Wave Propagating on Cord in Static Viscous Fluid under Non-Gravitational Field

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Abstract

In this paper, a two-dimensional wave equation for a cord (thin rope) that describes waves propagating on a cord in static viscous fluid under the non-gravitational field is derived. There, the viscous drag force is assumed to be proportional to the reverse velocity of motion. To solve this equation for a particular solution, micro-amplitude approximation is also assumed. As a result, unimodal and kink solitary wave solutions with damped velocities for this equation are found. It is denoted that progressive waves in the present system stop in nearly finite time. In appendix, the reason why the phase shifts between incident and reflective waves on connected two cords are inverse to the case of light waves is explained.

Keywords: 2D wave equation for cord, viscous fluid, viscous drag force, unimodal solitary wave, kink wave, damped velocity of motion

1 Introduction

Recently, we studied multiple mode of sufficiently small rotational stationary waves on a cord under the non-gravitational field [2]. In a previous paper, [3]
we derived a two-dimensional wave equation for a cord (thin rope) under the non-gravitational field and also its unimodal solitary wave solution propagating on a cord. There, we found that the wave equation for a cord reduces to an ordinary linear wave equation under the micro-amplitude approximation. This wave equation has the d’Lambert’s solution [1]. And then, two solitary waves propagating in the opposite directions are stable against a collision [3],[4].

In the present paper, we study the wave equation for a cord in static viscous fluid under non-gravity. We assume that the viscous drag force is proportional to a reverse velocity of motion. And, we solve the wave equation of a cord to obtain a parametric formed solution under the micro-amplitude approximation. Brief outline of this paper is denoted as follows. In Sect. 2, we derive the cord wave equation in static viscous fluid. In Sect.3, we find the unimodal or kink solitary wave solution with damped velocity through assuming a parametric form of a solution expressed by sech or sine-Gordon kink function. The last section is devoted to discussion. There, we discuss the experimental problem concerning the present system. In appendix, we study why the phase shifts of the cord waves are inverse [5] compared with electromagnetic waves.

2 Wave Equation for Cord

First, we assume that only viscous drag force as an external force is exerted on a cord and that stretching and contraction of a cord are negligible. And, we let a cord lie along the $x$-axis and a wave propagate on the $xy$-plane. Here, we consider an infinitesimal portion of the cord in the interval $[s, s+ds]$, where $s$ is an arclength along the cord as shown in Fig. 1. And let $T$ be a norm of the tension vector.

![Figure 1: Infinitesimal portion of cord in $[s, s+ds]$](image)

We define the viscous drag force $F$ to be

$$F = -kv,$$  \hspace{1cm} (2.1)
where \( k \) is a friction coefficient per unit length and \( v \) velocity of motion. Then, the equation of motion for the cord under the non-gravitational field reads

\[
\sigma ds \frac{\partial^2 z}{\partial t^2} = T(s + ds)e^{i\theta(s+ds)} - T(s)e^{i\theta(s)} - kds \frac{\partial z}{\partial t},
\]  

(2.2)

where \( z = x + iy \), \( t \) is time, \( \sigma \) a linear density of the cord. Taking the limit \( ds \to 0 \), we have

\[
\sigma \frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial s} \left( T \frac{\partial z}{\partial s} \right) - k \frac{\partial z}{\partial t},
\]  

(2.3)

Here, we should remark that

\[
z_s = e^{i\theta},
\]  

(2.4)

\[
x_s^2 + y_s^2 = 1,
\]  

(2.5)

where \((\cos \theta, \sin \theta)\) is a unit tangential vector, and subscript \( s \) denotes partial differentiation with respect to \( s \) here and hereafter. Next, we assume that the tension value \( T \) is constant. Then, we have

\[
\frac{\partial^2 z}{\partial t^2} + \frac{k}{\sigma} \frac{\partial z}{\partial t} - \frac{T}{\sigma} \frac{\partial^2 z}{\partial s^2} = 0.
\]  

(2.6)

This is the 2D wave equation for a cord in static viscous fluid under the non-gravitational field.

### 3 Solitary Waves Propagating on Cord

We shall introduce a parametric form of the wave solution \((x(s,t), y(s,t))\) for Eq. (2.6), namely,

\[
x(s,t) = s + f(\xi),
\]  

(3.1)

\[
y(s,t) = g(\xi),
\]  

(3.2)

where,

\[
\xi = s - h(t),
\]  

(3.3)

From Eqs. (3.1) and (3.2), we have

\[
x_t = f_{\xi} h_t, \quad x_{tt} = f_{\xi} h_{tt}^2 + f_{\xi} h_{tt},
\]  

(3.4)

\[
y_t = g_{\xi} h_t, \quad y_{tt} = g_{\xi} h_{tt}^2 + g_{\xi} h_{tt},
\]  

(3.5)

\[
x_s = 1 + f_{\xi}, \quad x_{ss} = f_{\xi},
\]  

(3.6)

\[
y_s = g_{\xi}, \quad y_{ss} = g_{\xi}.
\]  

