The Magnetized Relativistic Boltzmann Equation with a Hard Potential

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Abstract

A global existence theorem and uniqueness of solution of the coupled spatially homogeneous relativistic Maxwell-Boltzmann system is proved, in a Bianchi type I spacetime back-ground, in a hard potential case. The proof relies in the use of a particular form of Povzner inequality.

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1. INTRODUCTION

One of the most important models which rules the dynamic of dilute charged particles is expressed by the coupled relativistic Maxwell-Boltzmann system, in which particles interact with themselves through collisions and with their self consistence electromagnetic field. The study excludes the case of fast moving particles of gas being submitted to binary collisions. We restrict the study to homogeneous case, which means that the unknown in the equation depends only on time and velocity variables. In [5], Noutchegueme and R. Ayissi have
studied the Maxwell-Boltzmann system, in a Bianchi type I spacetime, with a bounded scattering kernel. One of the purpose of this article is to extend this result in a more physically relevant situation.

The nature of collisions between particles is determined by the scattering kernel. In the relativistic setting, a classification of the scattering kernel into hard and soft potential has been proposed in [2, 9]. As in [7], we consider a scattering kernel of hard potential type, which is more physically relevant.

The relativistic Boltzmann equation rules the dynamic of the considered charged particles which are subject to collide with themselves, by determining their distribution function, denoted \( f \), a non-negative real valued function of both the time and the momentum of particles.

The Maxwell equations are the equations of electromagnetism and determine the electromagnetic field \( F \) created by the fast moving and charged particles. We consider the case where this field is generated by a Maxwell current defined by the distribution function, a charge density \( e \) and a future pointing unit vector \( u \), tangent at any point to the temporal axis.

The main objective of the present paper is to extend the results of [5] in two points. Firstly, we consider a hard potential case, instead of bounded kernel. Secondly, we remove the assumption that the initial datum of the Boltzmann equation is invariant under a subgroup of \( O_3 \). But for the sake of method we first consider a bounded kernel in section 3, in order to extend to the hard potential case after obtaining an existence theorem. The main tool here is a particular form of Povzner inequality.

The paper organises as follows:
- In section 2, we introduce the equations.
- In section 3, we study the bounded case.
- In section 4, we study the hard potential case.

2. THE MAXWELL-BOLTZMANN SYSTEM IN A BIANCHI TYPE I SPACETIME

2.1. Notations. In a time oriented Bianchi type I spacetime, we consider the collisional evolution of fast moving massive and charged particles and denote by \( x^\alpha = (x^0; x^i) \) the usual coordinates in \( \mathbb{R}^4 \), where \( t = x^0 \) represents the time and \( (x^i) \) the space, \( g \) is the metric tensor of Lorentzian signature \((-;++++)\) which writes:

\[
g = -dt^2 + a^2(t)(dx^1)^2 + b^2(t)((dx^2)^2 + (dx^3)^2),
\]

where \( a \) and \( b \) are two differentiable increasing functions on \( \mathbb{R}^+ \) such that:

\[
a \leq b, \quad a(0) = a_0 \geq \frac{3}{2}.
\]

The expression of the Christoffel symbols of the Levi-Civita connection \( \nabla \) of \( g \) is:
\[ \Gamma_{\alpha \beta} = \frac{1}{2} g^{\lambda \mu} [\partial_\alpha g_{\mu \beta} + \partial_\beta g_{\alpha \mu} - \partial_\mu g_{\alpha \beta}] . \]  

(2.3)

As in [5], we require that there exists a constant \( C > 0 \) such that:

\[ \left| \frac{1}{a} \frac{da}{dt} \right| \leq C, \quad \left| \frac{1}{b} \frac{db}{dt} \right| \leq C. \]  

(2.4)

The particles are statistically described by their distribution function, denoted \( f = f(x^\alpha, p^\alpha) \) in which \( (x^\alpha) \) is the position \( (p^\alpha) = (p^0, \vec{p}) \) is the the 4-momentum of the particle. So:

\[ f : T (\mathbb{R}^4) \cong \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^+, \quad (x^\alpha, p^\alpha) \mapsto f(x^\alpha, p^\alpha). \]  

(2.5)

On \( \mathbb{R}^3 \) a scalar product is defined by

\[ \vec{p} \cdot \vec{q} = a^2 p^1 q^1 + b^2 (p^2 q^2 + p^3 q^3). \]  

(2.6)

The trajectories \( s \mapsto (x^\alpha(s), p^\alpha(s)) \) of particles solve the differential system:

\[ \frac{dx^\alpha}{ds} = p^\alpha, \quad \frac{dp^\alpha}{ds} = P^\alpha := -\Gamma_{\lambda \mu}^\alpha p^\lambda p^\mu + ep^\beta F_\beta^\alpha \]  

(2.8)

where \( e = e(t) \) denotes the charge density of particles.

With the covariant variables, the distribution function \( f \) will be seen sometimes as a function of \( t \) and \( \vec{v} = (v^1, v^2, v^3) \) as in [4], instead of \( \vec{p} \), where:

\[ \begin{cases} 
  v^1 = a^2 p^1, & v^2 = b^2 p^2, & v^3 = b^2 p^3, \\
  v^0 = \sqrt{1 + a^{-2} (p^1)^2 + b^{-2} ((p^2)^2 + (p^3)^2)}. \end{cases} \]  

(2.9)

2.2. The Maxwell system in \( F \). The Maxwell system in \( F \) can be written as:

\[ \nabla_\alpha F^{\alpha \beta} = J^\beta, \quad \nabla_\alpha F_{\beta \gamma} + \nabla_\beta F_{\gamma \alpha} + \nabla_\gamma F_{\alpha \beta} = 0 \]  

(2.10)

where \( \nabla_\alpha \) stands for the covariant derivative in \( g \), \( J^\beta \) represents the Maxwell current whose local expression is given by:

\[ J^\beta = \int_{\mathbb{R}^4} \frac{p^\beta f(t, \vec{p}) (det g) \frac{1}{2} d\vec{p}}{p^0} - eu^\beta, \quad u^0 = 1, \quad u^i = 0 \]  

(2.11)
in which $u = \left( u^\beta \right)$ is a unit futur pointing timelike vector tangent to the time axis at any point.

