On the Formulation of Space-like Wave Equations

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Abstract

This study investigates the formalism of conjugate reference frames to formulate space-like wave equations without introducing unphysical quantities. This approach allows us to reduce the transformation between superluminal reference frames to a standard Lorentz transformation between ordinary reference frames, circumventing modifying the mathematical apparatus on which the theory of relativity is based. The formulation of a generalised tachyonic wave equation for particles with an arbitrary spin is also investigated.

Keywords: Extended Lorentz transformations, Tachyons, Tachyonic Klein-Gordon equation, Tachyonic Dirac equation.

1 Introduction

Text The physics of tachyons is an intriguing and fascinating subject that has attracted the attention of physicists in both the pre-relativistic [1–3] and post-relativistic era [4–6]. The formulation of a comprehensive theory of tachyons dates back to the 1960s and the pioneering works of Sudarshan and Feinberg, which catalysed the study of superluminal phenomena in both classical and quantum physics [7–11]. Recently, the theory of tachyons is again centre stage in particle physics, with the hypothetical superluminal behaviour of the neutrino [12–16], as well as in quantum optics [17–18].

In this study, the theory of tachyons is formulated via conjugate reference frames. This method has previously been investigated by other researchers [9, 19], but without sufficient depth. This approach permits addressing space-like Lorentz transformations within the framework of the usual theory of relativity, circumventing modifying its mathematical apparatus. This method works by permuting the spatial components of relativistic quantities using temporal quantities, and vice versa. In this study, it is proved that this approach allows all
relativistic wave equations of massive particles to be extended to superluminal motions with simplicity.

The article is organised as follows: in Section 2, the conjugate reference frame is defined, highlighting its physical and geometrical meaning. In Section 3, an investigation of the equivalence between superluminal and subluminal reference frames and the consequent implications on the structure of the main equations of classical dynamics are presented. The formulation of the Klein-Gordon and Dirac equations for a massive tachyonic particle is presented in Section 4. Finally, a generalisation of the construction of a wave equation for space-like particles with an arbitrary spin is presented in Section 5.

2 Conjugate Reference Frames

Two reference frames $\mathcal{O}$ and $\mathcal{O}^*$ are defined as conjugated if their relative velocity is infinite. In this study, the asterisk (*) is used to denote a quantity associated with a conjugate reference frame, while the tilde (~) denotes a quantity associated with a superluminal reference frame.

Let be $\bar{u} > c$ be the velocity of the reference frame $\mathcal{O}$ in relation to a given subluminal reference frame $\mathcal{O}_0$. Then its conjugate velocity is $\bar{u}^* = c^2/\bar{u}$. To prove this statement, we need to recall the generalised addition formula for relativistic velocities [19]

$$w = \frac{\bar{u} + v}{1 + \bar{u}v/c^2}$$

(1)

where $v$ is the relative velocity between the reference frames $\mathcal{O}$ and $\mathcal{O}^*$. Computing the limit $v \to \infty$ of function (1), we obtain $w \to c^2/\bar{u}$, which is the conjugate velocity $\bar{u}^*$.

The formalism of conjugate reference frames allows us to extend the theory of relativity to superluminal motions in a simple manner. A space-like Lorentz transformation between two reference frames $\mathcal{O}$ and $\mathcal{O}$ is equivalent to an ordinary Lorentz transformation between subluminal reference frames $\mathcal{O}$ and $\mathcal{O}^*$. Next, we explore what it means, from a geometrical point of view, to transform a reference frame $\mathcal{O}$ into its conjugate $\mathcal{O}^*$. Thus, we consider the distance between two events connected by a space-like transformation [19] (for simplicity, we assume motion along the z-axis)

$$s^2 = c^2 \bar{t}^2 - \bar{z}^2 < 0$$

(2)

where the chosen coordinates of the first event are $(0, 0)$. Eq. (2) is a two-fold two-dimensional hyperboloid. Therefore, the transformation from $\mathcal{O}$ to $\mathcal{O}^*$ comprises a reflection of the coordinate axis relative to the asymptotes of the hyperboloid in Eq. (3). This reflection transforms the temporal coordinates $ct$ to spatial coordinates, or vice versa, without needing to compute the Lorentz matrix $\Lambda$ (which transforms the reference frame $\mathcal{O}$ into $\mathcal{O}$). Essentially, a superluminal Lorentz transformation $\Lambda$ can be reduced to an ordinary Lorentz transformation $\Lambda$ by performing the approp-
riate reflections relative to the asymptotes. This formalism yields a new approach to formulating the quantum theory within the framework of the generalised theory of relativity without complicating the mathematical apparatus on which the theory of the ordinary relativity is based.

