Non Linear Spin 1 Field Equation in Curved Space-Time: Separation and Solution in Static RW Metric

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Abstract

A non linear spin 1 field equation is proposed in a general curved space time. The non-linearity is represented by a reduced version of a torsion induced non linearity previously considered. The field equation is made explicit in the Robertson Walker space-time by means of the Newman Penrose formalism based on a given null tetrad frame. In case of static space time and time oscillating field, the angular and radial dependence are separated with a separation formalism borrowed from the linear case. By suitable assumption the separated angular equations are integrated and there results a cylindrical symmetry of the solutions. The radial dependence is reduced to the study of two coupled non linear radial equations. The solution is finally reported to the solution of a Bernoulli equation that is studied in every case of the curvature parameter.

Keywords: Non linear spin 1 field equation in GR: Robertson Walker metric; Variable separation; Solutions

1 Introduction

The formulation of the spin field equation in space time with torsion generally entails the addition of non linear terms to the equations (e.g., [4]). Physical
consequences of the interaction of torsion with matter field have been discussed (e.g., [10]). Modification of the equations by torsion is an effect that appears also in the standard Model and supersymmetric and supersymmetric string theory (e.g., [3, 6, 7]).

Due to the interest of the subject, it didn’t seem useless to have a general formulation of spinor field equations to include torsion. This was done in [13] in the line of the two spinor formulation of massive field equation of arbitrary spin in curved space time of Ref. [5]. The leading idea in such procedure was of generalizing what happens for the Dirac field (e.g., [12]). In such case the interaction of field and torsion amounts to the interaction of the Dirac field with its own current. Therefore the treatment of [13] does not cover the most general interaction of the field with torsion. It includes however a non linear Proca and a non linear Rarita-Schwinger equation. What lacks in that formulation is the physical interpretation of the general spin field equation with torsion. As to this problem it seems unavoidable to go back to a specific physical model. Another open problem of interest is of determining explicit solution of the field equation, at least for the lowest spin values, in some specific space time model. (For recent models see e.g., [8]; for the non linear Dirac equation see e.g., [15, 16, 17]).

In the present paper a reduced form of the nonlinear spin 1 field equation proposed in [13] is studied. The formulation is in terms of only two spinors $\phi, \chi$. The non linear interaction is obtained here by an ansatz on the original one of [13] that involves two other spinor fields satisfying the complex conjugate equation that $\phi, \chi$ satisfy.

The equation is studied in the RW space-time by the Newman Penrose formalism [9] and it is based on a null tetrad frame previously introduced. The equation is separated in the static RW space-time, for time oscillating spinor field, by mimicking the variable factorization assumptions of the linear case [12, 18].

The separated angular equation are integrated by a further assumption and there results a cylindrical symmetry of the solutions. The separated radial dependence is reduced to the solution of two coupled non linear equations. They can be disentangled so that the study is reduced to the study of the solution of a Bernoulli equation that is performed for flat, open and closed space time.

2 Assumptions and general results

The non linear spin 1 field equation is formulated here in a general curved space time by the two spinor formalism. Accordingly it is characterized by the spinors $\phi_{AB}(x) = \phi_{BA}(x)$ and $\chi_{AX}(x)$ satisfying the non linear coupled
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The equations

\[(\nabla^A_{X'} - U^A_{X'})\phi_{AB} = -\mu\chi_{AX'}\]
\[(\nabla^X_{A'} - U^X_{A'})\chi_{BX'} = \nu\phi_{AB}\]
\[U_{AX'} = p(\phi_{AC'X'} - \chi_{CX'}\bar{\phi}^{C'}_A), \quad p \in \mathbb{C}\]

with \(2\mu\nu = -m_o^2\), \(m_o\) the mass of the particle of the field. The scheme is borrowed from the general formulation of spin field equation with torsion in a general space-time [13]. Consequently the interaction term \(U\) is obtained by an ansatz from the one of [13]. Note that in (3) \(\bar{\phi}\) is assumed to be a complex conjugate copy of \(\phi\) with non-primate indices.

In view of the following consideration it is useful to assume

\[\phi_{00}(x) = \phi_{11}(x) = \phi_0(x), \quad \phi_{01}(x) = \phi_{10}(x) = \phi_1(x)\]
\[\chi_{00'}(x) = -\chi_{11'}(x) = \phi_1(x), \quad \chi_{01'}(x) = -\phi_0(x) = -\chi_{10'}(x)\]

With such assumptions the expressions

\[I_{BX'} = -U^A_{X'}\phi_{AB} = U_{AX'}\phi^A_B\]
\[L_{AB} = -U^X_{A'}\chi_{BX'} = U_{AX'}\chi^X_B\]

can be explicitly calculated. One obtains

\[I_{00'} = -I_{11'} = L_{01} = -L_{10} = 2p\phi_1(\phi_0\bar{\phi}_0 - \phi_1\bar{\phi}_1)\]
\[I_{10'} = -I_{01'} = L_{11} = -L_{00} = -2p\phi_0(\phi_0\bar{\phi}_0 - \phi_1\bar{\phi}_1)\]

The results (8), (9), once inserted into (1), (2), produce a cubic non linearity of the equations.