The particles are then spatially at rest. Now the identity

$$\nabla_\alpha \nabla_\beta F^{\alpha \beta} = 0$$

imposes, given (2.10) that

$$\nabla_\alpha J^\beta = 0. \quad (2.12)$$

2.3. The Boltzmann equation in $f$. The Boltzmann equation in a Bianchi type I spacetime writes:

$$\frac{p^\alpha}{p^0} \frac{\partial f}{\partial x^\alpha} + \frac{P^i}{P^0} \frac{\partial f}{\partial p^i} = Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} v_\phi \sigma(k, \theta) (f' f'_s - f f_s) a b^2 d\omega d\bar{q} \quad (2.13)$$

where

$$v_\phi = \frac{k \sqrt{\delta}}{p^0 q^0}, \quad f' = f \left( t, p' \right), \quad f'_s = f \left( t, \bar{q} \right), \quad f = f \left( t, p \right), \quad f_s = f \left( t, q \right).$$

Here $Q$ is the collision operator, $v_\phi$ the M\(\ddot{u}\)ller velocity, $\sigma$ the scattering kernel, $\theta$ the scattering angle, $\delta$ and $k$ are given by

$$\delta = -(p_\alpha + q_\alpha) (p^\alpha + q^\alpha), \quad k = \sqrt{(p_\alpha - q_\alpha) (p^\alpha - q^\alpha)},$$

and are called total energy and relative momentum respectively. In the instantaneous, binary and elastic scheme, if $p, q$ and $p', q'$ stand for the two momenta before and after shock, the collision operator $Q$ is defined by:

$$Q(f, h) = Q_+(f, h) - Q_-(f, h), \quad f, h : \mathbb{R}^3 \to \mathbb{R} \quad (2.14)$$

$$Q_+(f, h) = \int_{\mathbb{R}^3} \int_{S^2} a b^2 f \left( p' \right) h \left( q' \right) v_\phi \sigma(k, \theta) d\bar{q} d\omega, \quad (2.15)$$

$$Q_-(f, h) = \int_{\mathbb{R}^3} \int_{S^2} a b^2 f \left( p \right) h \left( \bar{q} \right) v_\phi \sigma(k, \theta) d\bar{q} d\omega. \quad (2.16)$$

The energy momentum conservation is written as

$$p^0 + q^0 = p'^0 + q'^0, \quad \bar{p} + \bar{q} = \bar{p}' + \bar{q}' \quad (2.17)$$

As suggested in [4] and [8], we parametrize the post-collisional momenta as follows: $p^\alpha$ and $q^\alpha$ being given, we first consider

$$n^\alpha = p^\alpha + q^\alpha, \quad t^\alpha = (n_\omega^i, -n_\omega), \quad \omega \in S^2 \quad (2.18)$$

then, the post-collisional momenta are represented by:
\[ p'_{\alpha} = \frac{p^\alpha + q^\alpha}{2} + \frac{k}{2} \frac{t^\alpha}{\sqrt{t_\beta t^\beta}}, \quad q'_{\alpha} = \frac{p^\alpha + q^\alpha}{2} - \frac{k}{2} \frac{t^\alpha}{\sqrt{t_\beta t^\beta}}. \] 

(2.19)

They satisfy the mass shell condition and energy momentum conservation.

As shown in [1], the Jacobian of the change of variable \( \left( \vec{p}, \vec{q} \right) \mapsto \left( \vec{p}', \vec{q}' \right) \) is

\[ \frac{\partial (\vec{p}', \vec{q}')}{\partial (\vec{p}, \vec{q})} = \frac{p'^0 q'^0}{p^0 q^0}. \] 

(2.20)

2.4. **Assumptions on the scattering kernel.** In this work, we assume that the scattering kernel has the form

\[ \sigma (k, \theta) = k^3 \sin^\gamma \theta, \quad -2 < \gamma \leq -1, \quad 0 \leq \beta < \gamma + 2. \] 

(2.21)

Since \( \frac{k}{2} \) is a bounded, a kernel of this form falls into the hard potential case.

2.5. **The Maxwell-Boltzmann system in \( (F, f) \).** Setting \( \beta = 0 \) in the first equation (2.10), we easily deduce that

\[ J^0 = 0. \] 

(2.22)

By (2.22), the expression (2.11) of \( J^\beta \) with \( \beta = 0 \), \( u^0 = 1 \) gives:

\[ e (t) = \int_{\mathbb{R}^3} f (t, \vec{p}) \, ab^2 \, d\vec{p}, \] 

(2.23)

and shows that \( f \) determines \( e \).

The second set of the Maxwell equations is identically satisfied and the first set reduces to \( \partial F_{ij} = 0 \), so:

\[ F_{ij} = F_{ij}(0) := \varphi_{ij}. \] 

(2.24)

(2.24) means that the magnetic part \( F_{ij} \) does not evolve during time.

It remains to determine the electric part \( F^0 i := E^i \).

