An Improved \((G'/G)\) - Expansion Method

for Solving Nonlinear Evolution Equations

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Abstract

In this paper, we apply improved \((G'/G)\)-expansion method to discover a strategy for the approximate solution of nonlinear evolution equations (NLEEs). The given NLEEs through substitution are converted into a nonlinear ordinary differential equation. The travelling wave solution is approximated by the \((G'/G)\) -expansion method with unknown parameters.

Keywords: Improved \((G'/G)\)-expansion method, Non-linear evolution equations, Traveling wave solutions

1. Introduction

Nonlinear evolution equations are encountered in the various areas of scientific discipline and have an important place in applied mathematics and physics. NLEEs are mathematical models of the physical phenomenon that arise in engineering, chemistry, gas dynamic, traffic flow, heat conduction, elasticity mechanics, meteorology, solid-state physics, fluid dynamics, plasma physics, ocean and atmospheric waves, mathematical biology, material science, etc. Exact solutions of NLEEs play an important role in the proper understanding of qualitative features of many phenomena and processes in the mentioned areas of natural science.
Expansion methods for finding solitary travelling-wave solutions to NLEE s have received considerable attention over the past twenty years or so. The basic \((G'/G)\)-expansion method was introduced in 2008 by Wang et al. [23] for obtaining the travelling wave solutions of various \textit{NLEEs}. Consequently, many researchers applied the \((G'/G)\)-expansion method to solve different kinds of \textit{NLEEs}. Examples are given in \([1, 7, 8, 10, 11, 14] \) and \([29]\).

In recent years, the solutions of \textit{NLEEs} have been investigated by many researchers who have used various numerical methods to work out the solution of NLEE s. With the development of solution theory, many powerful methods for obtaining the exact solutions of NLEE s have been presented by many authors. For instances, Tanh method \([2,3]\), F-expansion method \([4]\), Backlund transforms \([5]\), a numerical simulation and explicit solutions\([6]\), Adomian decomposition method \((ADM)\) \([3,9]\), variational iteration method\([3,12]\), the Jacobi elliptic function expansion method\([13]\), \((G'/G, 1/G)\)-expansion method\([15]\), inverse scattering transform \([20]\), Modified Pseudospectral method \([21]\), spectral collocation method \([22]\), an efficient algorithm method \([24]\), finite element analysis and numerical method \([25]\), \((G'/G^2)\)-expansion method \([26]\), Exp-function method \([28]\).

The aim of this paper is to discuss one such extension of the basic \((G'/G)\)-expansion method and to use the improved \((G'/G)\)-expansion method to find travelling wave solutions for the Burgers equation, Boussinesq equation, and Burgers-KDV equation. The Burgers equation is considered one of the fundamental model equations in fluid mechanics, the equation of burgers demonstrates the coupling between diffusion and convection processes, The Boussinesq equation describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice. The Burgers-KdV equation arises from many physical contexts such as the propagation of undular bores in shallow water, the flow of liquids containing gas bubbles, weakly nonlinear plasma waves with certain dissipative effect, theory of ferro electricity, nonlinear circuit, and the propagation of waves in an elastic tube filled with a viscous fluid.

This manuscript has organized as follows: Section 1 gives an introduction of the study. In Section 2, we provide the description of the method to be used in the paper. Section 3 presents the application schemes of the afore-mentioned method, explaining the techniques to be used in the analysis. Finally, the manuscript ended up with the conclusion in Section 4.

2. Description of the improved \((G'/G)\)-expansion method

The main steps of improved \((G'/G)\)-expansion method are as follows:
1. We consider the nonlinear partial differential equation for $u(x,t)$, has the form,

$$P(u,u_t,u_{xx},u_{tt},\cdots) = 0$$  \hspace{1cm} (1)

Seek travelling wave solutions of equation (1) by use transformation,

$$u(x,t) = U(\zeta), \quad \zeta = x - wt$$  \hspace{1cm} (2)

where $w$ is constant, convert the partial differential equation (1) in to an ordinary differential equation,

$$Q(u,u',u'',\cdots) = 0$$  \hspace{1cm} (3)

all terms of the resulting ordinary differential equation contain derivatives in $\zeta$.

2. We suppose that the solution of Eq.(3) can be expressed as the following polynomial in $(G'/G)$ such that:

$$u(\zeta) = \sum_{i=-m}^{m} \alpha_i (G'/G)^i$$  \hspace{1cm} (4)

where $G = G(\zeta)$ satisfies a second order linear ordinary differential equation in the form:

$$G'' + 2\lambda G' + \mu G = 0$$  \hspace{1cm} (5)

where $\alpha_i$, $\lambda$ and $\mu$ are constants. $m$ is a positive integer which will be determined. To determine the integer $m$ Substituting Eq. (4) along with Eq. (5) into Eq. (3) and then consider homogeneous balancing between the highest order derivatives and highest order nonlinear terms appearing in Eq. (3).

3. The results in step(ii) equating the coefficient of powers of $(G'/G)$, then setting each coefficient to zero, yield a system of algebra equation for the determination of the parameters $\alpha_i$, $\lambda$, $w$ and $\mu$.

4. Solve the system of algebraic equations with aid of maple and we obtain values for $\alpha_i$, $\lambda$, $w$ and $\mu$ then, substitute obtained values in Eq. (4) along with Eq. (5) with value of $m$ we obtained the solution of the nonlinear evolution Eq. (1).

5. By using the general solutions of Eq. (4), for $\lambda^2 - 4\mu > 0$, we obtained,

$$\frac{G'(\zeta)}{G(\zeta)} = \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[ \frac{A \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \zeta \right) + B \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \zeta \right)}{A \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \zeta \right) + B \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \zeta \right)} \right] - \frac{\lambda}{2}$$  \hspace{1cm} (6)
3. Applications of the improved \((G'/G)\)-expansion method:

In this section, we apply the \((G'/G)\)-expansion method to obtain some new and more general exact travelling wave solutions of the Burgers equation, Boussinesq equation, and Burgers-KDV equation.

