Minimal Quantities and Measurable Variant of Gravity II.
Strong Principle of Equivalence and Transition to High Energies

Alexander Shalyt-Margolin

Institute for Nuclear Problems, Belarusian State University
11 Bobruiskaya str., Minsk 220040, Belarus

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Abstract

This paper continues a study of gravity within the scope of the measurability notion introduced by the author in his previous works. Based on the earlier results, it is shown that the Strong Principle of Equivalence (SPE) of General Relativity may be reformulated in terms of measurable quantities and is valid in this case at low energies far from the Planck’s. Next, the possibility for generalization SPE of a measurable analog of gravity in the ultraviolet (Planck) energy region is analyzed.

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1 Introduction

This paper is a continuation of a study into a quantum theory and gravity in terms of the measurability notion, initiated in [1]–[9], with the aim to form the above-mentioned theories proceeding from the variations (increments) dependent on the existent energies.
These theories should not involve the infinitesimal variations $dt, dx_i, dp_i, dE, i = 1, ..., 3$ and, in general, any abstract small quantities $\delta t, \delta x_i, \delta E, \delta p_i, ....$

In work [10] in the general form it is demonstrated that all the basic ingredients of General Relativity (GR) have their measurable analogs, the way to derive every term in a measurable variant of the Einstein equations is presented. Passage of the measurable analog of GR to the ultraviolet (Planck) region is considered, showing that it is quite natural from the viewpoint of the methods and approaches developed in [10].

This paper directly follows from [10]. Here it is demonstrated that a measurable analog of the Strong Principle of Equivalence (SPE) is valid, i.e., SPE may be formulated entirely in terms of measurable quantities at low energies $E \ll E_p$.

Note, as in GR only low energy regions $E \ll E_p$ are considered, it is implied that SPE is valid in GR just in this energy region. The region of high energies $E \approx E_p$ belongs to Quantum Gravity that has not be formed by now. Nevertheless, in terms of the measurability notion, we can perform an initial analysis of the possible generalization of SPE to the Planck (quantum-gravity) region. This is the principal object of the work.

The structure of this paper is as follows. Section 2 briefly outlines the necessary preliminary information from [1]–[9]. At the same time, for better understanding, some aspects are elucidated and supplemented. In particular, of importance are Remark 2.3–Remark 2.5. In Section 3, proceeding from the results of Section 4 in [10], it is indicated that the Strong Principle of Equivalence (SPE) may be reformulated in terms of the measurability notion at low energies $E \ll E_p$ and is valid in this case.

In Section 4, within the scope of the space-time foam notion, the possibility for generalization of SPE for a measurable analog of gravity in the ultraviolet (Planck) region is analyzed.

# 2 Necessary Preliminary Information

Let us briefly consider the earlier results [1]–[9] laying the basis for this study. It is assumed that there is a minimal (universal) unit for measurement of the length $\ell$ corresponding to some maximal energy $E_\ell = \frac{\hbar c}{\ell}$ and a universal unit for measurement of time $\tau = \ell/c$. Without loss of generality, we can consider $\ell$ and $\tau$ at Plank’s level, i.e. $\ell = \kappa l_p, \tau = \kappa t_p$, where the numerical constant $\kappa$ is on the order of 1. Consequently, we have $E_\ell \propto E_p$ with the corresponding proportionality factor.

Then we consider a set of all nonzero momenta

$$P = \{p_x\}, i = 1, .., 3; |p_x| \neq 0.$$  

(1)
From this set we can isolate a set of the **Primarily Measurable** momenta characterized by the property

\[ p_{x_i} = p_{N_i} = \frac{\hbar}{N_i \ell}, \] (2)

where \( N_i \) is an integer number and \( p_{x_i} \) is the momentum corresponding to the coordinate \( x_i \).

From these formula it is not unreasonable to propose the following definition:

**Definition 1. Primary Measurability**

**1.1.** Any variation in \( \Delta x_i \) for the coordinates \( x_i \) and \( \Delta t \) of the time \( t \) is considered **primarily measurable** if

\[ \Delta x_i = N_{\Delta x_i} \ell, \Delta t = N_{\Delta t} \tau, \] (3)

where \( N_{\Delta x_i} \neq 0 \) and \( N_{\Delta t} \neq 0 \) are integer numbers.