Writing (2.11) for \( \beta = i, \) using (2.10) for \( \alpha = 0 \) and (2.3) gives:

\[ J^i = \int_{\mathbb{R}^3} \frac{p^i f (t, \vec{p}) \, ab^2}{p^0} \, d\vec{p}, \] 

(2.25)

\[ \dot{E}^i + \Gamma^i_{0j} E^j = \int_{\mathbb{R}^3} \frac{p^i f (t, \vec{p}) \, ab^2}{p^0} \, d\vec{p}. \] 

(2.26)

Since \( f = f (t, \vec{p}) \), the Boltzmann equation (2.13) can be written:

\[ \frac{\partial f}{\partial t} + \frac{P^i}{P^0} \frac{\partial f}{\partial P^i} = Q (f, f). \] 

(2.27)
Still using the letter $f$ instead of $f^#$ usually used in the standard notation, solving the non linear PDE $(2.27)$ is equivalent to solve the characteristic system:

$$\begin{align*}
\frac{dt}{1} = \frac{dp^1}{p^1} = \frac{dp^2}{p^2} = \frac{dp^3}{p^3} = \frac{df}{Q(f,f)} = ds,
\end{align*}$$

which allows to take $t$ as parameter. Now we obtain from $(2.8)$ and $(2.3)$:

$$\frac{p^i}{p^0} = -2\Gamma^i_{0j}p^j - e \left[ F^0 + g^{ik}p^k \varphi \right], \quad i = 1, 2, 3. \quad (2.29)$$

Using relations $(2.23), (2.25), (2.28)$ and $(2.29)$, the Maxwell-Boltzmann system transforms into a Maxwell-Boltzmann-Momentum system of the form:

$$\begin{align*}
\frac{\dot{E}^i}{a} &= -\Gamma^i_{0j}E^j + \int_{\mathbb{R}^3} \frac{q^i f(t,\mathbf{q})ab^2d\mathbf{q}}{q^0} \quad (a) \\
\frac{\dot{p}^i}{a} &= -2\Gamma^i_{0j}p^j - \left[ E^i + g^{ik}p^k \varphi \right] \int_{\mathbb{R}^3} f(t,\mathbf{q})ab^2d\mathbf{q} \quad (b) \\
\frac{df}{dt} &= Q(f,f) \quad (c) \\
F_{ij} &= F_{ij}(0) = \varphi_{ij} \quad (d)
\end{align*}$$

$(2.30)$

Now $f$ and $\bar{p}$ are independent variables for the integro-differential system $(2.30)$.

The collision operator expresses in terms of covariant variables using $(2.9)$ as

$$Q(f,f)(t,v) = a^{-1}b^{-2} \int_{S^2} d\omega \int d\mathbf{q} v_\sigma(k,\theta) \left[ f(t,v')f(t,\mathbf{u}') - f(t,\mathbf{v})f(t,\mathbf{u}) \right]$$

and the Boltzmann equation $(2.27)$ becomes:

$$\frac{\partial f(t,v)}{\partial t} = Q(f,f)(t,v). \quad (2.31)$$

Now we introduce some useful functional spaces.

### 2.6. Functional spaces.

The framework for the distribution function $f$ is $L^1_r(\mathbb{R}^3)$, the subspace of $L^1(\mathbb{R}^3)$ whose norm, denoted $\| . \|_{1,r}$, $r \geq 0$ is defined by:

$$L^1_r(\mathbb{R}^3) = \left\{ f \in L^1(\mathbb{R}^3) : \| f \|_{1,r} = \int_{\mathbb{R}^3} |f(\mathbf{p})| (p^0)^r d\mathbf{p} < +\infty \right\}.$$

We will denote $\| . \|_{1,1}$ by $\| . \|$ and we define:

$$|f(t)|_{1,r} = \int_{\mathbb{R}^3} |f(t,\mathbf{v})| < \mathbf{v} >^r d\mathbf{v}, \quad < \mathbf{v} >= \sqrt{1 + |\mathbf{v}|^2}.$$

Consequently, we have:
Now, we set for $r \in \mathbb{R}$, $r > 0$:

$$X_r = \{ f \in L^1_1(\mathbb{R}^3), f \geq 0 \ \text{a.e.} \ | f | \leq r \}.$$  

(2.33)

$X_r$ is a complete and connected metric space for the induced norm. For any real interval $I$, we set:

$$C ([I; L^1_1(\mathbb{R}^3)]) = \{ f : I \rightarrow L^1_1(\mathbb{R}^3), f \text{ continuous and bounded} \},$$

$$C ([I; X_r]) = \{ f \in C ([I; L^1_1(\mathbb{R}^3)]), f(t) \in X_r, \forall t \in I \}.$$  

(2.34)

$C ([I; X_r])$ is a complete metric space for the induced norm. The framework for $\mathcal{E}$ and $E$ is $\mathbb{R}^3$, with the norm $\| \cdot \|$ or $\| \cdot \|_{\mathbb{R}^3}$.

$$C ([I; \mathbb{R}^3]) = \{ m : I \rightarrow \mathbb{R}^3, m \text{ continuous and bounded} \}$$

is a Banach space for the norm $\| m \| = \sup \{ \| m(t) \|, \ t \in I \}$.

We define on $\mathbb{R}^3 \times \mathbb{R}^3 \times L^1_1(\mathbb{R}^3)$ and on $C ([I; \mathbb{R}^3]) \times C ([I; \mathbb{R}^3]) \times C ([I; L^1_1(\mathbb{R}^3)])$:

$$\| (\mathcal{E}, f) \| = \| \mathcal{E} \| + \| f \|,$$

(2.35)

$$\| (\mathcal{E}, f) \| = \| \mathcal{E} \| + \| f \|.$$  

(2.36)

3. THE MAXWELL-BOLTZMANN SYSTEM WITH A BOUNDED KERNEL

For technical purpose, in this section we change the scattering kernel $k\sqrt{\delta \sigma}$ into a bounded kernel $S (\mathcal{E}, \mathcal{E}, \mathcal{E}, \mathcal{E})$, a non-negative continuous real valued function of its arguments, and on which we additionally require that:

$$\begin{cases}
0 \leq S (\mathcal{E}, \mathcal{E}, \mathcal{E}, \mathcal{E}) \leq C_1 \\
S (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4) - S (\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4) \leq C_1 \| \mathcal{E}_1 - \mathcal{E}_2 \|,
\end{cases}$$

