Cosmologic Solution of Scalar and Spinorial Interactive Fields with the Dark Energy Pattern and a Magnetic Primordial not Perturbed Field in an Anisotropic Space-Time of Petrov D

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Abstract

In this document analyzes and obtains an exact solution to interactive and self-consistent spinorial and scalar fields with their interaction of fields, of the type dark energy, and a magnetic primordial field, where the magnetic field that do not induce electric currents or fields. It is obtained that the solution, from the geometrical point of view, is equivalent to the one obtained for a dark energy fluid and a primordial magnetic field. It is determined that the function of the scalar field $\varphi$ and the phase of the spinorial field $G$ are proportional in all the values of $t$, which establishes a kind of exclusive property of the dark energy under the conditions of the interactive and self-consistent spinorial and scalar fields. It is concluded that the influence of the magnetic field under this spinorial and scalar fields is relevant for short periods.

Keywords: cosmology, Einstein, exact, solution, quantum field

1 Introduction

The study of cosmic magnetic in the interstellar and intergalactic mediums is still current; it has been discussed about in [1, 2, 3]. Some of the exact solutions that have been studied are primordial not-perturbed magnetic fields
with fluids like the dark energy, the Hard Universe and the Ekpyrotic in an anisotropic space-time of the Petrov D. In [2], it is determined that for a group of fluids, if \( P = \lambda \mu \), for \( \lambda \in [1/3, 1] \), there is no solution in \( \mathbb{R} \); the same happens for a Zeldovich fluid (\( \lambda = 1 \)), and for a magnetic field with no fluid. On the other hand, the importance of studying spinorial fields and interactive scalars as possible matter sources has been discussed in [4], where it has been obtained that the phase of the spinor tends to be proportional to the function of the scalar field for great periods. This triggered the interest of studying the possibilities of interaction between said fields, such as the spinorial, scalar, and the primordial magnetic in an anisotropic space-time of Petrov D; this allows to evaluate the influence of one in another, including the gravitational field.

The present work considers the self-consistent interaction of the spinorial and scalar fields, which leads to pressure and volume energy density of these fields; it is equivalent to the results obtained for dark energy in [1, 5], along with a magnetic field that does not produce electric currents.

2 The symmetry, the Einstein tensor, the Electromagnetic field and the dark energy

2.1 The symmetry and the Einstein tensor

The anisotropic symmetry of the Petrov D has been considered in [5], in the following way

\[
ds^2 = F dt^2 - t^{2/3} K (dx^2 + dy^2) - \frac{t^{2/3}}{K^2} dz^2,
\]

where \( F \) and \( K \) are functions of \( t \).

The components of the Einstein tensor [6] \( (G^\beta_\alpha = R^\beta_\alpha - 1/2 \delta^\beta_\alpha R) \) different from zero, of (1), are

\[
G^0_0 = \frac{4 K^2 - 9 t^2 \dot{K}^2}{12 t^2 K^2 F}, \tag{2}
\]

\[
G^1_1 = -\frac{3 K t \dot{K} \left(2 F - \dot{F} t\right) + 3 Ft^2 \left(2 K \dot{K} - 5 \dot{K}^2\right) + 4 K^2 \left(\dot{F} t + F\right)}{12 t^2 K^2 F^2}, \tag{3}
\]

\[
G^2_2 = G^1_1, \tag{4}
\]

\[
G^3_3 = -\frac{6 K t \dot{K} \left(2 F - \dot{F} t\right) - 3 Ft^2 \left(4 K \dot{K} - \dot{K}^2\right) + 4 K^2 \left(\dot{F} t + F\right)}{12 t^2 K^2 F^2}, \tag{5}
\]

where the points over the functions represents derivatives over time.
2.2 The magnetic field