**1.2.** Let us define any physical quantity as **primary or elementary measurable** when its value is consistent with point **1.1** of this Definition.

So, from **Definition 1.** it directly follows that all the momenta satisfying 2) are the **Primarily Measurable** momenta.

Then we consider formula (2) and **Definition 1.** with the addition of the momenta \( p_{x_0} = p_{N_0} = \frac{\hbar}{N_0 \ell} \), where \( N_0 \) is an integer number corresponding to the time coordinate (\( N_{\Delta t} \) in formula (3)).

For convenience, we denote **Primarily Measurable Quantities** satisfying **Definition 1.** in the abbreviated form as \( \text{PMQ} \).

It is clear that \( \text{PMQ} \) is inadequate for studies of the physical processes. To illustrate, the space-time quantities

\[ \frac{\tau}{N_t} = p_{N_{tC}} \frac{\ell^2}{\hbar}, \]
\[ \frac{\ell}{N_i} = p_{N_i} \frac{\ell^2}{\hbar}, 1 = 1, ..., 3, \] (4)

where \( p_{N_i}, p_{N_{tC}} \) are **Primarily Measurable** momenta, up to the fundamental constants are coincident with \( p_{N_i}, p_{N_{tC}} \) and they may be involved at any stage of the calculations but, evidently, they are not \( \text{PMQ} \) in the general case.

Thus, it is reasonable to use **Definition 2.**

**Definition 2. Generalized Measurability**

We define any physical quantity at all energy scales \( E \leq E_\ell \) as **generalized measurable** or, for simplicity, **measurable** if any of its values may be obtained in terms of \( \text{PMQ} \) specified by points **1.1, 1.2** of **Definition 1.**
Remark 2.1 What is the main difference between Primarily Measurable Quantities (PMQ) and Generalized Measurable Quantities (GMQ)? PMQ defines variables which may be obtained as a result of an immediate experiment. GMQ defines the variables which may be calculated based on PMQ, i.e. based on the data obtained in previous clause.

The main target of the author is to form a quantum theory and gravity only in terms of measurable quantities (or of PMQ).

Now we consider separately the two cases.

A) Low Energies, \( E \ll E_\ell \).

In \( \mathbf{P} \) we consider the domain \( \mathbf{P}_{LE} \subset \mathbf{P} \) (LE is abbreviation of ”Low Energies”) defined by the conditions

\[
\mathbf{P}_{LE} = \{ p_{x_i} \}, \quad i = 1, \ldots, 3; \quad P_\ell \gg |p_{x_i}| \neq 0,
\]

where \( P_\ell = E_\ell / c \) - maximal momentum.

In this case the formula of (2) takes the form

\[
N_i = \frac{\hbar}{p_{x_i} \ell}, \text{ or } \quad p_{x_i} = \frac{\hbar}{N_i \ell},
\]

\[
|N_i| \gg 1,
\]

where the last row of the formula (6) is given by the requirement (5).

As the energies \( E \ll E_\ell \) are low, i.e. \( |N_i| \gg 1 \), primary measurable momenta are sufficient to specify the whole domain of the momenta to a high accuracy \( \mathbf{P}_{LE} \).

It is clear that

\[
[N_i] \leq N_i \leq [N_i] + 1,
\]

where \([n] \) defines the integer part of \( n \). Then \( |N_i|^{-1} \) falls within the interval with the finite points \( ||N_i||^{-1} \) and \( ||N_i|+1||^{-1} \) (which of the numbers is greater or smaller, depends on a sign of \( N_i \)). In any case we have

\[
|N_i|^{-1} - [N_i]^{-1} \leq |([N_i] + 1)^{-1} - [N_i]^{-1}| = |([N_i] + 1)[N_i]^{-1}|.
\]

Thus, the difference between \( p_{N_i} \) and \( p_{[N_i]} \) is negligibly small. Therefore, the primary measurable momenta \( p_{N_i}, (|N_i| \gg 1) \) are sufficient to specify the whole domain of the momenta to a high accuracy \( \mathbf{P}_{LE} \).

