Interacting Scalar Field and LTB Gravity: "Factorized" Solutions

Antonio Zecca

Dipartimento di Fisica dell’ Università degli Studi di Milano (Retired)
GNFM, Gruppo Nazionale per la Fisica Matematica Milano - Italy

Abstract

The problem of the interaction of massive complex scalar field with gravity is studied in the context of Lemaitre Tolman Bondi space time. Solution of a factorized form of the coefficients of the metric tensor are determined explicitly. For mass less (complex) scalar field the factorized solution amount to a Robertson Walker like space time. The corresponding time evolution is explicitly given for a special choice of the integration constant. This is coherent with previous results. The time evolution is further discussed in the general mass less field case. For massive real scalar field it is shown that no non trivial solution of the scheme do exist.

Keywords: LTB - Scalar field equation - Einstein field equation - Scalar field-gravity interaction

1 Introduction

A study of interest in General Relativity is the interaction of spin field and gravity. The problem can be formulated by coupling the spin field equation in a given metric with the Einstein field equation whose source is assumed to be the energy momentum tensor of the spin field. Besides mathematical interest [3], results can be related to the problem of collapsing field by gravity

1Address: Dipartimento di Fisica, Via Celoria, 16 - 20133 Milano, Italy
black holes formation and cosmological inflation [4, 7], and accelerated late expansion of the universe [1].

The object of the present paper is to reconsider the simple case given by a massive complex scalar field interacting with the Lemaitre-Tolman-Bondi gravity whose metric tensor $g_{\mu\nu}$ is given by [5]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - e^{\Gamma(r,t)}dr^2 - Y^2(r,t)\left(d\theta^2 + \sin\theta^2d\phi^2\right)$$

The problem results then in the study of the coupled equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = kT_{\mu\nu}(\phi) \quad (k = 8\pi G/c^4)$$

$$\nabla^\alpha\nabla_\alpha\phi + m_0^2\phi = 0$$

$m_0$ the mass of the scalar field and $T_{\mu\nu}(\phi) = (1/2)[\partial_\mu\phi\partial_\nu\bar{\phi} + \partial_\mu\bar{\phi}\partial_\nu\phi - g_{\mu\nu}\partial_\alpha\phi\partial_\alpha\bar{\phi} - m_0^2\phi\bar{\phi}]$ the energy momentum tensor of the complex scalar field $\phi$.

By taking trace, the equation (2) can be rearranged into the form

$$R_{\mu\nu} = (k/2)(\partial_\mu\bar{\phi}\partial_\nu\phi + \partial_\mu\phi\partial_\nu\bar{\phi} - m_0^2\phi\bar{\phi}g_{\mu\nu})$$

The system of equations (2), (3) has been recently studied in the real case in [9] and in the complex case in [11]. [When dealing with [11] the correction [12] should also be considered.] Solutions have been determined that, in the mass less case, are coherent with analogous results of [6]. To that end the preliminary integration step of [9] has been employed in [11].

Here the study for "factorized" solution is developed systematically in case of complex massive scalar field. Accordingly the separated radial solutions are determined in the $m_o = 0$ (complex) scalar field case. They again are coherent with the results of [6, 11] and amount to a Robertson Walker metric. The separated time equation is completely integrated in a special case and discussed in the general mass less scalar field case. In case $m_o \neq 0$ it is shown that no non trivial factorized solution do exist for real scalar field (coherently with the fact that a particular complex solution exists [11, 12]).

2 Explicit equations

According to the discussion in [11] the consistency condition of the Einstein field equation, $\nabla^\mu T_{\mu\nu}(\phi) = 0$, and the independence of the trace of (4) on $\theta, \varphi$ are automatically satisfied by the fact that $\phi$ is forced to depend only on the $r$ and $t$ variables. Accordingly eq. (4) becomes explicitly (e., g. [11]):

$$\frac{2\ddot{\Gamma} + \dot{\Gamma}^2}{4} + 2\frac{\dddot{Y}}{Y} = k(\phi\bar{\phi} - \frac{1}{2}m_0^2\phi\bar{\phi})$$

$$2YY'' - YT'Y \frac{Y''}{Y^2} - e^{\Gamma}(\frac{\dddot{\Gamma}}{4} + \frac{\dddot{Y}}{Y} + \frac{\dddot{\gamma}}{2}) = k(\phi'\bar{\phi}' + \frac{1}{2}e^{\Gamma}m_0^2\phi\bar{\phi})$$
Scalar field interacting with LTB gravity

\[ -1 + \frac{e^{-\Gamma}}{2} [2YY'' + 2Y'^2 - YY'] - YY'' - \frac{YY'}{2} = \frac{k}{2} m_0^2 Y^2 \phi \bar{\phi} \]  