**Example (1)**

Consider Burgers equation of the form

\[ u_t - 2uu_x - u_{xx} = 0 \]  

Will be converted the ordinary differential equation, by use transformation \( \zeta = x - wt \).

\[ wu' + 2uu' + u'' = 0 \]  

Setting the integration constant as zero yield

\[ wu + u^2 + u' = 0 \]  

Balance the nonlinear term \( u^2 \) with the highest order derivative \( u' \) gives \( m = 1 \). Therefore, the solution Eq. (7) turns out to be

\[ u(\zeta) = \sum_{i=1}^{1} \alpha_i \left( \frac{G'}{G} \right)^i \]  

where \( \alpha_i \) are arbitrary constants. Substituting Eq. (10) with Eq. (5) into Eq. (9), then collecting all terms with the same power of \( (G'/G) \) and setting each coefficient of power of \( (G'/G) \) to zero, we achieve a system of algebraic equations and solving the system with maple, we obtain the set of solutions.

Case 1:

\[ \alpha_1 = 1, \quad \alpha_0 = \frac{\lambda - w}{2}, \quad \alpha_{-1} = 0, \quad \mu = \frac{1}{4}(\lambda^2 - w^2) \]  

Case 2:

\[ \alpha_1 = 0, \quad \alpha_0 = -\frac{\lambda + w}{2}, \quad \alpha_{-1} = \frac{1}{4}(w^2 - \lambda^2), \quad \mu = \frac{1}{4}(w^2 - \lambda^2) \]  

Case 3:

\[ \alpha_1 = 1, \quad \alpha_0 = -\frac{w}{2}, \quad \alpha_{-1} = \frac{w^2}{16}, \quad \mu = -\frac{w^2}{16}, \quad \lambda = 0 \]  

Substituting Eq. (12) with Eq. (6) into Eq. (10) and simplify, we obtain the following

\[ u_1(\zeta) = \left( \frac{w}{2} \right) \left[ \frac{A \sinh \left( \frac{w}{2} \zeta \right) + B \cosh \left( \frac{w}{2} \zeta \right)}{A \cosh \left( \frac{w}{2} \zeta \right) + B \sinh \left( \frac{w}{2} \zeta \right)} \right] \frac{\lambda}{2} + \frac{\lambda - w}{2} \]  

Substituting Eq. (13) with Eq. (6) into Eq. (10) and simplify, we obtain the following
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\(u_2(\zeta) = -\frac{\lambda + w}{2} - \frac{1}{4}(w^2 - \lambda^2) \left[ \frac{w}{2} \left( \frac{A \sinh \left( \frac{w}{2} \zeta \right) + B \cosh \left( \frac{w}{2} \zeta \right)}{A \cosh \left( \frac{w}{2} \zeta \right) + B \sinh \left( \frac{w}{2} \zeta \right)} \right) - \frac{\lambda}{2} \right]^{-1} \) (16)

Substituting Eq. (14) with Eq. (6) into Eq. (10) and simplify, we obtain the following

\[ u_3(\zeta) = \frac{w}{4} \left[ \frac{A \sinh \left( \frac{w}{4} \zeta \right) + B \cosh \left( \frac{w}{4} \zeta \right)}{A \cosh \left( \frac{w}{4} \zeta \right) + B \sinh \left( \frac{w}{4} \zeta \right)} \right] - \frac{w}{2} \]

\[ + \left( \frac{w^2}{16} \right) \left[ \frac{w}{4} \left( \frac{A \sinh \left( \frac{w}{4} \zeta \right) + B \cosh \left( \frac{w}{4} \zeta \right)}{A \cosh \left( \frac{w}{4} \zeta \right) + B \sinh \left( \frac{w}{4} \zeta \right)} \right) \right]^{-1} \] (17)

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (15) can be written respectively as

\[ u_1(x, t) = -\frac{w}{2} \left[ \frac{1}{w} \tanh \left( \frac{w}{2} (x - wt) \right) \right], \quad w > 0, \] (18)

\[ u_1(x, t) = -\frac{w}{2} \left[ \frac{1}{w} \coth \left( \frac{w}{2} (x - wt) \right) \right], \quad w > 0, \] (19)

In particular, when setting \(B = 0\), and \(A \neq 0\), or \(A = 0\), and \(B \neq 0\), the solutions of Eq. (16) can be written respectively as

\[ u_2(\zeta) = -\frac{\lambda + w}{2} - \frac{1}{4}(w^2 - \lambda^2) \left[ \frac{w}{2} \tanh \left( \frac{w}{2} (x - wt) \right) - \frac{\lambda}{2} \right]^{-1} \] (20)

\[ u_2(\zeta) = -\frac{\lambda + w}{2} - \frac{1}{4}(w^2 - \lambda^2) \left[ \frac{w}{2} \coth \left( \frac{w}{2} (x - wt) \right) - \frac{\lambda}{2} \right]^{-1} \] (21)

In particular, when setting \(B = 0\), and \(A \neq 0\), or \(A = 0\), and \(B \neq 0\), the solutions of Eq. (17) are the same can be written as

\[ u_3(x, t) = -\frac{w}{4} \left[ 2 - \tanh \left( \frac{w}{4} (x - wt) \right) - \coth \left( \frac{w}{4} (x - wt) \right) \right] \] (22)

Graphical representation of the solutions of Burgers equation:

Solution \(u_1(x,y)\) is kink waves are traveling waves which arise from one asymptotic state to another. Figure (1-1) below shows the shape of the exact kink-type solution of the Burgers equation.
Figure (1-1) graph of the solution $u_{11}(x, y)$ for $w=1$ with $0 \leq t \leq 20, -20 \leq x \leq 20$. Soliton solution $u_{12}(x, y)$ is singular kink solution represented in the below figure (1-2) which shows the shape of the exact singular kink-type solution of the Burgers equation.

Figure (1-2) graph of the solution $u_{11}(x, y)$ for $w=1$ with $0 \leq t \leq 20, -20 \leq x \leq 20$.