This means that in the indicated domain a discrete set of primary measurable momenta \( p_{N_i}, (i = 1, \ldots, 3) \) from formula (6) varies almost continuously, practically covering the whole domain.
That is why further $P_{LE}$ is associated with the domain of \textbf{primary measurable} momenta, satisfying the conditions of the formula (5) (or (6)). Of course, all the calculations of point A) also comply with the \textbf{primary measurable} momenta $P_{N_{i}c} = P_{N_{0}}$ in formula (4). Because of this, in what follows we understand $P_{LE}$ as a set of the \textbf{primary measurable} momenta $P_{x_{\mu}} = P_{N_{i}}, (\mu = 0, \ldots, 3)$ with $|N_{\mu}| \gg 1$.

\textbf{Remark 2.2.} It should be noted that, as all the experimentally involved energies $E$ are low, they meet the condition $E \ll E_{\ell}$, specifically for LHC the maximal energies are $\approx 10 TeV = 10^{4} GeV$, that is by 15 orders of magnitude lower than the Planck energy $\approx 10^{19} GeV$. But since the energy $E_{\ell}$ is on the order of the Planck energy $E_{\ell} \propto E_{p}$, in this case all the numbers $N_{i}$ for the corresponding momenta will meet the condition $min|N_{i}| \approx 10^{15}$, i.e., the formula of (6). So, all the experimentally involved momenta are considered to be \textbf{primary measurable} momenta, i.e. $P_{LE}$ at low energies $E \ll E_{\ell}$.

In this way in the proposed paradigm at low energies $E \ll E_{p}$ any momentum with $p_{x_{\mu}}, \mu = 0, \ldots, 3$ takes the form $p_{x_{\mu}} = p_{N_{\mu}}$, where $N_{\mu}$ - integer with the property $|N_{\mu}| \gg 1$.

Further for the fixed point $x_{\mu}$ we use the notion $p_{x_{\mu}} = p_{N_{x_{\mu}}}$ or $p_{x_{\mu}} = p_{N_{x_{\mu}} + \Delta p_{x_{\mu}}}$. Naturally, the small variation $\Delta p_{x_{\mu}}$ at the point $p_{x_{\mu}} = p_{N_{x_{\mu}}}$ of the momentum space $P_{LE}$ is represented by the \textbf{primary measurable} momentum $p_{N'_{x_{\mu}}}$ with the property $|N'_{x_{\mu}}| \gg |N_{x_{\mu}}|$. The problem is as follows: is any possibility that $\Delta p_{x_{\mu}}$ is infinitesimal? For the special point $p_{x_{\mu}} = p_{N_{x_{\mu}}}$ the answer is negative.

Indeed, the "nearest" points to $p_{N_{x_{\mu}}}$ are $p_{N_{x_{\mu}} - 1}$ and $p_{N_{x_{\mu}} + 1}$.

It is obvious that

$$|p_{N_{x_{\mu}}} - p_{N_{x_{\mu}} - 1}| = |p_{N_{x_{\mu}}(N_{x_{\mu}} - 1)}|,$$

$$|p_{N_{x_{\mu}}} - p_{N_{x_{\mu}} + 1}| = |p_{N_{x_{\mu}}(N_{x_{\mu}} + 1)}|. \hspace{1cm} (8)$$

It is easily seen that the difference $|p_{N_{x_{\mu}}(N_{x_{\mu}} + 1)}| - |p_{N_{x_{\mu}}(N_{x_{\mu}} - 1)}|$ for $|N_{x_{\mu}}| \gg 1$ is infinitesimal, i.e., to within a high accuracy, we have $|p_{N_{x_{\mu}}(N_{x_{\mu}} + 1)}| = |p_{N_{x_{\mu}}(N_{x_{\mu}} - 1)}|$. And a small variation of $|\Delta p_{x_{\mu}}|$ at the point $p_{x_{\mu}} = p_{N_{x_{\mu}}}$ has a minimum that equals $|p_{N_{x_{\mu}}(N_{x_{\mu}} + 1)}|$. Clearly, with an increase in $|N_{x_{\mu}}|$, we can obtain no matter how small $|p_{N_{x_{\mu}}(N_{x_{\mu}} + 1)}|$