(3.1)

where $C_1$ is positive constant. The Boltzmann equation (2.13) changes as:

$$\frac{\partial f}{\partial t} + \frac{P^i}{p^0} \frac{\partial f}{\partial p^i} = \frac{1}{p^0} \overline{Q} (f, f),$$

(2.32)

$$\overline{Q} (f; h) = \overline{Q}_+ (f; h) - \overline{Q}_- (f; h),$$

(2.33)

$$\overline{Q}_+ (f; h) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{ab^2}{q^0} f (\mathcal{E}) h (\mathcal{E}) S (\mathcal{E}, \mathcal{E}, \mathcal{E}, \mathcal{E}) d\omega d\eta,$$

(3.4)
\[
\bar{Q}^- (f, h) = \int_{\mathbb{R}^2} \int_{S^2} \frac{a b^2}{q^3} f (\bar{p}) h (\bar{q}) S (\bar{p}, \bar{q}, \bar{p}', \bar{q}') \, d\omega d\bar{q}.
\] (3.5)

The Maxwell-Boltzmann-Momentum system (2.30) transforms in:

\[
\begin{aligned}
\dot{E}^i &= -\Gamma^i_{0j} E^j + \int_{\mathbb{R}^3} \frac{q^i f(t, \bar{p}) ab^2}{q^j} d\bar{q} = H_1 (t, \bar{p}, \bar{E}, f) (a) \\
\dot{p}^i &= -2\Gamma^i_{0j} \dot{p}^j - \left[ E^i + \eta_{i}^j \eta_{j}^p \varphi_{p} \right] \int_{\mathbb{R}^3} f (t, \bar{q}) ab^2 d\bar{q} = H_2 (t, \bar{p}, \bar{E}, f) (b) \\
\frac{d}{dt} = \frac{1}{\bar{p}^i} \bar{Q} (f, f, \bar{p}) = H_3 (t, \bar{p}, \bar{E}, f) (c) \\
F_{ij} &= F_{ij} (0) = \varphi_{ij} (d).
\end{aligned}
\] (3.6)

3.1. **Local existence of solutions.** In what follows and in the next, we briefly review the results of [5].

First, we estimate the differences in \( f, E, \bar{p} \) in \( L_1^1 \) and \( \mathbb{R}^3 \) norms:

**Proposition 1.** Let \( \bar{p}_1, \bar{p}_2, \bar{E}_1, \bar{E}_2 \in \mathbb{R}^3 \), \( f_1, f_2 \in L_1^1 (\mathbb{R}^3) \). Then:

\[
\begin{aligned}
\| H_1 (t, \bar{p}_1, \bar{E}_1, f_1) - H_1 (t, \bar{p}_2, \bar{E}_2, f_2) \|_{\mathbb{R}^3} &\leq C_2 \left( \| \bar{E}_1 - \bar{E}_2 \|_{\mathbb{R}^3} + \| f_1 - f_2 \| \right) (a) \\
\| H_2 (t, \bar{p}_1, \bar{E}_1, f_1) - H_2 (t, \bar{p}_2, \bar{E}_2, f_2) \| &\leq C_3 \left( \| \bar{p}_1 - \bar{p}_2 \| + \| \bar{E}_1 - \bar{E}_2 \|_{\mathbb{R}^3} + \| f_1 - f_2 \| \right) (b) \\
\| H_3 (t, \bar{p}_1, \bar{E}_1, f_1) - H_3 (t, \bar{p}_2, \bar{E}_2, f_2) \|_{\mathbb{R}^3} &\leq C_4 \left( \| \bar{p}_1 - \bar{p}_2 \| + \| f_1 - f_2 \| \right) (c)
\end{aligned}
\] (3.7)

where

\[
\begin{aligned}
C(t) &= 8\pi C_1 a b^2 (t), \quad C_2 = 3C + b^2 \\
C_3 &= 5 \left( 6C + 1 \right) \left( 1 + a + \frac{b^2}{a} \right) \left( 1 + \frac{a}{b} + \frac{b}{a} + \frac{1}{a} + \frac{1}{b} \right) \times \\
&\quad \left( 1 + \| f_2 \| \right) \left( 1 + ab^2 \right) \left( 1 + \| f_2 \| \right) \left( 1 + \| f_2 \| \right) (3.8)
\end{aligned}
\]

In order to state the local existence theorem, we first recall this useful theorem:

**Theorem 2.** Let \( t_0 \geq 0 \), \( (\bar{p}_{t_0}, \bar{E}_{t_0}, f_{t_0}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times L_1^1 (\mathbb{R}^3) \) be given. Then:

1. There exists a real number \( \tau > 0 \) such that the differential system (3.6) has a unique solution \( (\bar{p}, \bar{E}, f) \in C ([t_0, t_0 + \tau]; \mathbb{R}^3) \times C ([t_0, t_0 + \tau]; L_1^1 (\mathbb{R}^3)) \) satisfying \( (\bar{p}, \bar{E}, f) (t_0) = (\bar{p}_{t_0}, \bar{E}_{t_0}, f_{t_0}) \). Moreover \( f \) satisfies:

\[
\| f \| = \sup \{ \| f (t) \|, t \in [t_0, t_0 + \tau] \} \leq \| f_{t_0} \|. 
\] (3.9)
2. The Maxwell-Boltzmann system (2.10), (3.2) has a unique local solution \( (F, f) \) on \([t_0, t_0 + \tau] \) such that

\[
F^{0i}(t_0) = E^{0i}_{t_0}, \quad F_{ij} = F_{ij}(t_0) = \varphi_{ij}, \quad f(t_0) = f_0, \quad \|f\| \leq \|f_0\|.
\] (3.10)

We end by stating the following local existence theorem coming from theorem 2.

**Theorem 3.** Let \( \overline{p_0}, \overline{E_0} \in \mathbb{R}^3 \), \( f_0 \in L^1_1(\mathbb{R}^3) \), \( \varphi_{ij} \in \mathbb{R} \) be given.