Furthermore, the Soliton solution $u_{2}(x, y)$ is periodic waves of singular kink solution, given in Figure (1-3). This figure shows the shape of the exact singular kink-type solution of the Burgers equation.

Figure (1-3) graph of the solution $u_{2}(x, y)$ for $w=1, \lambda=0.5$ with $0 \leq t \leq 20, -20 \leq x \leq 20$.

Solution $u_{2}(x, y)$ is kink waves are traveling waves which arise from one asymptotic state to another. Figure (1-4) below presents the shape of the exact kink-type solution of the Burgers equation.
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Figure (1-4) graph of the solution \(u_2(x, y)\) for \(w=1, \lambda=0.5\) with \(0 \leq t \leq 20, -20 \leq x \leq 20\).

Soliton solution \(u_3(x, y)\) is singular kink waves solution is shown in Figure (1-5) which presents the shape of the exact singular kink-type solution of the Burgers equation.

Figure (1-5) graph of the solution \(u_3(x, y)\) for \(w=1\) with \(0 \leq t \leq 20, -20 \leq x \leq 20\).

Example (2)
The Boussinesq equation

\[
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - 3u^{(2)}_{xx} - u_{xxxx} = 0
\]  

(23)

Will be converted the ordinary differential equation, by use transformation \(\zeta = x - wt\), and integrating twice with constants as zero yield

\[
(w^2 - 1)u - 3u^2 - u'' = 0
\]

(24)

Balance the nonlinear term \(u^2\) with the highest order derivative \(u''\) gives \(m = 2\). Therefor, the solution of Eq. (23) is of the form

\[
u(\zeta) = \sum_{i=2}^{\infty} \alpha_i \left(\frac{G'}{G}\right)^i
\]

(25)

where \(\alpha_i\) are arbitrary constants. Substituting Eq. (25) with Eq. (5) into Eq. (24), then collecting all terms with the same power of \((G'/G)\) and setting each coefficient of power of \((G'/G)\) to zero, we achieve a system of algebraic equations and solving the system with maple, we obtain the set of solutions.
Case 1:
\[ \alpha_1 = 0, \; \alpha_2 = -2, \; \alpha_0 = \frac{1}{4} w^2 - \frac{1}{4}, \; \alpha_{-1} = 0, \; \alpha_{-2} = -\frac{1}{128} (w^2 - 1)^2, \]
\[ \mu = \frac{1}{16} \cdot \frac{1}{16} w^2, \; \lambda = 0, \; w = w \]  
(26)

Case 2:
\[ \alpha_1 = 0, \; \alpha_2 = -2, \; \alpha_0 = \frac{1}{12} w^2 - \frac{1}{12}, \; \alpha_{-1} = 0, \; \alpha_{-2} = -\frac{1}{128} (w^2 - 1)^2, \]
\[ \mu = -\frac{1}{16} + \frac{1}{16} w^2, \; \lambda = 0, \; w = w \]  
(27)

Case 3:
\[ \alpha_1 = 0, \; \alpha_2 = 0, \; \alpha_0 = -\frac{1}{2} \lambda^2 + \frac{1}{2} w^2 - \frac{1}{2}, \; \alpha_{-1} = \frac{1}{2} \lambda (w^2 - \lambda^2 - 1), \]
\[ \alpha_{-2} = -\frac{1}{8} (w^2 - \lambda^2 - 1)^2, \; \mu = \frac{1}{4} + \frac{1}{4} w^2 + \frac{1}{4} \lambda^2, \; \lambda = \lambda, \; w = w \]  
(28)

Case 4:
\[ \alpha_1 = 0, \; \alpha_2 = 0, \; \alpha_0 = -\frac{1}{2} \lambda^2 - \frac{1}{6} w^2 + \frac{1}{6}, \; \alpha_{-1} = -\frac{1}{2} \lambda (w^2 + \lambda^2 - 1), \]
\[ \alpha_{-2} = -\frac{1}{8} (w^2 + \lambda^2 - 1)^2, \; \mu = -\frac{1}{4} + \frac{1}{4} w^2 + \frac{1}{4} \lambda^2, \; \lambda = \lambda, \; w = w \]  
(29)

Case 5:
\[ \alpha_1 = \alpha_1, \; \alpha_2 = -2, \; \alpha_0 = -\frac{1}{2} \lambda^2 + \frac{1}{2} w^2 - \frac{1}{8} \alpha_1^2, \; \alpha_{-1} = 0, \]
\[ \alpha_{-2} = 0, \; \mu = \frac{1}{4} - \frac{1}{16} \alpha_1^2, \; \lambda = \frac{1}{2} \alpha_1, \; w = w \]  
(30)

Case 6:
\[ \alpha_1 = \alpha_1, \; \alpha_2 = -2, \; \alpha_0 = -\frac{1}{6} \lambda^2 - \frac{1}{6} w^2 - \frac{1}{8} \alpha_1^2, \; \alpha_{-1} = 0, \]
\[ \alpha_{-2} = 0, \; \mu = -\frac{1}{4} + \frac{1}{4} w^2 + \frac{1}{16} \alpha_1^2, \; \lambda = -\frac{1}{2} \alpha_1, \; w = w \]  
(31)

Substituting Eq. (26) with Eq. (6) into Eq. (10) and simplify, we obtain the following
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\[
\begin{align*}
{u}_1(\zeta) &= -2 \left[ \frac{\sqrt{w^2-1}}{4} \left( A \sinh \left( \frac{\sqrt{w^2-1}}{4} \zeta \right) + B \cosh \left( \frac{\sqrt{w^2-1}}{4} \zeta \right) \right)^2 \right. \\
  &\quad \left. - \frac{1}{128} \left( w^2-1 \right)^2 \left[ \frac{\sqrt{w^2-1}}{4} \left( A \sinh \left( \frac{\sqrt{w^2-1}}{4} \zeta \right) + B \cosh \left( \frac{\sqrt{w^2-1}}{4} \zeta \right) \right)^2 \right] \right] \\
  &\quad + \frac{1}{4} w^2 - \frac{1}{4} \\