3 Classical Tachyon Dynamics

Let us consider two conjugate reference frames $\mathcal{O}$ and $\mathcal{O}^*$ with velocities $\tilde{u} > c$ and $\tilde{u}^* = (c^2/\tilde{u}) < c$, respectively. Assuming the extended theory of relativity proposed by Sudarshan holds, then the respective Lorentz factors are given by

$$\gamma = \left(1 - \frac{\tilde{u}^2}{c^2}\right)^{-1/2} = \left(1 - \frac{c^2}{\tilde{u}^*^2}\right)^{-1/2} \quad \text{and} \quad \gamma^* = \left(1 - \frac{\tilde{u}^*^2}{c^2}\right)^{-1/2}$$

From Eq. (3), we see that $\gamma$ is purely imaginary, while $\gamma^*$ is real. Comparing $\gamma$ and $\gamma^*$, we see that they are connected by the following relation

$$\gamma = -i\beta \gamma^* \quad \text{with} \quad \beta = \tilde{u}^*/c$$

where $\beta$ is the usual relativistic factor. That $\gamma$ is purely imaginary implies that the mass of a particle moving with superluminal velocity must be imaginary. This guarantees the reality of all other dynamical quantities, a property necessary for the dynamical quantities to have physical meaning. We are now able to determine the relations between the energy and momentum measured by observers in $\mathcal{O}$ and $\mathcal{O}^*$.

Therefore, we calculate

$$E = \gamma (im)c^2 = -i\beta \gamma^*(im)c^2 = (\gamma^* m\tilde{u})c = p^*c$$

Eq. (5) states that the energy of the superluminal particle in $\mathcal{O}$ corresponds to the kinetic term in $\mathcal{O}^*$. Because energy and momentum, respectively, are the temporal and spatial components of the four-vector equation, $\mathbf{P} = (E/c, \mathbf{p})$, we can assume that the relation between $\mathcal{O}$ and $\mathcal{O}^*$, given by Eq. (5), comprises changing temporal and spatial quantities. This confirms what we established in Section 2. Let us now calculate

$$E^* = \gamma^* mc^2 = i\gamma mc^2 / \beta = \gamma (im)c^2 / \tilde{u}^* = \gamma (im)\tilde{u}^*c = \tilde{p}c$$

Predictably, Eq. (6) is the reciprocal of the relation given by Eq. (5), i.e., the energy of the subluminal particle corresponds to the kinetic term in $\mathcal{O}$. Thus, we can conclude that there is perfect reciprocity between a superluminal particle and its corresponding subluminal conjugate counterpart.

Next, we examine how the form of the relativistic invariant $E^2$ changes on passing from a subluminal reference frame to a conjugated superluminal one. Therefore, we calculate
where the second equation in Eq. (7) is obtained by substituting $E^*$ with $\tilde{p}c$ and $\tilde{p}^*$ with $\tilde{E}/c$ in the first equation, based on $E^* = \tilde{p}c$ and $\tilde{p}^* = \tilde{E}/c$ as derived from Eq. (5) and Eq. (6). The second equation in Eq. (7) is the typical energy-momentum relation of a tachyon but has been obtained from the ordinary relation by substituting the temporal quantity in $\tilde{\mathcal{O}}^*$ with the spatial quantity in $\tilde{\mathcal{O}}$, and vice versa, without needing to introduce an imaginary mass. Thus, the problem of the imaginary mass (which has no meaning in the real world) is solved, and the formalism of the theory of tachyons is recovered within the framework of ordinary physics. It can be proven that every relativistic invariant can be transformed into its superluminal counterpart by simply permuting the temporal and spatial terms, without need for an imaginary mass or other unphysical quantities.

In conclusion, to formulate a theory of tachyons by extending the ordinary theory of relativity [4–5, 9], then we must introduce imaginary quantities. However, to preserve the algebraic structure of the ordinary theory of relativity, the theory of tachyons is formulated using the same equations of subluminal particles, with the temporal components substituted with the spatial components.