Then there exists a real number \( T > 0 \) such that:

1. The differential system (3.6) has a unique solution \( (\overline{p}, \overline{E}, f) \in \mathcal{C}([0, T]; \mathbb{R}^3)^2 \times \mathcal{C}([0, T]; L^1_1(\mathbb{R}^3)) \) such that \( (\overline{p}, \overline{E}, f)(0) = (\overline{p_0}, \overline{E_0}, f_0) \). Moreover:

\[
\|f\| \leq \|f_0\|.
\] (3.11)

2. The Maxwell-Boltzmann system (2.10), (3.2) has a unique solution \( (F, f) \) satisfying \( F^{0i}(0) = E^{0i}_{0}, \quad F_{ij} = F_{ij}(0) = \varphi_{ij}, \quad f(0) = f_0 \).

3.2. **Global existence theorem.**

3.2.1. **The method.** To prove global existence, the authors in [5] used the following method:

Let \([0, T]\) be the maximal existence domain of solution of the system (3.6) denoted here by \((\overline{p}, \overline{E}, \hat{f})\) and given by theorem 3 with the initial data \((\overline{p}_0, \overline{E}_0, f_0) \in \mathcal{C}([0, T]; \mathbb{R}^3)^2 \times \mathcal{C}([0, T]; L^1_1(\mathbb{R}^3))\).

We want to prove that \( T = +\infty \).

(a) If we already have \( T = +\infty \), the problem of existence is solved.

(b) If we suppose \( 0 < T < +\infty \), then the solution \((\overline{p}, \overline{E}, \hat{f})\) can be extended beyond \( T \), which contradicts the maximality of \( T \).

Supposing \( 0 < T < +\infty \) and \( t_0 \in [0, T] \), it is shown in [5] that there exists a number \( \tau > 0 \) independent of \( t_0 \) such that the system (3.6) has a unique solution \((\overline{p}, \overline{E}, f)\) on \([t_0, t_0 + \tau]\), with the initial data \((\overline{p}_0, \overline{E}_0, f_0)\) at \( t = t_0 \). Then taking \( t_0 \) sufficiently close to \( T \), for example, \( t_0 \) such that \( \theta < T - t_0 < \frac{\tau}{2} \) and hence \( T < t_0 + \frac{\tau}{2} \), we can extend the solution \((\overline{p}, \overline{E}, \hat{f})\) to \([0, t_0 + \frac{\tau}{2}]\) which strictly contains \([0, T]\), and this contradicts the maximality of \( T \).

3.2.2. **Preliminary results.** The following preliminary results were used in [5]:

**Lemma 4.** The maps \( t \mapsto \overline{E}(t), \; t \mapsto \overline{p}(t) \) are uniformly bounded over \([0, T]\).

**Proposition 5.** Let \( t_0 \in [0, T[, \; \left(\overline{p}_{t_0}, E_{t_0}, f_{t_0}\right) \in \mathcal{C}([t_0, t_0 + \tau]; \mathbb{R}^3)^2 \times \mathcal{C}([t_0, t_0 + \tau]; L^1_1(\mathbb{R}^3)) \) be given. Then there exists a number \( \tau \in ]0, \frac{1}{4}[, \) independent of \( t_0 \), such that the
system (3.6) has a unique solution \((E, p, f) \in C([t_0, t_0 + \tau; \mathbb{R}^3]) \times C([t_0, t_0 + \tau; X_r])\) such that \((E, p, f) (t_0) = (\tilde{p}_0, \tilde{E}_0, \tilde{f}_0)\).

3.2.3. The global existence theorem: Based on the method detailed above and using preliminary results, the following global existence theorem was proved in [5]:

**Theorem 6.** Let \(\tilde{p}_0, \tilde{E}_0 \in \mathbb{R}^3, f_0 \in L^1_1(\mathbb{R}^3), \varphi_{ij} \in \mathbb{R}\) be given, such that \(\|f_0\| \leq r\) where \(r > 0\) is a given real number. Then:

1) The differential system (3.6) has a unique global solution \((E, p, f)\) defined all over the interval \([0, +\infty[\) and such that \((E, p, f) (0) = (E_0, p_0, f_0)\) and

\[\|f\| \leq \|f_0\|, \quad f(t) \geq 0, \quad t \in [0, +\infty[.\]

2) The Maxwell-Boltzmann system (2.10), (3.2) has a unique global solution \((F, f)\) on \([0, +\infty[\) satisfying:

\[F^{0i}(0) = E_0^i, \quad F_{ij} = F_{ij}(0) = \varphi_{ij}, \quad f(0) = f_0, \quad \|f\| \leq \|f_0\|.\]

4. THE MAXWELL-BOLTZMANN SYSTEM IN HARD POTENTIAL CASE

Here, we extend the result of theorem 6 to some hard potential case. We still consider the Maxwell-Boltzmann-Momentum system (2.30) with the collision operator now given by:

\[Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} v_\phi \sigma(k, \theta) (f_f - ff^*) ab^2 d\omega d\tilde{q} = a^{-1} b^{-2} \int_{S^2} d\omega \int df \bar{u}_v \phi \sigma(k, \theta) (f_f^* - ff^*).\]

4.1. The method. We construct a modified system by truncating a certain part of the collision kernel in the equation (2.30 – c). As in the truncated system the scattering kernel is bounded, global existence of solution is insured by theorem 6. Then we obtain a sequence of solutions of the truncated system, and showing that this is a Cauchy sequence we obtain a solution of the initial system (2.30).