So, in the proposed paradigm at low energies $E \ll E_{p}$ a set of the \textbf{primarily measurable} $P_{LE}$ is discrete, and in every measurement of $\mu = 0, \ldots, 3$ there is the discrete subset $P_{x_{\mu}} \subset P_{LE}$:

$$P_{x_{\mu}} \doteq \{\ldots, p_{N_{x_{\mu}} - 1}, p_{N_{x_{\mu}}}, p_{N_{x_{\mu}} + 1}, \ldots\}. \hspace{1cm} (9)$$
In this case, as compared to the canonical quantum theory, in continuous space-time we have the following substitution:

\[
\frac{\partial}{\partial p_\mu} \mapsto \frac{\Delta}{\Delta p_\mu}, \quad \frac{\partial F}{\partial p_\mu} \mapsto \frac{\Delta F(p_{N_{x_\mu}})}{\Delta p_\mu} = \frac{F(p_{N_{x_\mu}}) - F(p_{N_{x_\mu} + 1})}{p_{N_{x_\mu}} - p_{N_{x_\mu} + 1}}.
\] (10)

It is clear that for sufficiently high integer values of \( |N_{x_\mu}| \), formula (10) reproduces a continuous paradigm in the momentum space to any preassigned accuracy.

Similarly for sufficiently high integer values of \( |N_t| \) and \( |N_i| = N_{x_i}| \), the quantities \( \tau/N_t, \ell/N_{x_i} \) from formula (4) may be arbitrary small.

Hence, for sufficiently high integer values of \( |N_t| \) and \( |N_i| = N_{x_i}| \), the primarily measurable momenta \( P_{x_\mu} \) (formula (9)) represent a measurable analog of small (and infinitesimal) space-time increments in the space-time variety \( \mathcal{M} \subset \mathbb{R}^4 \).

Because of this, for sufficiently high integer values of \( |N_{x_\mu}| \), the space-time analog of formula (10) is as follows:

\[
\frac{\partial}{\partial x_\mu} \mapsto \frac{\Delta}{\Delta N_{x_\mu}}, \quad \frac{\partial F}{\partial x_\mu} \mapsto \frac{\Delta F(x_\mu)}{\Delta N_{x_\mu}} = \frac{F(x_\mu) - F(x_\mu + \ell/N_{x_\mu})}{\ell/N_{x_\mu}}.
\] (11)

Now we formulate the principle of correspondence to a continuous theory.

**Principle of Correspondence to Continuous Theory (PCCT).**

At low energies \( E \ll E_p \) (or same \( E \ll E_\ell \)) the infinitesimal space-time quantities \( dx_\mu; \mu = 0, ..., 3 \) and also infinitesimal values of the momenta \( dp_i, i = 1, 2, 3 \) and of the energies \( dE \) form the basic instruments (“construction materials”) for any theory in continuous space-time. Because of this, to construct the measurable variant of such a theory, we should find the adequate substitutes for these quantities.

It is obvious that in the first case the substitute is represented by the quantities \( \ell/N_{x_\mu} \), where \( |N_{x_\mu}| \) — no matter how large (but finite!) integer, whereas in the second case by \( p_{N_{x_i}} = \frac{h}{N_{x_i}\ell}; i = 1, 2, 3; \mathcal{E}_{N_{x_0}} = \frac{c}{N_{x_0}\ell} \), where \( N_{x_\mu} \) — integer with the above properties \( \mu = 0, ..., 3 \).

In this way in the proposed approach all the primary measurable momenta \( p_{N_{x_\mu}}, |N_{x_\mu}| \gg 1 \) are small quantities at low energies \( E \ll E_\ell \) and primary...
measurable momenta $p_{N_{\mu}}$ with sufficiently large $|N_{\mu}| \gg 1$ being analogous to infinitesimal quantities of a continuous theory.

As, according to Remark 2.2, all the momenta at low energies $E \ll E_p$, to a high accuracy, may be considered to be the primary measurable momenta, from formula (4) we derive that at low energies the primary measurable momenta $p_{N_{\mu}}$ generate measurable small space-time variations and at sufficiently high $|N_{\mu}|$ - infinitesimal variations.

B) High Energies, $E \approx E_p$.

In this case formula (2) takes the form

$$N_i = \frac{\hbar}{p_{x_i} \ell}, \text{ or } p_{x_i} \doteq p_{N_i} = \frac{\hbar}{N_i \ell}$$

where $N_i$ is an integer number and $p_{x_i}$ is the momentum corresponding to the coordinate $x_i$. The discrete set $p_{N_i} \doteq p_{N_{\mu}}$ is introduced as primary measurable momenta.