The magnetic field is determined by the following lagrangian

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + L_{sp,sc}, \]  

where

\[ L_{sp,sc} = \frac{i}{2} (\bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi + \frac{1}{2} \phi,_{\alpha} \phi^{,\alpha} \Phi(S), \]

and it will be considered that the only components of the tensor of the electromagnetic field \[6\] \( F_{\mu\nu} \) different from zero are \( F_{12} = -F_{21} = B_{0z} = \text{const} \), where it is evidenced that the invariant \( F_{\mu\nu} F^{\mu\nu} = 2B(t)^2 = 2B_{0z}^2 \pi/(t^{4/3}K^2) \), where \( B(t) = B_{0z} \pi^{1/2}/(t^{2/3}K) \) is the magnitude of the effective magnetic field. It does not generate currents or induced electric fields because the fluid of the magnetic field \( \Phi \) does not change with time,

\[ d\Phi = B(t)dA(t) = B(t)\sqrt{g_{11}g_{22}}dxdy = B_{0z} \pi^{1/2}dxdy. \]

The selection of the field, in the established way, allows the compliance to the equations of the field \( F^{\mu\nu}_0 = 0 \); moreover, the equivalence to zero of the divergence of stress-energy tensor \( _{em}T^{\mu\nu}_\mu = 0 \), where this tensor \( _{em}T^{\mu\nu} \) is that because of the electro-magnetic fields, whose only components, different from zero, for \( _{em}T^\mu_\nu \), are

\[ _{em}T^0_0 = -_{em}T^1_1 = -_{em}T^2_2 = _{em}T^3_3 = \frac{B_{0z}^2}{8t^{4/3}K^2}. \]

2.3 Spinorial and interactive scalars fields

The spinorial and interactive scalar fields of this study are determined by the lagrangian(6) where \( \psi \) is the spinor, \( S = \bar{\psi} \psi \) is the invariant scalar of the spinorial field, the function \( \Phi(S) \) is a non-linear function of the spinorial field and \( \varphi \) is the function of the scalar field; moreover, \( \bar{\psi} \) is the hermitian spinorial conjugated function of Dirac, and it is defined as \( \bar{\psi} = \psi^s \gamma^0 \) (later, it will be defined \( \bar{\psi}^0 \)) and where the matrices \( g_{\mu\nu} \gamma^\nu = \gamma_\mu \) can be defined, according to metric in the following way

\[ \gamma_\mu = e^\eta_\mu \pi_\eta, \quad g_{\mu\nu} = e^\eta_\mu e^\beta_\nu h_{\eta\beta}, \]

where \( h_{\eta\beta} \) is the tensor of Minkowski of the flat world with the signature\((+,-,-,-)\), and the matrices \( \pi_\eta \) are the matrices of Dirac used for the flat space-time,
\(e^\mu_\eta\), \(e^\rho_\mu\) is the set of tetrad of the fourth vector. The matrices \(\bar{\gamma}^\eta\) are taken from the pattern:

\[
\bar{\gamma}^0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \quad \bar{\gamma}^1 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}, \quad \bar{\gamma}^2 = \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
i & 0 & 0 & 0
\end{bmatrix}, \quad \bar{\gamma}^3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
\]

The covariant spinorial derivative \(\nabla_\mu\) is defined as

\[
\nabla_\mu\psi = \frac{\partial}{\partial x^\mu} \psi - \Gamma_\mu \psi,
\]

where \(\Gamma_\mu\) are the spinorial matrices of the related coefficient, which is related to the symbol of Christoffel in the following way

\[
\Gamma_\mu = \frac{1}{4} g^{\rho\sigma} \left( (e^\eta_\rho)_{,\mu} e^\rho_\mu - \Gamma^\rho_{\mu\sigma} \gamma^\delta \gamma^\sigma \right).
\]

From the function (6), \(\bar{\gamma} (7), \) and the metric(1) are obtained the equations of the spinorial field, the scalar field and the components of the stress-energy tensor because of the fields \((sp,sc) T^{\rho}_{\mu}\). These equations and components can be expressed as:

\[
\frac{i}{2} \gamma^\nu \nabla_\mu \psi + \frac{i}{2} \nabla_\mu \gamma_\nu \psi - m \psi + \frac{1}{2} \varphi_{,\alpha} \varphi^{,\alpha} \Psi'(S) \psi = 0, \quad \Psi'(S) = \frac{d\Psi}{dS},
\]