(3.7)
Substituting Eqs.(3.4)-(3.7) into Eq.(2.6), we obtain

\[ f_{\xi\xi} h_t^2 + f_\xi h_{tt} + \frac{k}{\sigma} f_\xi h_t - \frac{T}{\sigma} f_{\xi\xi} = 0, \]  
(3.8)

\[ g_{\xi\xi} h_t^2 + g_\xi h_{tt} + \frac{k}{\sigma} g_\xi h_t - \frac{T}{\sigma} g_{\xi\xi} = 0. \]  
(3.9)

Substituting Eqs.(3.6) and (3.7) into Eq.(2.5), we get

\[ f_\xi^2 + 2 f_\xi + g_\xi^2 = 0. \]  
(3.10)

Here, we assume that

\[ f_\xi \approx 0. \]  
(3.11)

We take Eq. (3.11) into account and neglect the second-order term, \( f_\xi^2 \) in Eq. (3.10), and then we have

\[ f_\xi = -\frac{1}{2} g_\xi^2, \quad f_{\xi\xi} = -g_\xi g_{\xi\xi}, \]  
(3.12)

\[ f(\xi) = -\frac{1}{2} \int^\xi g_\xi^2 du. \]  
(3.13)

Substituting Eq.(3.12) into Eq.(3.8) and from Eq.(3.9), we obtain

\[ g_\xi g_{\xi\xi} \left( h_t^2 - \frac{T}{\sigma} \right) + \frac{1}{2} g_\xi^2 \left( h_{tt} + \frac{k}{\sigma} h_t \right) = 0, \]  
(3.14)

\[ g_{\xi\xi} \left( h_t^2 - \frac{T}{\sigma} \right) + g_\xi \left( h_{tt} + \frac{k}{\sigma} h_t \right) = 0. \]  
(3.15)

These equations yield

\[ g_{\xi\xi} = 0, \quad \text{and} \quad h_{tt} + \frac{k}{\sigma} h_t = 0, \]  
(3.16)

or

\[ g_\xi = 0, \quad \text{and} \quad h_t^2 - \frac{T}{\sigma} = 0. \]  
(3.17)

### 3.1 Unimodal wave on cord

Now we are at the position to derive a unimodal solitary wave solution \((x, y)\) with a sufficiently small amplitude. Here, we assume that

\[ g(\xi) = \delta \text{sech}\xi, \quad (\delta \approx 0) \]  
(3.18)

This equation and Eq.(3.12) yield

\[ f_\xi = -\frac{\delta^2}{2} \tanh^2 \xi \text{sech}^2 \xi. \]  
(3.19)
Since $|\text{sech} \xi| \leq 1$ and $|\tanh \xi| < 1$, Eq. (3.11) holds. From Eqs. (3.13) and (3.19), we obtain

$$f(\xi) = -\frac{\delta^2}{6}(\tanh^3 \xi + C), \quad (3.20)$$

where $C$ is an arbitrary constant. Substituting Eq. (3.20) into Eq. (3.1), we have

$$x = s - \frac{\delta^2}{6}(\tanh^3 \xi + C), \quad (3.21)$$

Here, we neglect the $\delta^2$ term due to $|\tanh \xi| < 1$. Then $x = s$ holds.

At the end, we derive $h(t)$. Because of Eq. (3.18), the former equations in Eqs. (3.16) and (3.17) hold approximately. The latter equation of Eq. (3.17) yields the following result:

$$h(t) = ct + s_0, \quad (3.22)$$

$$c = \pm \sqrt{T/\sigma}, \quad (3.23)$$

where $s_0$ is an arbitrary constant. Accordingly, $\xi$ becomes $\xi = s - s_0 - ct$ in Eq. (3.3). Then, $x = s$ and Eq. (3.18) yield

$$y = \delta \text{sech}(x - s_0 - ct). \quad (3.24)$$

This solution is inadequate to the wave equation for a cord in static viscous fluid since the propagating velocity of the wave isn’t damped in spite of the friction force.

Next, the latter equation in Eq. (3.16) yields the following result:

$$h(t) = \frac{\sigma}{k} \{a \exp[-(k/\sigma)t] + b\}, \quad (3.25)$$

where $a$ and $b$ are arbitrary constants. Then, $x = s$ and Eq. (3.18) yield

$$y = \delta \text{sech} \left\{ x - s_1 - \frac{\sigma}{k} a \exp[-(k/\sigma)t] \right\}, \quad (3.26)$$

where $s_1 = b\sigma/k$. This solution is adequate since the velocity of motion is damped exponentially due to the friction force. We find that the larger the friction coefficient $k$ becomes and the smaller the linear density of a cord $\sigma$ becomes, the more rapidly the propagating velocity of the unimodal solitary wave is damped. The curve of the unimodal solitary wave solution for $\delta = 0.1$, $s_1 = 10.0$, $a = -5.0$, $\sigma = 3.0$, $k = 2.0$, is plotted in Fig. 2 when $t = 0, 2, 4, 6, \text{and} \ 8$. At the time $t = 6$ and $8$, the wave shapes nearly overlap. We can make sure that the progressive wave stops in practically finite time.