\end{align*}
\]

Substituting Eq. (27) with Eq. (6) into Eq. (10) and simplify, we obtain the following

\[
\begin{align*}
{u}_2(\zeta) &= -2 \left[ \frac{\sqrt{1-w^2}}{4} \left( A \sinh \left( \frac{\sqrt{1-w^2}}{4} \zeta \right) + B \cosh \left( \frac{\sqrt{1-w^2}}{4} \zeta \right) \right)^2 \right. \\
  &\quad \left. - \frac{1}{128} \left( w^2-1 \right)^2 \left[ \frac{\sqrt{1-w^2}}{4} \left( A \sinh \left( \frac{\sqrt{1-w^2}}{4} \zeta \right) + B \cosh \left( \frac{\sqrt{1-w^2}}{4} \zeta \right) \right)^2 \right] \right] \\
  &\quad + \frac{1}{12} w^2 - \frac{1}{12} \\

\end{align*}
\]

Substituting Eq. (28) with Eq. (6) into Eq. (10) and simplify, we obtain the following.

\[
\begin{align*}
{u}_3(\zeta) &= -\frac{1}{2} \lambda^2 + \frac{1}{2} w^2 - \frac{1}{2} \\
  &\quad + \frac{1}{2} \lambda (w^2 - \lambda^2 - 1) \left[ \frac{\sqrt{w^2-1}}{2} \left( A \sinh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) \right)^{-1} \right. \\
  &\quad \left. - \frac{1}{8} \left( w^2 - \lambda^2 - 1 \right)^2 \left[ \frac{\sqrt{w^2-1}}{2} \left( A \sinh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) \right)^{-1} \right] \right] \\

\end{align*}
\]
Substituting Eq. (29) with Eq. (6) into Eq. (10) and simplify, we obtain the following

\[
u_4(\zeta) = -\frac{1}{2} \lambda^2 - \frac{1}{6} w^2 + \frac{1}{6}
\]

\[
-\frac{1}{2} \lambda (w^2 + \lambda^2 - 1) \left\{ \frac{\sqrt{1-w^2}}{2} \begin{bmatrix} A \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \\ A \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \end{bmatrix} \right\}^{-1}
\]

\[
-\frac{1}{8} (w^2 + \lambda^2 - 1)^2 \left\{ \frac{\sqrt{1-w^2}}{2} \begin{bmatrix} A \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \\ A \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \end{bmatrix} \right\}^{-2}
\]

Substituting Eq. (30) with Eq. (6) into Eq. (10) and simplify, we obtain the following

\[
u_5(\zeta) = -\frac{1}{2} + \frac{1}{2} w^2 - \frac{1}{8} \alpha_i^2 + \alpha_i \left\{ \frac{\sqrt{w^2-1}}{2} \begin{bmatrix} A \sinh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) \\ A \cosh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) + B \sinh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) \end{bmatrix} \right\}
\]

\[-2 \left\{ \frac{\sqrt{w^2-1}}{2} \begin{bmatrix} A \sinh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) \\ A \cosh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) + B \sinh \left( \frac{\sqrt{w^2-1}}{2} \zeta \right) \end{bmatrix} \right\}^2
\]

Substituting Eq. (31) with Eq. (6) into Eq. (10) and simplify, we obtain the following

\[
u_6(\zeta) = \frac{1}{6} - \frac{1}{6} w^2 - \frac{1}{8} \alpha_i^2 + \alpha_i \left\{ \frac{\sqrt{1-w^2}}{2} \begin{bmatrix} A \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \\ A \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \end{bmatrix} \right\}
\]

\[-2 \left\{ \frac{\sqrt{1-w^2}}{2} \begin{bmatrix} A \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \\ A \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) \end{bmatrix} \right\}^{-1}
\]
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\[ -2 \left\{ \frac{\sqrt{1-w^2}}{2} \right\} \frac{A \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right)}{A \cosh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right) + B \sinh \left( \frac{\sqrt{1-w^2}}{2} \zeta \right)} \right\}^2 \]  

(37)

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (32) are same can be written as

\[ u_1^1 (\zeta) = \frac{w^2 - 1}{8} \left[ 2 - \tanh^2 \left( \frac{\sqrt{w^2 - 1}}{4}(x - wt) \right) - \coth^2 \left( \frac{\sqrt{w^2 - 1}}{4}(x - wt) \right) \right] \]  

(38)

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (33) are same can be written as

\[ u_2^1 (\zeta) = \frac{w^2 - 1}{24} \left[ 2 - 3 \tanh^2 \left( \frac{\sqrt{1-w^2}}{4}(x - wt) \right) - 3 \coth^2 \left( \frac{\sqrt{1-w^2}}{4}(x - wt) \right) \right] \]  

(39)

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (34) can be written respectively as

\[ u_3^1 (\zeta) = \frac{1}{2} \left( w^2 - \lambda^2 - 1 \right) \left[ 1 + \frac{\lambda}{\sqrt{w^2 - 1}} \coth \left( \frac{\sqrt{w^2 - 1}}{2}(x - wt) \right) \right] \]  

\[ - \frac{w^2 - \lambda^2 - 1}{w^2 - 1} \coth^2 \left( \frac{\sqrt{w^2 - 1}}{2}(x - wt) \right) \]  

\[ \left( w^2 - \lambda^2 - 1 \right) \]  

(40)

\[ u_3^1 (\zeta) = \frac{1}{2} \left( w^2 - \lambda^2 - 1 \right) \left[ 1 + \frac{\lambda}{\sqrt{w^2 - 1}} \tanh \left( \frac{\sqrt{w^2 - 1}}{2}(x - wt) \right) \right] \]  

\[ - \frac{w^2 - \lambda^2 - 1}{w^2 - 1} \tanh^2 \left( \frac{\sqrt{w^2 - 1}}{2}(x - wt) \right) \]  