(7)

\[ 2 \frac{\dot{Y}'}{Y} - \Gamma \frac{Y''}{Y} = \frac{k}{2} (\dot{\phi} \bar{\phi}' + \bar{\phi} \phi') \]  

(8)

\[ \ddot{\phi} - e^{-\Gamma} \phi'' + e^{-\Gamma} (\frac{\Gamma'}{2} - 2 \frac{Y'}{Y}) \phi' + (\frac{\dot{\Gamma}}{2} + 2 \frac{\dot{Y}}{Y}) \phi + m_0^2 \phi = 0 \]  

(9)

where \( \phi = \phi(r, t) \). The equations (5)-(8) correspond respectively to the cases \( R_{tt}, R_{rr}, R_{\theta\theta}, R_{tr} \) in (4). The scalar field equation (3) takes the form (9). (e. g., \([8]\)). Solutions of the last equations were already given in [6] in the mass less case. According to an homogeneous and an in-homogeneous assumption one obtains respectively the Robertson-Walker space-time or an inherently in-homogeneous metric tensor. Recently the scheme has been considered in the general case of a massive complex scalar field [11]. In the mass less case it gives results coherent with those [6]. There, the integration procedure uses a first general integration step discussed in [9].

Here the eqs. (5)-(9) are studied by the alternative factorized assumptions:

\[ Y(r, t) = y(r) \xi(t), \quad \Gamma(r, t) = a(r) + \tau(t), \quad \phi(r, t) = b(r) T(t) \]  

(10)

with \( y = \bar{y}, \xi = \bar{\xi}, \Gamma = \bar{\Gamma}, \phi \in \mathbb{C} \).

From (10), (5) \( b(r) \) is of the form \( b(r) = c_1 e^{i \beta(r)} \) (\( c_1 \) constant) \( \beta(r) \in \mathbb{R} \). Equation (6) implies \( \beta'(r) = 0 \) or \( \beta = \beta_1 \in \mathbb{R} \) and \( \beta_1 \) constant. Therefore

\[ \phi(r, t) = b_1 T(t), \quad b_1 \in \mathbb{C} \]  

(11)

From (8), (10), (11) one has

\[ \xi^2(t) = \tau_1 e^{\tau(t)}, \quad (\dot{\tau} = 2 \frac{\xi}{\xi}) \]  

(12)

\( \tau_1 \) an integration constant and \( y(r) \), so far, arbitrary. By putting (10) into (5) one obtains

\[ \frac{\dot{\tau}}{2} + \frac{\tau^2}{4} + 2 \frac{\dot{\xi}}{\xi} = |b_1|^2 (\dot{T} \bar{T} - \frac{m_0^2}{2} T \bar{T}) \]  

(13)

From (6) one gets the separated equations

\[ 2 y'' - y' a' = c_1 e^{a(r)} y \]  

(14)

\[ e^{\tau(t)} (\dot{\tau} \xi^2 + 6 \dot{\xi}^2 + 2 \xi^2 k |b_1|^2 m_0^2 T \bar{T}) = 2 c_1 \xi^2 \]  

(15)

and from (7) one obtains the separated equations:

\[ -\frac{e^{a(r)}}{\tau_1 y^2} + \frac{y''}{y} + \frac{y'^2}{y^2} - \frac{1}{2} \frac{y'}{2 y} a'(r) = c_2 e^{a(r)} \]  

(16)

\[ \xi \ddot{\xi} + 2 \dot{\xi}^2 + \pm \frac{m_0^2}{2} |b_1|^2 T \bar{T} \xi^2 = \tau_1 c_2 \]  

(17)
$c_1, c_2$ separation constants. The scalar field equation simplifies to:

$$\ddot{T} + 3 \frac{\dot{T}}{\xi} \dot{T} + m_o^2 T = 0 \tag{18}$$

Comparing (14) and (16) one obtains:

$$e^{a(r)} = \frac{2\tau_1 y^2}{2 + (2c_2 - c_1)\tau_1 y^2} \tag{19}$$

Accordingly one gets:

$$ds^2 = dt^2 - \tau_1 e^{\tau(t)} \left[ \frac{e^{a(r)}}{\tau_1} dr^2 - y^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{20}$$

$$= dt^2 - \xi(t)^2 \left[ \frac{dy^2}{1 + (c_2 - \frac{c_1}{2})\tau_1 y^2} - y^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{21}$$

With a suitable choice of the integration constant eq. (21) represents a Robertson-Walker like metric in the radial coordinate $y$. What must now be possibly determined is the time evolution of the space-time background.