3 Separation in static RW space-time

The object is now to look for possible solutions of eqs. (1-2). To that end the study is performed in the Robertson-Walker space-time of metric tensor \(g_{\mu\nu}\):

\[ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R(t)^2\left[\frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2}\right], a = 0, \pm 1\]

The linear part of eqs. (1-2) can be developed by the Newman-Penrose formalism based on the null tetrad frame introduced in [11]. By making explicit the expressions of the covariant spinor derivatives in terms of tabulated spin coefficients and directional derivatives [9, 1], the equations (1), (2) take the
form \((D = \partial_{00'}, \Delta = \partial_{11'}, \delta = \partial_{01'}, \delta^* = \partial_{10'})\):

\[
\begin{align*}
(D - 2\rho)\phi_{10} - (\delta^* - 2\alpha)\phi_{00} + I_{00'} &= i\mu_s\chi_{00'} \quad (11) \\
(D - \rho + 2\epsilon)\phi_{11} - \delta^*\phi_{10} + I_{10'} &= i\mu_s\chi_{10'} \quad (12) \\
(\Delta + \mu - 2\gamma)\phi_{00} - \delta\phi_{01} + I_{01'} &= -i\mu_s\chi_{01'} \quad (13) \\
(\Delta + 2\mu)\phi_{10} - (\delta + 2\beta)\phi_{11} + I_{11'} &= -i\mu_s\chi_{11'} \quad (14) \\
(D - \rho)\chi_{01'} - \delta\chi_{00'} + L_{00} &= -i\mu_s\phi_{00} \quad (15) \\
(D - \rho + 2\epsilon)\chi_{11'} - (\delta + 2\beta)\chi_{10'} + \mu\chi_{00'} + L_{10} &= -i\mu_s\phi_{10} \quad (16) \\
(\delta^* + 2\beta)\chi_{01'} - (\Delta + \mu - 2\gamma)\chi_{00'} + \rho\chi_{11'} + L_{01} &= -i\mu_s\phi_{01} \quad (17) \\
\delta^*\chi_{11'} - (\Delta + \mu)\chi_{10'} + L_{11} &= -i\mu_s\phi_{11} \quad (18)
\end{align*}
\]

with now \(\mu = \nu = i\mu_s, \mu_s\sqrt{2} = m_o\). The equations (11)-(18) can be separated in the static RW space time by setting

\[
\phi_j(t, r, \theta, \varphi) = e^{ikt}e^{im_o\theta}S_j(\theta)\phi_j(r), \quad S_j(\theta)S_j(\theta) = 1, \quad j = 0, 1 \quad (19)
\]

with \(m = 0, \pm1, \pm2, \ldots\) Indeed for \(R(t) = R_o = 1\) the directional derivatives and spin coefficients of the assumed null tetrad frame are [11]

\[
\begin{align*}
\sqrt{2} D &= \partial_t + \sqrt{1 - ar^2}\partial_r, \quad \sqrt{2}\Delta = \partial_t - \sqrt{1 - ar^2}\partial_r \\
r\sqrt{2}\delta &= \partial_\theta + i\csc\theta\partial_\varphi, \quad \delta^* = \delta, \quad \epsilon = \gamma = 0 \\
r\sqrt{2}\rho &= r\sqrt{2}\mu = -\sqrt{1 - ar^2}, \quad 2r\sqrt{2}\beta = -2r\sqrt{2}\alpha = \cot\theta
\end{align*}
\]

Then, by combining all the assumptions, the equations (11)-(14) separate with corresponding separation constant \(\lambda_h, h = 1, 2, 3, 4\). The separated angular equations are

\[
L^-_h S_0 = \lambda_h S_0, \quad L^-_h S_0 = \lambda_2 S_0, \quad L^+_h S_1 = \lambda_3 S_0, \quad L^+_h S_0 = \lambda_4 S_1 \quad (23)
\]

where it has been set \(L^\pm_n = \partial_{\theta} \mp m\csc\theta + n\cot\theta\)