(41)

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (35) can be written, respectively, as
\[ u_{4i}(\zeta) = \frac{-1}{2} \lambda^2 - \frac{1}{6} w^2 + \frac{1}{6} \frac{\lambda(w^2 + \lambda^2 - 1)}{\sqrt{1-w^2}} \coth \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]
\[ - \frac{1}{2} \left( w^2 + \lambda^2 - 1 \right)^2 \coth^2 \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]

\[ u_{4i}(\zeta) = \frac{-1}{2} \lambda^2 - \frac{1}{6} w^2 + \frac{1}{6} \frac{\lambda(w^2 + \lambda^2 - 1)}{\sqrt{1-w^2}} \tanh \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]
\[ - \frac{1}{2} \left( w^2 + \lambda^2 - 1 \right)^2 \tanh^2 \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]

In particular, when setting \( B = 0 \) and \( A \neq 0 \), or \( A = 0 \) and \( B \neq 0 \), the solutions of Eq. (36) can be written respectively as

\[ u_{5i}(\zeta) = \frac{-1}{2} + \frac{1}{2} w^2 - \frac{1}{8} \alpha_i^2 + \alpha_i \frac{\sqrt{w^2 - 1}}{2} \tanh \left( \frac{\sqrt{w^2 - 1}}{2} (x - wt) \right) \]
\[ - \frac{w^2 - 1}{2} \tanh^2 \left( \frac{\sqrt{w^2 - 1}}{2} (x - wt) \right) \]

\[ u_{6i}(\zeta) = \frac{-1}{2} + \frac{1}{2} w^2 - \frac{1}{8} \alpha_i^2 + \alpha_i \frac{\sqrt{w^2 - 1}}{2} \coth \left( \frac{\sqrt{w^2 - 1}}{2} (x - wt) \right) \]
\[ - \frac{w^2 - 1}{2} \coth^2 \left( \frac{\sqrt{w^2 - 1}}{2} (x - wt) \right) \]

In particular, when setting \( B = 0 \) and \( A \neq 0 \), or \( A = 0 \) and \( B \neq 0 \), the solutions of Eq. (37) can be written respectively as

\[ u_{6i}(\zeta) = \frac{-1}{6} \frac{1}{6} w^2 - \frac{1}{8} \alpha_i^2 + \alpha_i \frac{\sqrt{1-w^2}}{2} \tanh \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]
\[ - \frac{1-w^2}{2} \tanh^2 \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]

\[ u_{6i}(\zeta) = \frac{-1}{6} \frac{1}{6} w^2 - \frac{1}{8} \alpha_i^2 + \alpha_i \frac{\sqrt{1-w^2}}{2} \coth \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]
\[ - \frac{1-w^2}{2} \coth^2 \left( \frac{\sqrt{1-w^2}}{2} (x - wt) \right) \]
Graphical representation of the solutions of the Boussinesq equation:

Soliton solution $u_1(x, y)$ is singular kink wave solution. In Figure (2-1), we show the shape of the exact singular kink-type solution of the Boussinesq equation.

**Figure (2-1)** graph of the solution $u_1(x, y)$ for $w=2, \lambda=0$ with $0 \leq t \leq 20, -20 \leq x \leq 20$.

Soliton solution $u_3(x, y)$ is singular kink solution presented in Figure (2-2) which shows the shape of the exact singular kink-type solution of the Boussinesq equation.

**Figure (2-2)** graph of the solution $u_3(x, y)$ for $w=0.5, \lambda=0$ with $0 \leq t \leq 20, -20 \leq x \leq 20$.

Soliton solution $u_3(x, y)$ is singular kink wave solution given in Figure (2-3). This figure gives the shape of the exact singular kink-type solution of the Boussinesq equation.

**Figure (2-3)** graph of the solution $u_3(x, y)$ for $w=2, \lambda=1$ with $0 \leq t \leq 20, -20 \leq x \leq 20$. 


Solution \( u_3(x, y) \) is kink waves are traveling waves which arise from one asymptotic state to another. This is given in Figure (2-4) for showing the shape of the exact kink-type solution of the Boussinesq equation.

**Figure (2-4)** graph of the solution \( u_3(x, y) \) for \( w=2, \lambda=1 \) with \( 0 \leq t \leq 20, -20 \leq x \leq 20. \)

Figure (2-5) represents the solution \( u_4(x, y) \) of the Burgers-KDV equation exact one singular kink solution.

**Figure (2-5)** graph of the solution \( u_4(x, y) \) for \( w=0.5, \lambda=1 \) with \( 0 \leq t \leq 20, -20 \leq x \leq 20. \)

Figure (2-6) below shows the shape of the exact kink-type solution of the Boussinesq equation when the solution \( u_4(x, y) \) is kink waves are traveling waves which arise from one asymptotic state to another.

**Figure (2-6)** graph of the solution \( u_4(x, y) \) for \( w=0.5, \lambda=1 \) with \( 0 \leq t \leq 20, -20 \leq x \leq 20. \)
In Figure (2-7) below, we provide the shape of the exact kink-type solution of the Boussinesq equation for the solution $u_{5_1}(x,y)$ is kink waves are traveling waves which arise from one asymptotic state to another.

**Figure (2-7)** graph of the solution $u_{5_1}(x,y)$ for $w=2$, $\lambda=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.

Soliton solution $u_{5_1}(x,y)$ is singular kink wave solution given in Figure (2-8) which explains the shape of the exact singular kink-type solution of the Boussinesq equation.

**Figure (2-8)** graph of the solution $u_{5_2}(x,y)$ for $w=2$, $\lambda=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.

Solution $u_{6_1}(x,y)$ is kink waves are traveling waves which arise from one asymptotic state to another. This case is displayed in Figure (2-9) below for showing the shape of the exact kink-type solution of the Boussinesq equation.

**Figure (2-9)** graph of the solution $u_{6_1}(x,y)$ for $w=0.5$, $\lambda=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$. 
Solution $u_{b_2}(x, y)$ of the Burgers-KDV equation exact one singular kink solution is given in Figure (2-10) below.