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left[ \sqrt{-g} g^{\nu\mu} \{ \varphi_{,\mu} \Phi(S) \} \right] = 0,
\]

\[
sp,sc T^{\rho}_{\mu} = \frac{i}{4} g^{\rho\nu} \left[ \overline{\psi} \gamma_\mu \nabla_\nu \psi - \nabla_\mu \overline{\psi} \gamma_\nu \psi + \overline{\psi} \gamma_\nu \nabla_\mu \psi - \nabla_\nu \overline{\psi} \gamma_\mu \psi \right] + \varphi_{,\mu} \varphi^{,\rho} \Psi(S) - \delta^\rho_\mu \mathcal{E}_{sp,sc}.
\]

From (10), el set of tetrad can be selected in the following way:

\[
e^0_0 = \frac{1}{e^0_0} = \sqrt{F}, \quad e^1_1 = \frac{1}{e^1_1} = e^2_2 = \frac{1}{e^2_2} = t^{1/3} \sqrt{K}, \quad e^3_3 = \frac{1}{e^3_3} = \frac{t^{1/3}}{K}.
\]
3 The Einstein equations and the solutions of the magnetic field and the scalar and spinorial fields of dark energy

It will be assumed that the spinor $\psi$ and the scalar $\phi$ depend only on $t$, therefore, from the equation of the scalar field (16), it is obtained that

$$\varphi = \int \frac{C_{sc} \sqrt{F}}{t \Phi} dt + \varphi_0. \quad (19)$$

The equation of the spinorial field takes the pattern

$$\frac{\dot{\psi}_r}{\sqrt{F}} + \frac{\psi_r}{2t \sqrt{F}} + i \left( m - \frac{C_{sc}^2 \Phi, S}{2t^2 \Phi^2} \right) \psi_r = 0, \quad r = 1, 2, \quad (20)$$

where the point represents the partial derivative by $t$. From (20), it is obtained that

$$\overline{\psi \psi} = S = \frac{C_0}{t}, \quad (21)$$

where $C_0$ is a constant of integration.

From (21), and considering

$$\Phi = \frac{C_{sc}^2 S^2}{C_0^2 \left( \sigma - 2mC_0S \right)}, \quad (22)$$

where $\sigma$ is constant. The non-linear interaction in the lagrangian (7) takes the pattern

$$\frac{1}{2} \varphi, \alpha \varphi^\alpha \Phi(S) = \sigma - mS. \quad (23)$$

The physical sense of the constant $\sigma$ will be determined through energetic concepts that will be analyzed later. The components of the spinor of (20) can be expressed as

$$\psi_r = \frac{\alpha_r e^{-iG}}{\sqrt{t}}, \quad \psi_{r+2} = \frac{\alpha_{r+2} e^{iG}}{\sqrt{t}}, \quad r = 1, 2, \quad (24)$$

where $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ are constant related to $C_0$ in the sense $C_0 = \alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2$ and the phase of the spinor $G$ is

$$G = \int \left( m \sqrt{F} + \frac{\sqrt{F} C_{sc}^2 \Phi}{2 \Phi^2 C_0} \right) dt + G_0, \quad (25)$$
where $G_0$ is a constant.

The no-null components of the stress-energy tensor (17), in relation to the scalar and spinorial fields, which represent the volumetric density of the energy and the pressure for said fields, take the pattern

$$
\epsilon_{sp,sc} =_{sp,sc} T_0^0 = \frac{\dot{\phi}^2 \Phi}{2F} + \frac{mC_0}{t} = \sigma,
$$

$$
-P_{sp,sc} =_{sp,sc} T_1^1 =_{sp,sc} T_2^2 =_{sp,sc} T_3^3 = -\frac{C_{sc}^2 \Phi}{2t \Phi^2} - \frac{\dot{\phi}^2 \Phi}{2F} = \sigma,
$$

and meet the equivalence $sp,sc T^\mu_\nu = 0$.