Here, the reason why we apply $\text{sech} \xi$ to Eq. (3.2) as $g(\xi)$ is because the unimodal shape of a wave propagating on the cord is one of the most simple shapes among various kinds of waves and because Eq. (3.13) can be integrated into the fundamental function.
3.2 Kink wave on cord

In this subsection, we derive a kink solitary wave solution \((x, y)\) with a sufficiently small amplitude. Here, we assume that

\[ g(\xi) = \delta \tan^{-1}[\exp(\xi)], \quad (\delta \approx 0) \]  

(3.27)

This equation and Eq.(3.12) yield

\[ f_\xi = -\frac{\delta^2}{2} \text{sech}^2 \xi. \]  

(3.28)

Since \(|\text{sech}\xi| \leq 1\), Eq.(3.11) holds. From Eqs.(3.13) and (3.28), we obtain

\[ f(\xi) = -\frac{\delta^2}{2} (\tanh \xi + C), \]  

(3.29)

where \(C\) is an arbitrary constant. Substituting Eq.(3.29) into Eq.(3.1), we have

\[ x = s - \frac{\delta^2}{2} (\tanh \xi + C), \]  

(3.30)

Here, we neglect the \(\delta^2\) term due to \(|\tanh \xi| < 1\), then \(x = s\) holds.

Regarding \(h(t)\), we make use of the result obtained in the previous subsection. Because of Eq.(3.27), the former equations in Eqs.(3.16) and (3.17) hold approximately. First, we refer to the case of the latter equation in Eq.(3.17):

\[ h(t) = ct + s_0, \]  

(3.31)

\(\xi\) becomes \(\xi = s - s_0 - ct\) in Eq. (3.3). Then, \(x = s\) and Eq.(3.27) yield

\[ y = \delta \tan^{-1}[\exp(x - s_0 - ct)]. \]  

(3.32)
This wave solution propagating at constant velocity is inadequate.

Second, we refer to the case of the latter equation in Eq. (3.16):

\[ h(t) = \frac{\sigma}{k} \{ a \exp[-(k/\sigma)t] + b \}, \quad (3.33) \]

\[ x = s \text{ and Eq.}(3.27) \text{ yield} \]

\[ y = \delta \tan^{-1} \left\{ \exp \left[ x - s_1 - \frac{\sigma}{k} a \exp \left( -\frac{k}{\sigma} t \right) \right] \right\}. \quad (3.34) \]

This solution is adequate since the velocity of motion is damped exponentially due to the friction force. The curve of the kink wave solution for \( \delta = 0.1, s_1 = 10.0, a = -5.0, \sigma = 3.0, k = 2.0, \) is plotted in Fig. 3 when \( t = 0, 2, 4, 6, \) and \( 8. \) At the time \( t = 6 \) and \( 8, \) the wave shapes nearly overlap.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Kink waves on cord.}
\end{figure}

### 4 Discussion

Regarding a cosmic experiment to verify the present theory of the unimodal or kink solitary wave with the damped velocity, it may be difficult to execute it under non-gravity due to the capacity problem of viscous fluid in a space station. Instead of it, we have tried to perform such an experiment on the earth. We have swung a water-wetted cotton cord horizontally in water with which a bathtub is filled. The reason why we swing a cord horizontally is to avoid influence of the gravity. Then, the cord has formed a unimodal shape, but it has hardly progressed at all. Probably, the linear density of the wet cord will be too small compared with the friction coefficient of water. We had better try a heavier chain than a wet cotton cord. We wonder if any material exists with an intermediate value of a viscous coefficient between water and air.
A Connected two cords with different linear densities

In Ref. [5], we have found that there occurs the inverse phase between the incident wave and the reflective wave at the boundary of the cords with the different linear densities $\sigma_1$ (incident) and $\sigma_2$ (transmitted). This fact is different from the case of electromagnetic waves. We try to explain this reason. First, we shall introduce two impedances $\eta_1$ (incident) and $\eta_2$ (transmitted), and a reflective coefficient $R_f$ and a transmission coefficient $T_r$ as follows

$$R_f = (\eta_2 - \eta_1)/(\eta_1 + \eta_2), \quad T_r = 2\eta_2/(\eta_1 + \eta_2),$$  \hspace{1cm} (A.1)

where $\eta = \sqrt{\mu/\epsilon} \propto \sqrt{1/\epsilon}$. Hence in case of light waves, when the medium density is large, the impedance $\eta_1$ becomes small and $R_f > 0$ (the identical phase). Here, the intensity $I$ of a plane wave is defined as $I = |E|^2/(2\eta) \propto \int v|E|^2 dx = (2/3)T\delta^2\alpha$ [5]. And, taking into account $|E|^2 \propto \delta^2$ and $\alpha = \sqrt{T/\sigma}$, [5] we have $\eta \propto \sqrt{\sigma}$. Accordingly in case of cord waves, when the linear density $\sigma_1$ is large, the impedance $\eta_1$ becomes large and $R_f < 0$ (the inverse phase). Regarding the phase shift between the incident wave and the reflective wave, thus we have been able to explain the reason of the opposite phenomena here between electromagnetic waves and cord waves.

References


Received: August 20, 2020; Published: September 3, 2020