**Figure (2-10)** graph of the solution $u_{b_2}(x, y)$ for $w=0.5$, $\lambda=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.

**Example (3)**
Burgers-KDV equation,

$$u_t + 6uu_x + \beta u_{xxx} + u_{xxxx} = 0$$  \hspace{1cm} (48)

will be converted the ordinary differential equation, by use transformation $\zeta = x - wt$.

$$-wu + \beta u' + 3u^2 + u'' = 0$$  \hspace{1cm} (49)

Balance the nonlinear term $u^2$ with the highest order derivative $u''$ gives $m = 2$, then,

$$u(\zeta) = \sum_{i=0}^{2} \alpha_i \left(\frac{G'}{G}\right)^i$$  \hspace{1cm} (50)

where $\alpha_i$ are constants. Substituting Eq. (50) with Eq. (5) into Eq. (49), then collecting all terms with the same power of $(G'/G)$ and setting each coefficient of power of $(G'/G)$ to zero, we achieve a system of algebraic equations and solving the system with Maple, we obtain the set of solution.

Case 1:

$$\begin{align*}
\alpha_1 &= \alpha_1, \quad \alpha_2 = -2, \quad \alpha_0 = -\frac{1}{8}\alpha_1^2, \quad \alpha_{-1} = 0, \quad \alpha_{-2} = 0, \\
\mu &= -\frac{1}{20}\beta\alpha_1 + \frac{1}{16}\alpha_1^3, \quad \lambda = \frac{1}{2}\alpha_1 + \frac{1}{5}\beta, \quad w = -\frac{6}{25}\beta^2
\end{align*}$$

$$w=0.5, \lambda=1 \text{ with } 0 \leq t \leq 20, \quad -20 \leq x \leq 20.$$
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Case 2:

\[
\alpha_1 = \alpha_2 = -2, \, \alpha_0 = -\frac{1}{8} \alpha_1^2 + \frac{2}{25} \beta^2, \, \alpha_{-1} = 0, \, \alpha_{-2} = 0, \quad \mu = -\frac{1}{20} \beta \alpha_1 + \frac{1}{16} \alpha_1^2, \, \lambda = -\frac{1}{2} \alpha_1 + \frac{1}{5} \beta, \, w = \frac{6}{25} \beta^2
\]

Case 3:

\[
\alpha_1 = 0, \, \alpha_2 = 0, \, \alpha_0 = \frac{3}{50} \beta^2 - \frac{1}{5} \beta \lambda - \frac{1}{2} \lambda^2, \, \alpha_{-1} = \frac{1}{50} \beta^2 \lambda - \frac{1}{2} \lambda^3 + \frac{1}{250} \beta^3 - \frac{1}{10} \lambda^2 \beta, \\
\alpha_{-2} = -\frac{1}{5000} (\beta^2 - 25 \lambda^2)^2, \, \mu = -\frac{1}{100} \beta^2 + \frac{1}{4} \lambda^2, \, \lambda = \lambda, \, w = \frac{6}{25} \beta^2
\]

Case 4:

\[
\alpha_1 = 0, \, \alpha_2 = 0, \, \alpha_0 = -\frac{1}{50} \beta^2 - \frac{1}{5} \beta \lambda - \frac{1}{2} \lambda^2, \, \alpha_{-1} = \frac{1}{50} \beta^2 \lambda - \frac{1}{2} \lambda^3 + \frac{1}{250} \beta^3 - \frac{1}{10} \lambda^2 \beta, \\
\alpha_{-2} = -\frac{1}{5000} (\beta^2 - 25 \lambda^2)^2, \, \mu = -\frac{1}{100} \beta^2 + \frac{1}{4} \lambda^2, \, \lambda = \lambda, \, w = \frac{6}{25} \beta^2
\]

Case 5:

\[
\alpha_1 = \frac{2}{5} \beta, \, \alpha_2 = -2, \, \alpha_0 = \frac{1}{20} \beta^2, \, \alpha_{-1} = \frac{1}{1000} \beta^3, \quad \alpha_{-2} = -\frac{1}{80000} \beta^4, \, \mu = -\frac{1}{400} \beta^2, \, \lambda = 0, \, w = \frac{6}{25} \beta^2
\]

Case 6:

\[
\alpha_1 = \frac{2}{5} \beta, \, \alpha_2 = -2, \, \alpha_0 = -\frac{3}{100} \beta^2, \, \alpha_{-1} = \frac{1}{1000} \beta^3, \quad \alpha_{-2} = -\frac{1}{80000} \beta^4, \, \mu = -\frac{1}{400} \beta^2, \, \lambda = 0, \, w = -\frac{6}{25} \beta^2
\]

Substituting Eq. (51) with Eq.(6) into Eq. (50) and simplify, we obtain the following

\[
u_1(\xi) = \alpha_1 \left\{ \frac{\beta}{10} \left[ \frac{A \sinh \left( \frac{\beta}{10} \left( x - \frac{6}{25} \beta^2 \xi \right) \right)}{A \cosh \left( \frac{\beta}{10} \left( x - \frac{6}{25} \beta^2 \xi \right) \right)} + B \cosh \left( \frac{\beta}{10} \left( x - \frac{6}{25} \beta^2 \xi \right) \right) \right] + \frac{1}{4} \alpha_1 - \frac{1}{10} \beta \right\}
\]
\[
-2 \left\{ \frac{\beta}{10} \left[ \frac{A}{10} \sinh \left( \frac{\beta}{10} \left( x - \frac{6}{25} \beta^2 t \right) \right) + B \cosh \left( \frac{\beta}{10} \left( x - \frac{6}{25} \beta^2 t \right) \right) \right] + \frac{1}{4} \alpha_1 - \frac{1}{10} \beta \right\}^2 - \frac{1}{8} \alpha_1^2
\]

(57)