4.2. Preliminary results. We start by the following useful lemmas.

**Lemma 7.** The following inequalities hold

\[k \leq \sqrt{\delta}, \quad k \leq 2 \sqrt{u^0 v^0}, \quad \sqrt{\delta} \leq 2 \sqrt{u^0 v^0}, \quad k \leq a^{-1} |v - u|.\]  

**(4.1)**

**Proof.** The results are obtained by simple calculations. \(\square\)

**Lemma 8.** For the collision operator, the following property holds:
For any measurable function $h$ depending only of $k, \delta$ and $\omega$, we have:

$$
\int \int \int \frac{h(k, \delta, \omega)}{p^0 q^0} \left( f' f' - f f' \right) \left( p^0 \right)^r d\omega dq dp \\
= \int \int \int \frac{h(k, \delta, \omega)}{p^0 q^0} f f' \left( \left( p^0 \right)^r + \left( q^0 \right)^r - \left( p^0 \right)^r - \left( q^0 \right)^r \right) d\omega dq dp.
$$

Proof. See [4]

**Lemma 9.** Consider the collisional process in the Bianchi type I spacetime. Let $(p^\alpha, q^\alpha)$ and $(p'^\alpha, q'^\alpha)$ be pre and post collisional momenta respectively.

For $r > 1$, consider:

$$
G = \left( p^0 \right)^r + \left( q^0 \right)^r - \left( p^0 \right)^r - \left( q^0 \right)^r.
$$

Then $G$ satisfies

$$
G \leq C_r \left( \left( p^0 \right)^{r-1} q^0 + p^0 \left( q^0 \right)^{r-1} \right).
$$

(4.2)

If $\omega$ is restricted to the subset $\left\{ \omega \in S^2 : |n.\omega| \leq \frac{a^2(t)}{\sqrt{2b^2(t)}} |n| \right\}$, then:

$$
G \leq C_r \left( \left( p^0 \right)^{r-1} \frac{1}{2} (q^0)^{\frac{1}{2}} + \left( p^0 \right)^{\frac{1}{2}} \left( q^0 \right)^{r-\frac{1}{2}} \right) - c_r \left( \left( p^0 \right)^r + \left( q^0 \right)^r \right)
$$

(4.3)

where $C_r$ and $c_r$ are two different non-negative constants depending on $r$.

Proof. By the energy momentum conservation, we have

$$
p^0 + q^0 = p'^0 + q'^0, \ \text{for each } p^0 \text{ and } q^0.
$$

Let $p^\alpha$ and $q^\alpha$ be given. By the inequality

$$
\left\{ \begin{array}{l}
\alpha^r + \beta^r \leq (\alpha + \beta)^r \leq \alpha^r + \beta^r + C_r (\alpha^{r-1} \beta + \alpha \beta^{r-1}) \\
\alpha, \beta \geq 0; \ r > 1,
\end{array} \right.
$$

(4.4)

we deduce that

$$
\left( p^0 \right)^r + \left( q^0 \right)^r \leq \left( p^0 \right)^r + \left( q^0 \right)^r + C_r \left( \left( p^0 \right)^{r-1} \left( q^0 \right) + \left( p^0 \right) \left( q^0 \right)^{r-1} \right).
$$

Then

$$
G \leq C_r \left( \left( p^0 \right)^{r-1} q^0 + p^0 \left( q^0 \right)^{r-1} \right).
$$

(4.5)

For the second result, suppose that $|n.\omega| \leq \frac{a^2}{\sqrt{2b^2}} |n|$ and $p'^0 \geq q'^0$.

Then $p'^0$ is estimated as

$$
p'^0 \leq \frac{p^0 + q^0}{2} + \frac{k}{2} \frac{|a^2(t)n^1 \omega^1 + b^2 (n^2 \omega^2 + n^3 \omega^3)|}{\sqrt{(a^2(t)n^1 \omega^1 + b^2 (n^2 \omega^2 + n^3 \omega^3))^2}}.
$$
And we notice that

\[
\left| a^2(t)n^1\omega^1 + b^2(n^2\omega^2 + n^3\omega^3) \right| \leq 1
\]

if and only if

\[
2\left( a^2n^1\omega^1 + b^2(n^2\omega^2 + n^3\omega^3) \right)^2 \leq \left( n^0\right)^2\left( a^2(\omega^1)^2 + b^2((\omega^2)^2 + (\omega^3)^2) \right)
\]

Now using the facts that

\[
a \leq b, \quad \delta = (n^0)^2 - (a^2(n^1)^2 + b^2((n^2)^2 + (n^3)^2)) \geq 0
\]

we easily deduce that:

\[
|n.\omega| \leq \frac{a^2}{\sqrt{2}b^2} |n| \Rightarrow 2b^4(n.\omega)^2 \leq a^4|n|^2
\]

\[
\Rightarrow 2\left( a^2(n^1\omega^1)^2 + b^2((n^2\omega^2)^2 + (n^3\omega^3)^2) \right)
\]

\[
\leq a^2(a^2(n^1)^2 + b^2((n^2)^2 + (n^3)^2))
\]

Then \(|n.\omega| \leq \frac{a^2}{\sqrt{2}b^2} |n|\) implies, using lemma 10 that:

\[
p^n_{\nu} \leq \frac{p^0 + q^0}{2} + \frac{k}{2} \leq \frac{\left( \sqrt{p^0} + \sqrt{q^0} \right)^2}{2}
\]

So \(G\) is estimated as:

\[
G \leq 2(p^0)^r - (p^0)^r - (q^0)^r \leq \frac{1}{2^{r-1}} \left( \sqrt{p^0} + \sqrt{q^0} \right)^{2r} - (p^0)^r - (q^0)^r
\]

\[
\leq \frac{(p^0)^r}{2^{r-1}} + \frac{(q^0)^r}{2^{r-1}} + C_r \left( (p^0)^{r-\frac{1}{2}} (q^0)^{\frac{1}{2}} + (p^0)^{\frac{1}{2}} (q^0)^{r+\frac{1}{2}} \right) - (p^0)^r - (q^0)^r
\]

\[
\leq C_r \left( (p^0)^{r-\frac{1}{2}} (q^0)^{\frac{1}{2}} + (p^0)^{\frac{1}{2}} (q^0)^{r+\frac{1}{2}} \right) - C_r (p^0)^r + (q^0)^r
\]

where (4.4) is used and \(C_r, c_r > 0\) are constants depending on \(r\).