## 3 Separated time equations

Since $\xi(t)$ and $\tau(t)$ are related by (12) there follows that $T(t)$ and anyone of $\xi(t), \tau(t)$ are required to satisfy (13), (15), (17), (18). Equations (15), (17) are indeed the same equation. By making things explicit one has:

$$c_1 \equiv c_2 \tag{22}$$

There are then two time functions subjected to three conditions.

**3.1** Suppose first $m_o = 0$. Here one is left with the equations

$$\dot{\tau} = 2\dot{\xi}/\xi, \quad \ddot{T} + 3 \frac{\dot{T}}{\xi} \dot{T}/\xi = 0 \tag{23}$$

$$3\dot{\xi}/\xi = k|b_1|^2 \ddot{\phi}, \quad \ddot{\phi} + 2\dot{\phi} = c_1 \tau_1 \tag{24}$$

that correspond to (12), (13), (17), (18). From (23)one has

$$\ddot{T} = c_3 e^{-3\tau(t)} \equiv c_3 \tau_1^{3/2} \xi^{-3} \tag{25}$$

$c_3$ an integration constant. The last equation in (24) gives then

$$\dot{\xi} = \left( \frac{c_4}{\xi^4} + \frac{c_1 \tau_1}{2} \right)^{1/2}, \quad c_4 \in \mathbb{R} \tag{26}$$
Hence one finally has, in the a special case \( c_1 \)
\[
\xi = \left( \pm 3\sqrt{c_4}t + 3c_5 \right)^{1/3} \quad (c_1 = 0)
\]
\[
T = \frac{c_3\tau_1^2}{\pm 3\sqrt{c_4}} \log |\pm 3\sqrt{c_4}t + 3c_5| \quad (c_1 = 0)
\]
\[
\tau = (2/3)\log |\pm 3\sqrt{c_4}t + 3c_5| \quad (c_1 = 0)
\]
while the first equation in (24) implies a relation among \( c_4, |b_1|, c_5, \tau_1 \). The results are analogous to those of [11].

3.2 Suppose now \( m_o \neq 0 \), \( c_1 \neq 0 \) and real \( T \). Under the present assumptions it is possible to see that the system of the separated time equations (12), (13), (15), (17) does not have meaningful solution for real \( T(t) \). To show this, two different equations in the function \( T(t) \) are derived that do not admit of common non trivial solutions. The procedure is schematized in three steps.

i) A first equation for \( T \) can be obtained by in the following way. By deriving (15), the exponential \( \exp \tau(t) \) can be eliminated, by using again (15), thus obtaining (\( \dot{x} = \frac{dx}{dt} \)):
\[
2 \ddot{\tau} + 4\dot{\tau}\ddot{\tau} + 3\dot{\tau}^3 = -4k|b_1|^2 m_o^2 \left[ \frac{d}{dt}T^2 + \dot{T}T^2 \right]
\]
By comparing with the time derivative of (13) were \( \ddot{\xi}/\xi \) is expressed in terms of \( \dot{\tau}, \ddot{\tau} \), one gets:
\[
\dot{\tau} \left( \frac{3}{2} \ddot{\tau} + \frac{3}{4} \dot{\tau}^2 \right) = -k|b_1|^2 \left[ \frac{5}{6} \frac{m_o^2}{m_o^2} \frac{d}{dt}T^2 + m_o^2 \dot{\tau} + \frac{1}{3} \frac{d}{dt}\ddot{T} \right]
\]
By comparing (31) with (13) with again \( \ddot{\xi}/\xi \) expressed in terms of \( \dot{\tau}, \ddot{\tau} \) one obtains
\[
\dot{\tau} = - \left[ 6\ddot{T}^2 + 3m_o^2 T^2 \right]^{-1} \left[ 5m_o^2 \frac{d}{dt}T^2 + 2 \frac{d}{dt}\ddot{T} \right]
\]
that must be identified with \( \dot{\tau} \) of
\[
3\dot{\tau}\ddot{T} = -2 (\ddot{T} + m_o^2 T)
\]
obtained from (18). From (32), (33) there follows a closed equation for \( T(t) \) that, by setting \( y(T) = \ddot{T} \), and then \( z = y^2 \) reads
\[
T^2 z' - 6z = -2m_o^2 \dot{T}^3 \quad (z = y^2)
\]
The \( z \) equation could exactly be integrated, but it is not necessary for the present purposes.