Substituting Eq. (52) with Eq.(6) into Eq. (50) and simplify, we obtain the following

\[
u_3(\zeta) = \alpha_1 \left\{ \frac{\beta}{10} \left[ \frac{A}{10} \sinh \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) + B \cosh \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) \right] + \frac{1}{4} \alpha_1 - \frac{1}{10} \beta \right\}
\]

(58)

Substituting Eq. (53) with Eq.(6) into Eq. (50) and simplify, we obtain the following

\[
u_3(\zeta) = \frac{3}{50} \beta^2 - \frac{1}{5} \beta \lambda - \frac{1}{2} \lambda^2
\]

\[
+ \left( \frac{1}{50} \beta^2 \lambda - \frac{1}{2} \lambda^3 + \frac{1}{250} \beta^3 - \frac{1}{10} \lambda^2 \beta \right) \left\{ \frac{\beta}{10} \left[ \frac{A}{10} \sinh \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) + B \cosh \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) \right] \right\}^{-1}
\]

\[
- \frac{1}{5000} (\beta^2 - 25\lambda^2)^2 \left\{ \frac{\beta}{10} \left[ \frac{A}{10} \sinh \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) + B \cosh \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) \right] \right\}^{-2}
\]

(59)

Substituting Eq. (54) with Eq.(6) into Eq. (50) and simplify, we obtain the following
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\[ u_4(\zeta) = -\frac{1}{50} \beta^2 - \frac{1}{5} \beta \lambda - \frac{1}{2} \lambda^2 \]

\[ + \left( \frac{1}{50} \beta^2 \lambda - \frac{1}{2} \lambda^2 + \frac{1}{250} \beta^3 - \frac{1}{10} \lambda^2 \beta \right) \left[ \frac{\beta}{10} \left[ 2 \sinh \left( \frac{\beta}{10} x + \frac{6}{25} \beta^2 t \right) \right] + \frac{\beta}{10} \left[ 2 \cosh \left( \frac{\beta}{10} x + \frac{6}{25} \beta^2 t \right) \right] \right]^{-1} \]

\[ - \frac{1}{5000} \left( \beta^2 - 25 \lambda^2 \right)^2 \left[ \frac{\beta}{10} \left[ 2 \sinh \left( \frac{\beta}{10} x + \frac{6}{25} \beta^2 t \right) \right] + \frac{\beta}{10} \left[ 2 \cosh \left( \frac{\beta}{10} x + \frac{6}{25} \beta^2 t \right) \right] \right]^{-2} \]  

(60)

Substituting Eq. (55) with Eq.(6) into Eq. (50) and simplify, we obtain the following

\[ u_5(\zeta) = 2 \beta \left[ \frac{\beta}{20} \left[ 2 \sinh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] + \frac{\beta}{20} \left[ 2 \cosh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] \right]^{-1} \]

\[ - 2 \left[ \frac{\beta}{20} \left[ 2 \sinh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] + \frac{\beta}{20} \left[ 2 \cosh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] \right] \]

\[ + \frac{\beta^2}{20} + \frac{\beta^3}{1000} \left[ \frac{\beta}{20} \left[ 2 \sinh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] + \frac{\beta}{20} \left[ 2 \cosh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] \right]^{-1} \]

\[ - \frac{\beta^4}{8000} \left[ \frac{\beta}{20} \left[ 2 \sinh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] + \frac{\beta}{20} \left[ 2 \cosh \left( \frac{\beta}{20} x + \frac{6}{25} \beta^2 t \right) \right] \right]^{-2} \]  

(61)

Substituting Eq. (56) with Eq.(6) into Eq. (50) and simplify, we obtain the following
\[
\begin{align*}
\frac{u_6(\zeta)}{2} &= \frac{2}{5} \beta \left[ \frac{A \sinh \left( \frac{6}{25} \beta^2 \right) + B \cosh \left( \frac{6}{25} \beta^2 \right)}{A \cosh \left( \frac{6}{25} \beta^2 \right) + B \sinh \left( \frac{6}{25} \beta^2 \right)} \right] \\
-2 \left[ \frac{A \sinh \left( \frac{6}{25} \beta^2 \right) + B \cosh \left( \frac{6}{25} \beta^2 \right)}{A \cosh \left( \frac{6}{25} \beta^2 \right) + B \sinh \left( \frac{6}{25} \beta^2 \right)} \right]^2
\end{align*}
\]

In particular, when setting \( B = 0 \) and \( A \neq 0 \), or \( A = 0 \) and \( B \neq 0 \), the solutions of Eq. (57) can be written respectively as

\[
\begin{align*}
u_1(\zeta) &= -\frac{\beta^2}{50} \left[ 1 - 2 \tanh \left( \frac{6}{25} \beta^2 \right) \right] + \tanh^2 \left( \frac{6}{25} \beta^2 \right) \\
u_2(\zeta) &= -\frac{\beta^2}{50} \left[ 1 - 2 \coth \left( \frac{6}{25} \beta^2 \right) \right] + \coth^2 \left( \frac{6}{25} \beta^2 \right)
\end{align*}
\]

In particular, when setting \( B = 0 \) and \( A \neq 0 \), or \( A = 0 \) and \( B \neq 0 \), the solutions of Eq. (58) can be written respectively as

\[
\begin{align*}
u_3(\zeta) &= \frac{3}{50} \beta^2 - \frac{1}{5} \beta \lambda - \frac{1}{2} \lambda^2 \\
+ \left( \frac{1}{50} \beta^2 + \frac{1}{250} \beta^2 - \frac{1}{10} \lambda^2 \beta \right) \left[ \frac{1}{10} \beta \left( \frac{6}{25} \beta^2 \right) \right] - \frac{\lambda^2}{2}
\end{align*}
\]
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\[
-u_3(\zeta) = \frac{3}{50} \beta^2 - \frac{1}{5} \beta \lambda - \frac{1}{2} \lambda^2
\]

\[
+ \left[ \frac{1}{50} \beta^2 \lambda - \frac{1}{2} \lambda^2 + \frac{1}{25 \beta^3 - \frac{1}{10} \lambda^2} \right] \left[ \frac{1}{2} \coth \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) - \frac{\lambda}{2} \right]^{-1}
\]

\[
- \frac{1}{5000} \left( \beta^2 - 25 \lambda^2 \right)^2 \left[ \frac{1}{2} \coth \left( \frac{\beta}{10} \left( x + \frac{6}{25} \beta^2 t \right) \right) - \frac{\lambda}{2} \right]^2
\]  