Let \(m\) be any positive integer: Now we modify the Maxwell-Boltzmann-Momentum system (2.30) by setting:

\[
\begin{align*}
\dot{E}_i^m &= -\Gamma^i_0 E_j^m + \int_{\mathbb{R}^3} g^i f_m(t, \tau) \frac{ab}{q^0} d\tau \\
\dot{p}_m^i &= -\Gamma^i_0 p_m^j - \left[ E_m^i + \frac{g^i p_m^j}{p_m^r} \dot{\varphi} \right] \int_{\mathbb{R}^3} f_m(t, \tau) \frac{ab}{q^0} d\tau \\
\frac{df_m}{dt} &= ab^2 \int_{\mathbb{R}^2} v_{\phi, m} \left( k_m^3 \sigma_{0, m}(\omega)(f_{m*} f_m - f_m f_{m*}) \right) d\omega d\tau = Q_m(f_m, f_{m*}) \\
F_{ij} &= F_{ij}(0) = \varphi_{ij}, \quad E_m(0) = E_0, \quad p_m(0) = p_0, \quad f_m(0) = f_0
\end{align*}
\]

where \(v_{\phi, m} := \min \{ k \sqrt{\tau}, m \}, k_m := \min \{ k, m \}, \sigma_{0, m} := \min \{ \sigma_0(\omega), m \} \).
Lemma 10. For any $m$ such that if $\|f_0\|_{1,r}$ is bounded, then:

$$\sup_{m} \sup_{t \in [0,T]} |f_m(t)|_{1,r} + \| f_m(t) \|_{1,r} \leq C_r. \tag{4.7}$$

Proof. We first estimate $\|f_m(t)\|_{1,r}$ and then obtain the result using (2.32).

By theorem 6 we have

$$\sup_{t \in [0,T]} \| f_m(t) \|_{1,r} \leq C$$

where $C = \|f_0(t)\|_{1,1}$ does not depend on $m$ for $0 \leq r \leq 1$

because for $r \leq s \leq 1$, $\|f_m(t)\|_{1,r} \leq \|f_m(t)\|_{1,s}$. Now we assume that $r > 1$.

Since $v^0$ decreases with time for each $\varpi$, using (2.2) and (2.9), we have:

$$\partial_t v^0 = - \left( \frac{\dot{a}(t)}{a^3(t)} (v^1)^2 + \frac{\dot{b}(t)}{b^3(t)} (v^2)^2 + (v^3)^2 \right) \frac{1}{v^0} \leq 0.$$

By direct calculation using equation (2.31), we have:

$$\frac{d}{dt} |f_m(t)|_{1,r} =$$

$$a^{-1} b^{-2} \int \int v_{\phi,m}(k_m)^3 \sigma_{0,m}(\omega) (f_m f_{m*} - f_m f_{m*}) (v^0)^r d\omega d\varpi d\varpi + \int f_m(t,\varpi) \partial_t v^0 d\omega$$

and the second integral is negative. Hence, we may only consider:

$$\frac{d}{dt} |f_m(t)|_{1,r} \leq a^{-1} b^{-2} \int \int v_{\phi,m}(k_m)^3 \sigma_{0,m}(\omega) (f_m f_{m*} - f_m f_{m*}) (v^0)^r d\omega d\varpi d\varpi.$$

By lemma 8, we have:

$$\frac{d}{dt} |f_m(t)|_{1,r} \leq$$

$$a^{-1} b^{-2} \int \int v_{\phi,m}(k_m)^3 \sigma_{0,m}(\omega) f_m f_{m*} [(v^0)^r + (u^0)^r - (v^0)^r - (u^0)^r] d\omega d\varpi d\varpi.$$

Using the fact that $a^{-1} b^{-2}$ is bounded, we apply lemma 9 and some calculations of lemma 3.6 in [4] to obtain
\[
\sup_m \sup_{t \in [0,T]} |f_m(t)|_{1,r} \leq C_r.
\]
By (2.32), we obtain the desired result, and the proof is completed. \(\square\)

**Lemma 11.** Consider the sequence \(\{f_m\}\) on any interval \([0,T]\). For each small number \(\varepsilon > 0\), there exists a positive integer \(M\) such that if \(k, m \geq M\):

\[
\sup_{t \in [0,T]} \left| f_k(t) - f_m(t) \right|_{1,1} \leq \varepsilon. \quad (4.8)
\]

**Proof.** The proof is similar to the proof of the previous lemma. We first estimate \(\|f_k(t) - f_m(t)\|_{1,r}\) then by (2.32), the result follows.

Using the relation (2.31), we have:

\[
\frac{d}{dt} \|f_k(t) - f_m(t)\|_{1,r} =
\]

\[
= \int Sgn (f_k - f_m) (Q_k(f_k, f_k) - Q_m(f_m, f_m)) v^0 d\bar{v}
\]

\[
- \int \left( \frac{\dot{a}(t)}{a^3(t)} (v_1)^2 + \frac{\dot{b}(t)}{b^3(t)} ((v_2)^2 + (v_3)^2) \right) \left| f_k(t, \bar{v}) - f_m(t, \bar{v}) \right| \frac{1}{v_0} d\bar{v}
\]

\[
\leq \int Sgn (f_k - f_m) (Q_k(f_k, f_k) - Q_m(f_m, f_m)) v^0 d\bar{v}.
\]

It remains to follow the proof of **lemma 3.7** in [4] and obtain a positive integer \(N\) such that if \(k, m \geq N\) then:

\[
\sup_{t \in [0,T]} \|f_k(t) - f_m(t)\|_{1,1} \leq \varepsilon.
\]

Thus, the desired result is obtained by (2.32) and the proof is completed. \(\square\)