ii) A second equation for \( T(t) \) can be obtained from (13) in the form
\[
6\ddot{\tau} + 3\dot{\tau}^2 = k|b_1|^2 (4\ddot{T}^2 - 2m_o^2 T^2)
\]
where \( \dot{\tau}, \ddot{\tau} \) are given by (33) and its derivative. By further setting \( y(T) = \dot{T} \) and then \( y^2 = z \) in the resulting equation one obtains \( (z' = dz/dT) \):

\[
3zz'' - 5m_o^2 T z' - 2z'^2 - 3m_o^2 z(T - 2) - 2m_o^4 T^2 + 6k|b_1|^2 z^2 = 0
\]  

(36)

iii) One can now compare (34) and (36). By using also the derivative of (34) with respect to \( T \) it is possible to eliminate both \( z' \) and \( z'' \) in (36) to get

\[
[6(T - 1) - k|b_1|^2 T^4] z^2 + m_o^2[3T^3 + T^5/2] z = 0
\]  

(37)

that is \( z = 0 \) or

\[
z = \left[m_o^2(3T^3 + T^5/2)\right] \left[6(1 - T) + k|b_1|^2 T^4\right]^{-1}
\]  

(38)

From (34) and (38) one finally obtains

\[
z' = 2m_o^2 \left[3T + 6T^2 + (3/2)T^3 - k|b_1|^2 T^5\right] \left[6(1 - T) + k|b_1|^2 T^4\right]^{-1}
\]  

(39)

to be compared with the \( T \) derivative of (38). One gets then an algebraic equation of the form \( T \cdot P_8(T) = 0 \) where \( P_8(T) \) is a polynomial of degree 8. Hence the possible \( T \)'s are of the trivial form \( T = constant \). The same conclusion can be obtained if \( c_1 = 0 \) and \( T \) is real.

4 Remarks and comments

The factorized solution of the interacting scheme of equations (5-9) has been studied by distinguishing according to the value of the mass \( m_o \).

In case \( m_o = 0 \) the result obtained holds for complex field. It is similar to the ones already obtained in [6, 11] (see also [12]) by different procedures. It may be of interest the fact that there results a Robertson-Walker like space time picture. In such case the time evolution of the scheme has been given in (27)-(29) for a special choice of an integration constant. It is worth noting that, taking derivative of (26), \( \xi(t) \) in general satisfies the equation

\[
\ddot{\xi}\xi^5 = -2c_4
\]  

(40)

so that \( \ddot{\xi} \) is positive for positive \( \xi(t) \) if \( c_4 < 0 \). On the other hand, by taking derivative of (27), there is the evidence in that special case of an initial inflationary cosmological evolution. There is therefore the open problem of establishing whether a positive global solution of the time equations exists that shows both an initial inflationary phase and a late accelerated expansion. Unfortunately those properties, that would be of interest for cosmological applications, do not seem to be simultaneously possible. Indeed he equation (26) can be reported to

\[
\frac{1}{2} \int \frac{dx}{\sqrt{a^2 + bx}} = t + t_o, \quad x = \xi^2, \quad a = c_4, \quad b = c_1 \tau_1
\]  

(41)
By considering the series expansion near $x = 0$, the equation (41) gives

$$\frac{1}{3} \frac{x^3}{\sqrt{a}} = \frac{\xi^3}{3\sqrt{a}} \simeq t + t_o$$

(42)

Since $a = c_4 < 0$ to ensure later cosmological accelerated expansion, there follows that solutions for $\xi(t) \rightarrow 0$ cannot be real.

In case $m_o \neq 0$ it has been shown that no non trivial time solutions exist for real scalar field $T = \bar{T}$. This is coherent with the results of [11]. It would be of interest, both on mathematical and physical point of view, to determine possible solutions of the interacting scheme that do not obey to the factorization assumption (10).

Finally one can note that the result of the paper represents, in the LTB metric, the counterpart for the scalar field of what happens in the Robertson Walker metric, for massive spin field. Indeed, in the Robertson Walker space time, "standard" factorized solutions for the coupled Einstein-massive spinor field equation are not possible [10].

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