11].

5 Comments

In the previous Sections the massless field equations have been considered in the LTB space time and then in the special case of the LTB cosmology. The space being of type D, the number of nonzero field components is strongly reduced and one has to distinguish between fermions and bosons. Moreover, contrarily to the case of massive field equations, anyone of the expanded field equations involves one only component of the field. This appears both from eqs. (5)-(8) and from the explicit solutions in the previous Section. In particular there is the particular situation that every component of the field evolves independently. This could be an interesting property that however cannot be checked because there do not seem to be an experimental evidence of massless particles of higher spin.

Finally, from a theoretical point of view there is a general relation, for \( s = 2 \), between \( \phi_2 \) and \( \Psi_2 \) [2] that can reduce to a proportionality relation as in Kerr metric (e., g., [11]). In the present situation, however, in general \( \phi_2 \not\propto \Psi_2 \) (see (20) and (23)). If however \( M(r) = \text{constant} \), then \( M' = YY'd = 0 \) and \( d(t,r) = 0 \). Hence \( M = M_o \) and from (23), (20) \( \phi_2 \propto \Psi_2 \) and the cosmological model is not trivial, the function \( E(r) \) being still arbitrary.

References

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