The main difference of the case B) High Energies from the case A) Low Energies is in the fact that at High Energies the primary measurable momenta are inadequate for theoretical studies at the energy scales $E \approx E_p$.

This is easily seen when we consider, e.g., the Generalized Uncertainty Principle (GUP) [11]–[20], that is an extension of Heisenberg’s Uncertainty Principle (HUP) [22],[21], to (Planck) high energies

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p \sqrt{\frac{\Delta p}{\hbar}}$$

where $\alpha'$ is a constant on the order of 1.

Obviously, (13) leads to the minimal length $\ell$ on the order of the Planck length $l_p$

$$\Delta x_{min} = 2\sqrt{\alpha' l_p} \doteq \ell.$$  

In his earlier works [7],[9] the author, using simple calculations, has demonstrated that for the equality in (13) at high energies $E \approx E_p, (E \approx E_i)$ the primary measurable space quantity $\Delta x = N_{\Delta x} \ell$, where $N_{\Delta x} \approx 1$ is an integer number, results in the momentum $\Delta p(N_{\Delta x}, \text{GUP})$:

$$\Delta p \doteq \Delta p(N_{\Delta x}, \text{GUP}) = \frac{\hbar}{1/2(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1})\ell'}.$$

It is clear that for $N_{\Delta x} \approx 1$ the momentum $\Delta p(N_{\Delta x}, \text{GUP})$ is not a primary measurable momentum.
On the contrary, at low energies $E \ll E_p$, the primary measurable space quantity $\Delta x = N_{\Delta x}\ell$, where $N_{\Delta x} \gg 1$ is an integer number, due to the validity of the limit

$$\lim_{N_{\Delta x} \to \infty} \sqrt{N_{\Delta x}^2 - 1} = N_{\Delta x},$$

leads to the momentum $\Delta p(N_{\Delta x}, HUP)$:

$$\Delta p \approx \Delta p(N_{\Delta x}, HUP) = \frac{\hbar}{1/2(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1})\ell} \approx \frac{\hbar}{N_{\Delta x} \ell} = \frac{\hbar}{\Delta x}.$$  \hspace{1cm} (17)

It is inferred that, for sufficiently high integer values of $N_{\Delta x}$ the momentum $\Delta p(N_{\Delta x}, HUP)$ within any high accuracy may be considered to be the primary measurable momentum.

This example illustrates that primary measurable momenta are insufficient for studies in the high-energy domain $E \approx E_p$ and we should use the generalized measurable momenta.

As noted above, the main target of the author is to construct a quantum theory at all energy scales $E \leq E_\ell$ in terms of measurable quantities. In this theory the values of the physical quantity $G$ may be represented by the numerical function $\mathcal{F}$ in the following way [8]:

$$G = \mathcal{F}(N_i, N_t, \ell) = \mathcal{F}(N_i, N_t, G, \hbar, c, \kappa), \hspace{1cm} (18)$$

where $N_i, N_t$–integers for general form from the formula (2) and at high energies $E \approx E_\ell$ from the formula (12) and $G, \hbar, c$ are fundamental constants. The last equality in (18) is determined by the fact that $\ell = \kappa l_p$ and $l_p = \sqrt{G\hbar/c^3}$.

If $N_i \neq 0, N_t \neq 0$ (nondegenerate case), then it is clear that (18) can be rewritten as follows:

$$G = \mathcal{F}(N_i, N_t, \ell) = \mathcal{F}((N_i)^{-1}, (N_t)^{-1}, \ell) \hspace{1cm} (19)$$

Then at low energies $E \ll E_\ell$, i.e. at $|N_i| \gg 1, |N_t| \gg 1$, the function $\mathcal{F}$ is a function of the variables changing practically continuously, though these variables cover a discrete set of values. Naturally, it is assumed that $\mathcal{F}$ varies smoothly (i.e. practically continuously). As a result, we get a model, discrete in nature, capable to reproduce the well-known theory in continuous space-time to a high accuracy, as it has been stated above.