**Lemma 12.** Consider the sequences \(\{E_m\}\) and \(\{p_m\}\) on any finite interval \([0, T]\). For any small number \(\varepsilon > 0\), there exists a positive integer \(M\) such that if \(k, m \geq M\), then

\[
\sup_{t \in [0,T]} \|E_k(t) - E_m(t)\| \leq \varepsilon, \quad (4.9)
\]

\[
\sup_{t \in [0,T]} \|p_k(t) - p_m(t)\| \leq \varepsilon. \quad (4.10)
\]
Proof. We consider the relations \((3.6 - a), (3.7 - a)\) to deduce that:

\[
\left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| \leq C_2 \left( \left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| + \left\| f_k(t) - f_m(t) \right\| \right).
\]

Using the expression of \(C_2\) given by \((3.8)\), relations \((2.4)\), we easily deduce that there exists a positive absolute constant \(C_6\) such that:

\[
C_2 \leq C_6 = C_6 (a_0, b_0, T, C_1).
\]

Then

\[
\left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| \leq C_6 \left( \left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| + \left\| f_k(t) - f_m(t) \right\| \right).
\]

Integrating over \([0, t]\), we obtain:

\[
\left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| \leq C_6 \left( T \sup_{t \in [0, T]} \left\| f_k(t) - f_m(t) \right\| + \int_0^t \left\| \overline{E}_k(s) - \overline{E}_m(s) \right\| ds \right),
\]

\(t \in [0, T]\). By Cronwall inequality, we obtain:

\[
\left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| \leq TC_6 \sup_{t \in [0, T]} \left\| f_k(t) - f_m(t) \right\| e^{C_6T}.
\]

Then \((2.32)\) and lemma 11 allow to conclude.

Using the same scheme, invoking this time \((3.6 - b), (3.7 - b)\) we obtain:

\[
\left\| \overline{p}_k(t) - \overline{p}_m(t) \right\| \leq C_3 \left( \left\| \overline{p}_k(t) - \overline{p}_m(t) \right\| + \left\| f_k(t) - f_m(t) \right\| + \left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| \right).
\]

Using the expression of \(C_3\) given by \((3.8)\), relations \((2.4)\), invoking lemma 4 and theorem 6 to bound \(\left\| \overline{E}_m(t) \right\|\) and \(\left\| f_m(t) \right\|\), we easily deduce that there exists a positive absolute constant \(C_7\) such that:

\[
C_3 \leq C_7 = C_7 (a_0, b_0, \left\| f_0 \right\|, \left\| \overline{E}_0 \right\|, T, C_1, C).
\]

Then

\[
\left\| \overline{p}_k(t) - \overline{p}_m(t) \right\| \leq C_7 \left( \left\| \overline{p}_k(t) - \overline{p}_m(t) \right\| + \left\| f_k(t) - f_m(t) \right\| + \left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| \right).
\]

Integrating over \([0, t]\) and using the Gronwall lemma, we obtain:

\[
\left\| \overline{p}_k(t) - \overline{p}_m(t) \right\| \leq TC_7 \sup_{t \in [0, T]} \left( \left\| \overline{E}_k(t) - \overline{E}_m(t) \right\| + \left\| f_k(t) - f_m(t) \right\| \right) e^{C_7T}.
\]

Then \((2.32)\), lemma 11 and the inequality \((4.9)\) give the relation \((4.10)\). So, the proof is completed.
4.3. The global existence theorem. Now we can state the main result of this work.

**Theorem 13.** Let \( \overline{p}_0, E_0 \in \mathbb{R}^3 \), \( \varphi_{ij} \in \mathbb{R} \), \( f_0 \in L^1_r(\mathbb{R}^3) \) be given, with \( r > 1 + \frac{\beta}{2} \) and \( f_0 \geq 0 \). Suppose that the scattering kernel has the form \((2.21)\).

Then the equivalent Maxwell-Boltzmann-Momentum system \((3.6)\) has a unique global solution \((F, \overline{p}, f)\) such that \( f \in C([0, +\infty[, L^1_1(\mathbb{R}^3)) \) with \( f(t) \geq 0 \) and satisfying \( F^{(0)} := F^{(0)}(0) = E_0, F_{ij} = F_{ij}(0) = \varphi_{ij}, f(0, \cdot) = f_0 \).

\( (F, f) \) is the unique global solution of the Maxwell-Boltzmann system \((2.10), (2.13)\).

**Proof.** Lemmas 11 and 12 show that the sequence \( \{(E_m, \overline{p}_m, f_m)\} \) is a Cauchy sequence in the Banach space \((\mathbb{R}^3)^2 \times L^1_1(\mathbb{R}^3)\). Hence there exists \((E, \overline{p}, f)\) a solution of the system \((2.30)\) with initial condition \((E_0, \overline{p}_0, f_0)\). The initial condition \( f_0 \in L^1_r(\mathbb{R}^3) \) with \( r > 1 + \frac{\beta}{2} \) comes from lemma 3.7 in [4] and the non-negativity of \( f \) is guaranteed by the same lemma. The uniqueness is obtained by the proof of lemmas 11 and 12. This complete the proof. \( \square \)

5. CONCLUSION

This work was devoted to extend the result of [5] who considered the homogeneous relativistic Maxwell- Boltzmann system for a bounded scattering kernel with an additional hypothesis of invariance under a subgroup of \( O_3 \). In the present work, we discarded this hypothesis. After presenting the background spacetime, the unknowns and the equations, we considered the Maxwell-Boltzmann system for a bounded kernel and we briefly recalled the results of [5]. The same system has been considered in case of hard potential kernel. The method followed was the one used in [4], relying in the use of a particular form of Povzner inequality, but in a more difficult situation, because the Boltzmann equation was coupled with the Maxwell equations. Some energy estimates allowed us to obtain global existence theorem and uniqueness of mild solutions.

In our future investigations, we will consider an inhomogeneous magnetized Boltzmann equation for both bounded and hard potential cases.

**References**


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