4 Tachyonic Klein-Gordon and Dirac Equations

In ordinary quantum mechanics, the Klein-Gordon equation is obtained by performing the substitutions $E \rightarrow i\hbar \partial_t$ and $p \rightarrow -i\hbar c \partial_r$ on the classical energy-momentum relation. Based on the preceding sections, it is expected that the tachyonic Klein–Gordon equation can always be obtained from the ordinary energy–momentum relationship by substituting the operators. Therefore, we can formulate

$$\tilde{E} \rightarrow -i\hbar c \partial_r \text{ and } \tilde{p} \rightarrow i\hbar \partial_t$$

Thus, by performing the substitutions in Eq. (8) in the relation $E^2 = p^2c^2 + m^2c^4$, we obtain

$$\left(h^2 \partial_t^2 - h^2c^2 \partial_r^2 - m^2c^4\right)\varphi(t,r) = 0$$

which is the tachyonic Klein-Gordon equation usually formulated by assuming a particle with an imaginary mass in Eq. [7], an unphysical assumption that is not required in the theory of tachyons being investigated.

Next, we progress to the formulation of the tachyonic Dirac equation, which is complicated because we cannot perform a direct substitution with the operators given by Eq. (8) in the ordinary Dirac equation. Let us formulate the tachyonic Dirac equation as follows

$$\left(i\hbar \partial_t + i\hbar cA^\mu \partial_\mu - Bmc^2\right)\varphi(t,r) = 0$$
where \( \mu = 1, 2, 3 \), and \( A^\mu \) and \( B \) are \( 4 \times 4 \) matrices to be determined. Because standard \( \varphi(t, r) \) is a two-spinor and not a scalar function (as occurs for the tachyonic Klein-Gordon equation), to be covariant, the square of Eq. (10) must return the tachyonic Klein-Gordon equation. Performing the calculation, we obtain

\[
\begin{align*}
\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 c^2 \left( \sum_{i, j=1}^{3} \frac{A_i A_j + A_j A_i}{2} \frac{\partial^2}{\partial x_i \partial x_j} \right) + B^2 m^2 c^4 & \\
\quad - i \hbar c^3 m \left( \sum_{i=1}^{3} (A_i B + BA_i) \frac{\partial}{\partial x_i} \right) \varphi(t, r) = 0
\end{align*}
\]

Forcing Eq. (11) to be equal to Eq. (9), the following constraints are obtained

\[
(A_i B + BA_i) = 0; \quad (A_i A_j + A_j A_i) = 0 \text{ for } i \neq j; \quad B^2 = -1; \quad A^2 = 1 \tag{12}
\]

Using Eq. (12), setting \( \Gamma^0 = B \) and \( \Gamma^\mu = BA^\mu \), and considering a motion along the \( z \)-axis for simplicity, we obtain

\[
\Gamma^0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad \Gamma^3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{13}
\]

Compared to the ordinary Dirac matrix \( \gamma^0 \), \( \Gamma^0 \) has no components on the principal diagonal, and of the non-zero values, two are positive and two negative. Similarly, unlike the Dirac matrix \( \gamma^3 \), the non-zero components of \( \Gamma^3 \) lie on the principal diagonal, two of which are positive and two negative. The difference between these two matrices is due to the role substitution of energy operators with impulse operators, as established in the preceding sections. By explicitly formulating the ordinary and tachyonic Dirac equations (the tachyonic Dirac equation is obtained by substituting the matrices in Eq. (13) into the ordinary Dirac equation), we obtain

\[
\begin{align*}
\left( \begin{array}{cccc}
\hbar \partial_t - mc^2 & 0 & i \hbar c \partial_z & 0 \\
0 & \hbar \partial_t - mc^2 & \frac{i}{\hbar c} \partial_z & \frac{1}{\hbar c} \\
-\frac{i}{\hbar c} \partial_z & \frac{i}{\hbar c} \partial_z & \hbar \partial_t - mc^2 & 0 \\
0 & -\frac{i}{\hbar c} \partial_z & 0 & \hbar \partial_t - mc^2
\end{array} \right) \varphi = 0
\end{align*}
\]

\[
\frac{1}{\hbar} \left( \begin{array}{cccc}
\hbar \partial_t - mc^2 & 0 & i \hbar c \partial_z & 0 \\
0 & \hbar \partial_t - mc^2 & \frac{i}{\hbar c} \partial_z & \frac{1}{\hbar c} \\
-\frac{i}{\hbar c} \partial_z & \frac{i}{\hbar c} \partial_z & \hbar \partial_t - mc^2 & 0 \\
0 & -\frac{i}{\hbar c} \partial_z & 0 & \hbar \partial_t - mc^2
\end{array} \right) \tilde{\varphi} = 0 \tag{15}
\]