Obviously, at low energies $E \ll E_\ell$ the formula (19) is as follows:

$$G = \mathcal{F}(N_i, N_t, \ell) = \mathcal{F}((N_i)^{-1}, (N_t)^{-1}, \ell) = \mathcal{F}_p(p_{N_i}, p_{N_t}, \ell), \hspace{1cm} (20)$$
where \( p_{N}, p_{N_{x}} \) are primary measurable momenta.

**Remark 2.3.** What is the main point of this Section?
At low energies \( E \ll E_{p} \) we replace the abstract small and infinitesimal quantities \( \delta x_{\mu}, dx_{\mu}, \delta p_{\mu}, dp_{\mu} \) incomparable with each other, by the specific small quantities \( \ell/N_{x_{\mu}}, p_{N_{x_{\mu}}} \), which may be made however small at sufficiently high \( |N_{x_{\mu}}| \), still being ordered and comparable. It is very important that the quantities \( \ell/N_{x_{\mu}}, p_{N_{x_{\mu}}} \) are directly associated with the existing energies; for \( |N'_{x_{\mu}}| > |N_{x_{\mu}}| \) the momentum \( p_{|N'_{x_{\mu}}|} < p_{|N_{x_{\mu}}|} \) and \( p_{|N'_{x_{\mu}}|} \) corresponds to lower energy than \( p_{|N_{x_{\mu}}|} \). The same is true for the space variations \( \ell/N'_{x_{\mu}}, \ell/N_{x_{\mu}} \).

**Remark 2.4.**
At low energies \( E \ll E_{p} \) we should emphasize the difference between the primary measurable momenta \( p_{N_{x_{\mu}}}, p_{N_{x_{\mu}}} \in P_{LE} \) and the space-time quantities \( \ell/N_{x_{\mu}} \) corresponding to them in accordance with formula (4).
The first, that is \( p_{N_{x_{\mu}}}, \) in accordance with **Remark 2.2.** represent the whole set of the momenta \( P_{LE} \) at low energies \( E \ll E_{p} \) in terms of measurable quantities, whereas the second, \( \ell/N_{x_{\mu}} \), represent only the measurable small variations of space-time quantities. Because of this, any point \( p_{N_{x_{\mu}}}, p_{N_{x_{\mu}}} \in P_{LE} \) is associated with the fixed measurable minimal variation \( \Delta p_{N_{x_{\mu}}} \) from formula (10). At the same time, for a point with the space-time coordinates \( x_{\mu} \) such measurable minimal variation is dependent on the number \( |N_{x_{\mu}}| \) according to formula (11).

**Remark 2.5.**
Finally, according to **Definition 1.**, in the relativistic case the primary measurable energy is of the form

\[
E = \frac{\hbar c}{N_{0} \ell} N_{0} \equiv N_{x_{0}},
\]

where \( N_{0} \) is an integer number, and at low energies \( E \ll E_{p} \) it is obvious that \( N_{0} \gg 1 \).

Then at low energies \( E \ll E_{p} \) from **Remark 2.2.** it follows naturally that primary measurable energies, to a high accuracy, cover the whole low-energy spectrum. Then, considering that the formula

\[
E^2 = p^2 c^2 + m^2 c^4
\]

low energies \( E \ll E_{p} \) to a high accuracy is valid in terms of measurable quantities and all components of the vector \( p \) are the primary measurable momenta, we can found the mass \( m \) in terms of the measurability notion as follows:

\[
m^2 = \frac{\hbar^2}{c^2} \left( \frac{1}{N_{0}^{2} \ell^{2}} - \sum_{1 \leq i \leq 3} \frac{1}{N_{i}^{2} \ell^{2}} \right).
\]
3 Space-Time Metrics in Measurable Format and Strong Principle of Equivalence at Low Energy

The principal result of this section is based on Section 4 in [10] and we give all the required information from [10].

According to the above-mentioned results, the measurable variant of gravity should be formulated in terms of the small measurable space-time quantities \( \ell/N_{\Delta x_{\mu}} \) or same primary measurable momenta \( p_{N_{\Delta x_{\mu}}} \).