As to the separated radial equations one is left with

\[
\begin{align*}
&ik + \sqrt{1 - ar^2}\frac{\phi'_1}{\phi_1} + \frac{2}{r}\sqrt{1 - ar^2} - \frac{\lambda_1}{r}\frac{\phi_0}{\phi_1} - [im_o + \Phi] = 0 \quad (24) \\
&ik + \sqrt{1 - ar^2}\frac{\phi'_0}{\phi_0} + \frac{1}{r}\sqrt{1 - ar^2} - \frac{\lambda_2}{r}\frac{\phi_1}{\phi_0} - [im_o + \Phi] = 0 \quad (25) \\
&ik - \sqrt{1 - ar^2}\frac{\phi'_0}{\phi_0} - \frac{1}{r}\sqrt{1 - ar^2} - \frac{\lambda_3}{r}\frac{\phi_1}{\phi_0} - [im_o - \Phi] = 0 \quad (26) \\
&ik - \sqrt{1 - ar^2}\frac{\phi'_1}{\phi_1} - \frac{2}{r}\sqrt{1 - ar^2} - \frac{\lambda_4}{r}\frac{\phi_0}{\phi_1} - [im_o - \Phi] = 0 \quad (27) \\
&\Phi = 2\sqrt{2}\rho(|\phi_1|^2 - |\phi_0|^2)
\end{align*}
\]
Compatibility of (24)-(27) requires
\[ k = m_o, \quad \lambda_1 = -\lambda_4, \quad \lambda_3 = -\lambda_2 \] (28)
so that only two of the equations (24)-(27) are independent. For what concerns the separation of the other equations (15)-(18) one can proceed in a similar way with separation constants \( \lambda_i, \ i = 5, 6, 7, 8 \) thus obtaining again the equations (23) with identifications \( \lambda_1 = \lambda_7, \ \lambda_2 = \lambda_8, \ \lambda_4 = \lambda_6 \) and the radial equations (24)-(27) in different order.

## 4 Integration of the separated equations

The angular equations (23), once integrated under the constraints (28), give
\[ m = 0, \quad S_0 = c_o / \sin \theta, \quad S_1 = c_1 \] (29)
Consistency requires then:

i) if \( c_0 = c_1 = 0 \) then \( \lambda_1, \lambda_2 \) arbitrary complex numbers

ii) if \( c_0 \neq 0, \ c_1 = 0 \) then \( \lambda_2 = 0, \ \lambda_1 \) arbitrary complex number

iii) if \( c_0 = 0, \ c_1 \neq 0 \) then \( \lambda_1 = 0, \ \lambda_2 \) arbitrary complex number

iv) if \( c_0 \neq 0, \ c_1 \neq 0 \) then \( \lambda_1 = \lambda_2 = 0 \)

It is convenient to choose in the following \( c_0 \neq 0, \ c_1 \neq 0 \) so that \( \lambda_1 = \lambda_2 = 0 \). The expressions (29) do not satisfy the requirement \( S_j \overline{S_j} = 1 \). The problem can be bypassed by setting \( S_h(\theta) = \exp(is_h(\theta)) \), \( s_h \in \mathbb{R} \) with \( s_0 = \cos^{-1}(c_0 / \sin \theta), \ s_1 = \cos^{-1} c_1, \ c_0, c_1 \in \mathbb{R} \). This is possible because, the angular equations being real equations for real \( \lambda \)'s, both \( Re S_h \) and \( Im S_h \) satisfy the same equation the \( S_h \)'s satisfy.

By combining all the previous considerations and results one is left with the non linear radial equations:

\[ \sqrt{1-ar^2} \frac{\phi_1'}{\phi_1} + \frac{2}{r} \sqrt{1-ar^2} = 2\sqrt{2}p(|\phi_1|^2 - |\phi_0|^2) \] (30)
\[ \sqrt{1-ar^2} \frac{\phi_0'}{\phi_0} + \frac{1}{r} \sqrt{1-ar^2} = 2\sqrt{2}p(|\phi_1|^2 - |\phi_0|^2) \] (31)