Evidently, the matrices \( \gamma^3 \) and \( \Gamma^3 \) differ in the reversed positions of operators \( i \hbar \partial_t \) and \( -i \hbar \partial_z \), confirming the formalism introduced in Section 2. Again, using an imaginary mass was unnecessary. A more detailed analysis shows that

\[
\Gamma^0 = \gamma^5 \gamma^0 \quad ; \quad \Gamma^3 = \gamma^5 \gamma^3 \tag{16}
\]
Therefore, the tachyonic Dirac equation for motion along any direction becomes

\[(i\hbar \gamma^5 \gamma^0 \partial_t - i\hbar c \gamma^5 \gamma^\mu \partial_\mu - mc^2)\varphi(t, r) = 0\]  \hspace{1cm} (17)

This is the Chodos equation \[20\] (a Dirac-like tachyonic equation obtained from the Tanaka Lagrangian \[21\]) used to study the superluminal behaviour of the neutrino. The approach proposed in this study allows us to achieve the same result in a simpler and more direct manner. Before generalising this result, it is noteworthy that the matrix \(\gamma^5\) transforms the spin operators by moving the external non-zero elements on the principal diagonal, and \textit{vice versa}. Because the off-diagonal elements in the ordinary Dirac equation are associated with impulses and those on the diagonal with the energy, we can conclude that the action of \(\gamma^5\) substitutes spatial components with temporal components.

5 Generalized tachyonic wave equations

Following the same approach used to formulate the tachyonic Dirac equation, we can obtain a generalised tachyonic wave equation for particles with an arbitrary spin. For this purpose, we can use the Bhabha relativistic wave equation \[22\]

\[(i\hbar \gamma^0 \partial_t - i\hbar c \gamma^\mu \partial_\mu - \chi)\varphi(t, r) = 0\]  \hspace{1cm} (18)

where \(\chi\) is the mass energy term while \(\gamma^0\) and \(\gamma^\mu\) are spin matrices, which coincide with those of the Dirac only for half-integer spin particles. From Eq. (18), we obtain the Kemmer–Duffin equation for spin one particles \[23\] and the Rarita–Schwinger equation for spin three halves particles \[24\]. To formulate the corresponding generalised tachyonic wave equation we have to reformulate Eq. (18) using the new spin matrices. The explicit form of the matrix is, as usual, calculated by forcing the square equation to be equal to Eq. (9). However, in formulating the tachyonic Dirac equation, we have seen that these new matrices are nothing more than the product of a matrix operator that permutes the spatial and temporal components and the same spin matrices of the ordinary equation. This approach must also hold for the Bhabha equation because the same Dirac equation is merely a particular case of the Bhabha equation. Therefore, the tachyonic Bhabha-like equation can be formulated as

\[(i\hbar \Omega \gamma^0 \partial_t - i\hbar c \Omega \gamma^\mu \partial_\mu - \chi)\varphi(t, r) = 0\]  \hspace{1cm} (19)

where \(\Omega\) is a matrix whose structure acts to substitute the operatorial component \(i\hbar \partial_t\) with the operator \(-i\hbar c \partial_\mu\), and \textit{vice versa}. For half-integer spin particles, the matrix \(\Omega\) coincides with the matrix \(\gamma^5\).
6. Conclusion

In this study, a relativistic theory extended to superluminal motions has been formulated by exploiting the definition of conjugate reference frames. This formalism describes the superluminal Lorentz transformations within the framework of the ordinary theory of relativity without modifying the algebraic apparatus on which it is based. The mechanism of action employed in this theory involves permuting the temporal and spatial components of the relativistic quantities under consideration. Consequently, all the dynamical quantities are obtained without introducing unphysical variables, e.g., an imaginary mass. The power and elegance of this formalism are demonstrated in the reformulation of the quantum wave equations. In particular, the tachyonic equations that describe the dynamics of particles with non-zero spin maintain the same covariant structure as the ordinary equations, with energy and momentum operators that simply substitute each other. With this approach, the hypothesis that at least one of the neutrinos may have a superluminal behaviour is made less speculative because its study would still be within the framework of the ordinary relativistic theory of particles.

References


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