Let us consider the case of the random metric \( g_{\mu\nu} = g_{\mu\nu}(x) \) [25],[26], where \( x \in R^4 \) is some point of the four-dimensional space \( R^4 \) defined in measurable terms. The phrase ”some point of the four-dimensional space \( R^4 \) defined in measurable terms” means that all variations at the indicated point are determined in terms of measurable quantities (formula (18)–(20)). Specifically, as mentioned above, all small measurable variations, according to formula (4), take the form \( \ell/N_{\Delta x_{\mu}} \propto p_{N_{\Delta x_{\mu}}} \), where \( p_{N_{\Delta x_{\mu}}} \) are primary measurable momenta and \( |N_{\Delta x_{\mu}}| \gg 1 \).

Now, any such point \( x = \{x^\chi\} \in R^4 \) and any set of integer numbers \( \{N_{\Delta x_{\chi}}\} \) dependent on the point \( \{x^\chi\} \) with the property \( |N_{\Delta x_{\chi}}| \gg 1 \) may be correlated to the bundle with the base \( R^4 \) as follows:

\[
B_{N_{\Delta x_{\chi}}} = \{x^\chi, \ell/N_{\Delta x_{\chi}}\} \mapsto \{x^\chi\}. \tag{23}
\]

It is clear that \( \lim_{|N_{\Delta x_{\chi}}| \to \infty} B_{N_{\Delta x_{\chi}}} = R^4 \).

Then as a canonically measurable prototype of the infinitesimal space-time interval square [25],[26]

\[
ds^2(x) = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{24}
\]

we take the expression

\[
\Delta s^2_{\{N_{\Delta x_{\chi}}\}}(x) = g_{\mu\nu}(x,\{N_{\Delta x_{\chi}}\}) \frac{\ell^2}{N_{\Delta x_{\mu}}N_{\Delta x_{\nu}}}. \tag{25}
\]

Here \( g_{\mu\nu}(x,\{N_{\Delta x_{\chi}}\}) \) – metric \( g_{\mu\nu}(x) \) from formula (32) with the property that minimal measurable variation of metric \( g_{\mu\nu}(x) \) in point \( x \) for coordinate \( \chi \) has form

\[
\Delta g_{\mu\nu}(x,\{N_{\Delta x_{\chi}}\}) = g_{\mu\nu}(x+\ell/N_{\Delta x_{\chi}},\{N_{\Delta x_{\chi}}\}) - g_{\mu\nu}(x,\{N_{\Delta x_{\chi}}\}), \tag{26}
\]

Let us denote by \( \Delta_{\chi} g_{\mu\nu}(x,\{N_{\Delta x_{\chi}}\}) \) quantity

\[
\Delta_{\chi} g_{\mu\nu}(x, N_{\Delta x_{\chi}}) = \frac{\Delta g_{\mu\nu}(x, N_{\Delta x_{\chi}})_\chi}{\ell/N_{\Delta x_{\chi}}}. \tag{27}
\]
It is obvious that in the case under study the quantity \( \Delta g_{\mu\nu}(x, \{N_{\Delta x}\})_\chi \) is a **measurable** analog for the infinitesimal increment \( dg_{\mu\nu}(x) \) of the \( \chi \)-th component \( (dg_{\mu\nu}(x))_\chi \) in a continuous theory, whereas the quantity \( \Delta_\chi g_{\mu\nu}(x, N_{\Delta x}) \) is a **measurable** analog of the partial derivative \( \partial_\chi g_{\mu\nu}(x) \).

In this manner we obtain the (23)-formula induced bundle over the metric manifold \( g_{\mu\nu}(x) \):

\[
B_{g,N_{\Delta x}} \equiv g_{\mu\nu}(x, \{N_{\Delta x}\}) \mapsto g_{\mu\nu}(x). \tag{28}
\]

Referring to formula (4), we can see that (25) may be written in terms of the primary measurable momenta \( (p_{N_{\Delta x}^\mu}, p_{N_{\Delta x}^\nu}) \equiv p_{N_{\Delta x}} \) as follows:

\[
\Delta^2 s_{N_{\Delta x}}(x) = \frac{\ell^4}{\hbar^2} g_{\mu\nu}(x, \{N_{\Delta x}\}) p_{N_{\Delta x}^\mu} p_{N_{\Delta x}^\nu}. \tag{29}
\]

Considering that \( \ell \propto l_P \) (i.e., \( \ell = \kappa l_P \)), where \( \kappa = \text{const} \) is on the order of 1, in the general case (29), to within the constant \( \ell^4/\hbar^2 \), we have

\[
\Delta^2 s_{N_{\Delta x}}(x) = g_{\mu\nu}(x, \{N_{\Delta x}\}) p_{N_{\Delta x}^\mu} p_{N_{\Delta x}^\nu}. \tag{30}
\]

As follows from the previous formulae, the **measurable** variant of General Relativity should be defined in the bundle \( B_{g,N_{\Delta x}} \).