The Einstein equations follow the pattern [?] $G^\alpha_\beta = \kappa T^\alpha_\beta$, where $T^\alpha_\beta =_{em} +_{sp,sc} T^\beta_\alpha$. From (2-5, 9, 26) has the following, independent from each other, equation system,

$$
\frac{4 K^2 - 9 t^2 \dot{K}^2}{12 t^2 K^2 F} - \frac{B_{0z}^2 + 8 \sigma t^{4/3} K^2}{8 t^{4/3} K^2} = 0,
$$

(27)

$$
-3K t \dot{K} \left( 2F - \dot{F} t \right) + 3F t^2 \left( 2K \ddot{K} - 5 \dot{K}^2 \right) + 4K^2 \left( \dot{F} t + F \right)
- \frac{12 t^2 K^2 F^2}{12 t^2 K^2 F^2}
- \frac{-t^{2/3} B_{0z}^2 + 8 \sigma t^2 K^2}{8 K^2 t^2} = 0,
$$

(28)

$$
-6K t \dot{K} \left( 2F - \dot{F} t \right) - 3F t^2 \left( 4K \ddot{K} - \dot{K}^2 \right) + 4K^2 \left( \dot{F} t + F \right)
- \frac{12 t^2 K^2 F^2}{12 t^2 K^2 F^2}
- \frac{t^{2/3} B_{0z}^2 + 8 \sigma t^2 K^2}{8 K^2 t^2} = 0,
$$

(29)

From the equation (27), it is obtained that

$$
F = \frac{2 \left( 4 K^2 - 9 t^2 \dot{K}^2 \right)}{3 t^{2/3} (B_{0z}^2 + 8 t^{4/3} \sigma K^2)}.
$$

(30)

Considering (30) in (28) and (29), it is obtained that both equations meet if

$$
\left( -27t^3 \dddot{K}^3 + 48K^2 \dot{K} \dddot{K} + 36 \dddot{K} K^2 \dot{t}^2 - 16K^3 \right) B_{0z}^2 +
+72t^{4/3} K^2 \sigma \left( -9 t^3 \dddot{K}^3 - 4 t^2 \dot{K} \dddot{K}^2 + 4 \dddot{K} K^2 \dot{t}^2 + 8 K^2 t \dddot{K} \dot{t} \right) = 0.
$$

(31)

The no complex found solution of the equation (31) follows the pattern

$$
K = \frac{A(t)}{4 \sigma t^{2/3}} + \frac{B_{0z}^2}{2A(t) t^{2/3}},
$$

(32)
where $A(t)$ is

$$A(t) = A = \sqrt[3]{\left(4t^2 + 2\sqrt{\frac{-2B_0^6 + 4t^4\sigma}{\sigma}}\right)^2}, \quad (33)$$

or the pattern

$$K = \left(\sqrt[3]{t^2 + \sqrt{\frac{-B_{0,z}^6 + 2t^4\sigma}{2\sigma}}} + \sqrt[3]{t^2 - \sqrt{\frac{-B_{0,z}^6 + 2t^4\sigma}{2\sigma}}}\right) t^{-2/3}. \quad (34)$$

The solution (34), can be re-expressed for being used in a flexible way as an interval where $|t| < B_{0,z}^{3/2}/(2\sigma)^{1/4}$, such as

$$K = B_{0,z} \cos \left(\frac{1}{3} \arctan \left(\frac{\sqrt{B_{0,z}^6 - 2t^4\sigma}}{2\sigma} t^{-2}\right)\right) \frac{1}{\sqrt{2\sigma}} t^{-2/3}. \quad (35)$$

From (33) and (32), the function $F$ in (30) can be expressed as

$$F = \frac{2 \left(t^2 (B_2^2 - A(t)^2)^2 + \sqrt{(-2B_{0,z}^6 + 4t^4\sigma)/(\sigma)} (-A(t)^4 + B_2^4)\right)}{3 \left(B_{0,z}^6 - 2t^4\sigma\right) (3B_2^2 A(t)^2 + A(t)^4 + B_2^4)} \quad (36)$$

where $B_2 = B_{0,z}\sqrt{2\sigma}$.