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (60) can be written respectively as

\[
u_1(\zeta) = -\frac{1}{50} \beta^2 - \frac{1}{5} \beta \lambda - \frac{1}{2} \lambda^2
\]

\[
+ \left[ \frac{1}{50} \beta^2 \lambda - \frac{1}{2} \lambda^2 + \frac{1}{25 \beta^3 - \frac{1}{10} \lambda^2} \right] \left[ \frac{1}{2} \coth \left( \frac{\beta}{10} \left( x - \frac{6}{25} \beta^2 t \right) \right) - \frac{\lambda}{2} \right]^{-1}
\]

\[
- \frac{1}{5000} \left( \beta^2 - 25 \lambda^2 \right)^2 \left[ \frac{1}{2} \coth \left( \frac{\beta}{10} \left( x - \frac{6}{25} \beta^2 t \right) \right) - \frac{\lambda}{2} \right]^2
\]  

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (61) are the same can be written as

\[
u_5(\zeta) = \frac{\beta^2}{200} \left( 5 + 4 \coth \left( \frac{\beta}{20} \left( x + \frac{6}{25} \beta^2 t \right) \right) \right)
\]

\[
- \tan \left( \frac{\beta}{20} \left( x + \frac{6}{25} \beta^2 t \right) \right)
\]

\[
+ \frac{\beta^2}{200} \left( 5 + 4 \coth \left( \frac{\beta}{20} \left( x + \frac{6}{25} \beta^2 t \right) \right) \right)
\]

In particular, when setting \(B = 0\) and \(A \neq 0\), or \(A = 0\) and \(B \neq 0\), the solutions of Eq. (62) are the same can be written as

\[
u_6(\zeta) = -\frac{\beta^2}{200} \left( 3 - 4 \coth \left( \frac{\beta}{20} \left( x - \frac{6}{25} \beta^2 t \right) \right) \right)
\]

\[
- \tan \left( \frac{\beta}{20} \left( x - \frac{6}{25} \beta^2 t \right) \right)
\]

\[
+ \frac{\beta^2}{200} \left( 3 - 4 \coth \left( \frac{\beta}{20} \left( x - \frac{6}{25} \beta^2 t \right) \right) \right)
\]
Graphical representation of the solutions of The Burgers-KDV equation:
Solution $u_{11}(x,y)$ is kink waves are traveling waves which arise from one asymptotic state to another. This is given in Figure (3-1) which shows the shape of the exact kink-type solution of Burgers-KDV equation.

Figure (3-1) graph of the solution $u_{11}(x,y)$ for $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.
Solution $u_{11}(x,y)$ of the Burgers-KDV equation exact one singular kink solution, Figure(3-2) below shows

Figure (3-2) graph of the solution $u_{12}(x,y)$ for $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.
Solution $u_{12}(x,y)$ is kink waves are traveling waves which arise from one asymptotic state to another. This is presented in Figure (3-3) to show the shape of the exact kink-type solution of Burgers-KDV equation.

Figure (3-3) graph of the solution $u_{21}(x,y)$ for $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$. 
Solution $u_2(x, y)$ of the Burgers-KDV equation exact one singular kink solution is provided in Figure (3-4) below.

![Figure (3-4) graph of the solution $u_2(x, y)$ for $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.](image)

Solution $u_3(x, y)$ is kink waves are traveling waves which arise from one asymptotic state to another, shown in Figure (3-5) which provides the shape of the exact kink-type solution of Burgers-KDV equation.

![Figure (3-5) graph of the solution $u_3(x, y)$ for $\lambda=1$, $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.](image)

Soliton solution $u_3(x, y)$ is singular kink solution given in Figure (3-6) to show the shape of the exact singular kink-type solution of the Burgers-KDV equation.

![Figure (3-6) graph of the solution $u_3(x, y)$ for $\lambda=1$, $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.](image)

Solution $u_4(x, y)$ is kink waves are traveling waves which arise from one asymptotic state to another, shown in Figure (3-7) which provides the shape of the exact kink-type solution of Burgers-KDV equation.
Figure (3-7) graph of the solution $u_4(x, y)$ for $\lambda = 1, \beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.

Soliton solution $u_4(x, y)$ is singular kink solution, presented in Figure (3-8) to give the shape of the exact singular kink-type solution of the Burgers-KDV equation.

Figure (3-8) graph of the solution $u_4(x, y)$ for $\lambda = 1, \beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.

Solution $u_5(x, y)$ of the Burgers-KDV equation exact one singular kink solution is provided in Figure (3-9) below.

Figure (3-9) graph of the solution $u_5(x, y)$ for $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$.

Solution $u_6(x, y)$ of the Burgers-KDV equation exact one singular kink solution is shown in Figure (3-10) below.

Figure (3-10) graph of the solution $u_6(x, y)$ for $\beta=1$ with $0 \leq t \leq 20$, $-20 \leq x \leq 20$. 
4. Conclusions

In this paper, the \((G'/G)\)-expansion method has been applied to find some exact solutions of the three equations, Burgers equation, Boussinesq equation, and Burgers-KDV equation. Abundant exact traveling wave solutions are constructed for these equations by the proposed method. It is noteworthy to observe that our solutions are more general and contain further arbitrary constants and the arbitrary constants imply that these solutions have rich local structures. This method is effective and has a successful use in finding exact solutions of nonlinear evolution equations.

References


[16] M. A. Akbar, M. N. Alam and M. G. Hafez, Applications of the novel(G'/G)-expansion method to construct traveling wave solutions to
An improved \((G'/G)\) – expansion method


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