Comparing (30), (31) and integrating one first obtains:
\[ \phi_0 = a_o r \phi_1 \quad (a = 0) \] (32)
\[ \phi_0 = a_o \phi_1 \left( \frac{1+r^2}{\sqrt{1+r^2}} \right)^{\frac{1}{2}} \exp \sqrt{1+r^2}, \quad (a = -1) \] (33)
\[ \phi_0 = a_o \phi_1 \left( \frac{1-r^2}{\sqrt{1-r^2}} \right)^{\frac{1}{2}} \exp \sqrt{1-r^2}, \quad (a = 1) \] (34)
Hence (30), (31) can be disentangled into Bernoulli equations. The problem can be further studied in case of real spinor fields to obtain:

\[
\begin{align*}
\phi'_0 &= -\frac{1}{r}\phi_0 + \phi^3_0 (a_0^2 r^2 - 1) \quad (a = 0) \\
\phi'_1 &= -\frac{2}{r}\phi_1 + \frac{b\phi^3_1}{\sqrt{1 + r^2}} \left[ 1 - a_0^2 \frac{\sqrt{1 + r^2} - 1}{\sqrt{1 + r^2} + 1} \exp 2\sqrt{1 + r^2} \right] \quad (a = -1) \\
\phi'_1 &= -\frac{2}{r}\phi_1 + \frac{b\phi^3_1}{\sqrt{1 - r^2}} \left[ 1 - a_0^2 \frac{\sqrt{1 - r^2} - 1}{\sqrt{1 - r^2} + 1} \exp 2\sqrt{1 - r^2} \right] \quad (a = 1)
\end{align*}
\]

The flat space-time case \(a = 0\) can be integrated exactly:

\[
\begin{align*}
\phi_0 &= (Cr^2 - 2ba^2_0 r^3 - 2br)^{-\frac{1}{2}} \\
\phi_1 &= a_0 r (Cr^2 - 2ba^2_0 r^3 - 2br)^{-\frac{1}{2}}
\end{align*}
\]

\(C\) integration constant, \(b = 2\sqrt{2}p\). Note that for \(b < 0\), \(\phi_1 \sim r^{-\frac{3}{2}}\) for large \(r\).

For what concerns the integration of the radial equation (36), it can be studied for \(r \to \infty\). In that situation it can be approximated by:

\[
\phi'_1 = -2r^{-1}\phi_1 - ba^2_0 r^{-1} e^{2\sqrt{r}} \phi^3_1
\]

that can be integrated exactly:

\[
\begin{align*}
\phi_1 &= (Y_1 + Y_2)^{-\frac{1}{2}} \\
Y_1 &= C 2^{-4} x^4, \quad x = 2r \\
Y_2 &= -a_0 b \left( \frac{1}{2} + \frac{x}{6} + \frac{x^2}{12} + \frac{x^3}{12} \right) \exp \left( 2a_0 x^4 Ei(x) \right)
\end{align*}
\]

\(Ei(x)\) the integral exponential function [2]. One has \(\phi_1 \to (x^3 \exp x)^{-1/2}\) for \(r \to \infty\). Near \(r = 0\) the equation (36) can be approximated by \(\phi'_1 = -(2/r)\phi_1 + b\phi^3_1\) so that one has the approximation

\[
\phi_1 \underset{r \to 0}{\to} (Cr^4 + 2br/3)^{-\frac{1}{2}}
\]

As to equation (37), it can be similarly integrated near \(r = 0\).

Finally the solutions of the system (1), (2) is of the form

\[
\phi_h(t, r, \theta, \phi) = e^{imt} S_h(\theta) \phi_h(r), \quad h = 0, 1
\]

with the \(S_h\)'s and \(\phi_h\)'s previously obtained and show a cylindrical symmetry.
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5 Remarks and Comments

In the previous Sections the problem of the solution of the non linear spin 1 field equation has been considered in the static RW space-time. The starting equation has been induced from the formulation of the spin 1 field equation with torsion in a general curved space-time obtained in [13]. In that scheme, besides the spinors $\phi, \chi$ of equations (1), (2), two other spinors $\xi, \theta$ were involved satisfying the complex conjugation equation satisfied by $\phi, \chi$. Therefore also the interaction between field with torsion has been approximated by the expression (3). This open the problem of the interpretation of the field equation (1), (2), (3) that cannot be immediately interpreted as a non linear Proca field equation. Indeed the Proca field interpretation is possible by setting $\theta = \chi$ while $\xi$ is determined by the equation with $\theta$. This makes the interaction term of [13] much more involved than what assumed here in (3). Therefore, on physical ground, it seems risky to interpret the present scheme as a possible non linear form of the Proca equation. It could only be possibly ascribed to some non yet specified boson of spin 1.

On mathematical ground solutions of the equation has been proposed in the static RW space-time by adapting the method of variable separation employed for the separation of the arbitrary spin field equation in RW space-time [14]. There result solutions of cylindrical symmetry, oscillating time dependence and radial solution satisfying, in every case, non linear equations of Bernoulli type.

It would be interesting to know whether separated solution exist different from those proposed here in the static or in the non static RW space-time. Also the knowledge of non factorized solution would be desirable.

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Received: July 11, 2019; Published: July 30, 2019