### 4 Analysis of the solution

The function $F$ in (36) and $K$ in (34) or (35) are equivalent to the obtained for the case of the magnetic field with a dark energy fluid in [1]; therefore, the conclusions regarding the lack of singularities, the change from an anisotropic to an isotropic regime with time increase, and the meaning of the magnetic field in the solution, are the same.

For the solution found for the spinorial, and the interactive and self-consistent fields can be determined directly that from (19), (21), (22) and (25), it is obtained that

$$G = \frac{C_{sc}}{C_0} \varphi + \text{const}, \quad (37)$$

where the constant in (37) can be taken as null because it only produces changes in the constants $\alpha_\beta$, in (24), but not in $C_0$ since $\alpha^2 = \alpha^* \alpha$. The relation (37) means that the analyzed model of the spinorial and scalar interactive and self-consistent fields, for a kind of congruent interaction with the concept
of dark energy, is that, the scalar field plays the role of the phase of the components of the spinor, of the spinorial field, where it determines, at least for this particular case, some characteristics of the matter that behaves such as dark matter.

The solution of the scalar field for the lagrangian (7), when lacking the magnetic field, can be determined in an analogue way to [5]; for this reason, the metric of the function $F = g_{00}$ takes the pattern $F = \frac{1}{1 + 3\sigma t^2}$; therefore, from (19), it is obtained that

$$\varphi_{B=0} = -\frac{2}{3} \frac{mC_0 \ln \left(t\sqrt{3}\sigma + \sqrt{1 + 3\sigma t^2}\right) \sqrt{3} - \sqrt{1 + 3\sigma t^2} \sqrt{\sigma}}{C_{esc} \sqrt{\sigma}} + \varphi_{0B=0},$$

(38)

where $\varphi_{0B=0}$ is a constant of integration and the relation (37) is still valid for this case. For great values of $t (t \to \infty)$ the influence of the magnetic field is refused and $\varphi \to \varphi_{B=0} + V_0$, where $V_0$ is a constant, but for shorts values of $t$,

$$\lim_{t \to 0} \varphi \to \frac{t(-2mC_0 + \sigma t)}{C_{esc} \sqrt[6]{\sigma}} + \varphi_{0}$$

(39)

and

$$\lim_{t \to 0} \varphi_{B=0} \to \frac{t(-2mC_0 + \sigma t)}{C_{esc}} + \varphi_{1B=0},$$

(40)

where $\varphi_{1B=0} = \varphi_{0B=0} + 2/(3C_{sc})$ and where it is determined that the role of the magnetic field over the scalar and spinorial fields is relevant at the beginning (when $B_{0,z}^2 \sigma^{-1/3} t^{-4/3} \to 0$), but it decreases, with the increase of $t$. The function of the scalar, and therefore, the phase of the spinor, have a minimum in $t = mC_0/\sigma$, if $C_{sc} > 0$ it is considered, or as a maximum on the contrary.

5 Conclusions

It was obtained an exact solution to the Einstein equations in a cosmologic anisotropic symmetry of Petrov D for the case of a non-lineal and self-consistent interaction of scalar and spinorial fields, which is congruent with the concept of dark energy and the presence of a primordial not-perturbed magnetic field that does not induce to electric currents or fields. From this solution, it is determined that it is geometrically equivalent to the one obtained in [1], when it is considered a dark energy fluid; therefore, the conclusions that can be determined through geometry (metric interval) are the same. In relation to the scalar and spinorial fields, it is determined that the function of the scalar field is proportional to the phase of the spinor, which establishes an exclusive characteristic of dark energy from the perspective of the matter, under the
context of the studied fields. It is established that the magnetic field plays a relevant role in relation to the behavior of the scalar field; therefore, in the phase of the spinor, for early periods (when $B_{0,z}^2\sigma^{-1/3}t^{-4/3} \rightarrow 0$), and it decreases its influence as the time increases over it and the geometry of the space-time. As well as in [1, 5], the anisotropic symmetry tends to become isotropic with time increase, and the scalar and the spinorial fields tend to behave the same way as it the magnetic field would not exist.

References


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