**Remark 3.1**

According to (25)–(27), a **measurable** analog of the metric \( g_{\mu\nu}(x, \{N_{\Delta x}\}) \) is differing from \( g_{\mu\nu}(x) \) by the value of a "minimal" interval and by minimal variations of \( g_{\mu\nu}(x, \{N_{\Delta x}\}) \). However, the components \( g_{\mu\nu}(x, \{N_{\Delta x}\}) \) themselves are coincident with \( g_{\mu\nu}(x) \).

For convenience, apart from formula (25), we use the equivalent formula

\[
\Delta^2 s_{\{N_{\Delta x}\}}(x) = g^{\mu\nu}(x, \{N_{\Delta x}\}) \frac{\ell^2}{N_{\Delta x}^\mu N_{\Delta x}^\nu}, \tag{31}
\]

that is a **measurable** analog of the formula

\[
ds^2(x) = g^{\mu\nu}(x) dx_\mu dx_\nu. \tag{32}
\]

Since it has been demonstrated that the metric components in continuous and **measurable** cases are the same, they may be used to raise and to lower the indices in the **measurable** case as well. Specifically, instead of a set of the quantities \( g_{\mu\nu}(x, \{N_{\Delta x}\}), N_{\Delta x}, \ell/N_{\Delta x}^\nu, p_{N_{\Delta x}^\mu} \), we can use the set \( g^{\mu\nu}(x, \{N_{\Delta x}\}), N_{\Delta x}, \ell/N_{\Delta x}^{\mu}, p_{N_{\Delta x}^\nu} \).
Measurability and Strong Principle of Equivalence in Low Energies

We can easily show that because the energies are low \((E \ll E_p\) or same \(\vert N_{\Delta x_\chi}\gg 1\)), the **Strong Principle of Equivalence (SPE)** ([27],p.69) is valid in terms of measurable quantities.

Indeed, let \(x^0 = (x_\mu^0), \mu = 0,...,3\) be some fixed point of the space-time variety \(M \subset \mathbb{R}^4\), with the metric \(g^{\mu\nu}(x)\) i.e. \(x^0 \in M\).

According to SPE, in continuous space-time the point \(x^0\) has a sufficiently small neighborhood, where the metric \(g^{\mu\nu}(x)\) is equivalent to the Minkowskian metric \(\eta^{\mu\nu}(x); ||\eta^{\mu\nu}|| = \text{Diag}(-1,1,1,1)\).

We denote this neighborhood as \(X^0(g^{\mu\nu})\).

Without loss of generality, we can calculate \(X^0(g^{\mu\nu})\) for each of the coordinates \(\mu = 0,...,3\) symmetric relative to \(x^0\), i.e., we have

\[
X^0(g^{\mu\nu}) \doteq \{ (x_\mu^0 - a_\mu < x_\mu < x_\mu^0 + a_\mu) \doteq \vert x_\mu - x_\mu^0 \vert < a_\mu, \quad \mu = 0,...,3; a_\mu > 0 \}.
\]

Then we can easily find integer \(N_{\Delta x_\chi}; \vert N_{\Delta x_\chi}\gg 1\) sufficiently high in absolute value so that

\[
\vert x_\mu - x_\mu^0 \vert = \frac{\ell}{\vert N_{\Delta x_\chi}\vert} \ll a_\mu.
\]

As noted above, for sufficiently high \(\vert N_{\Delta x_\chi}\vert\), the metric \(g^{\mu\nu}(x)\), to however high accuracy, is considered to be the measurable metric \(g^{\mu\nu}(x,\{N_{\Delta x_\chi}\})\).

As with an increase in \(\vert N_{\Delta x_\chi}\vert\) the quantity \(\ell/\vert N_{\Delta x_\chi}\vert\) is varying practically continuously, the metric \(g^{\mu\nu}(x)\) to however high accuracy could be considered the measurable metric for

\[
\vert x_\mu - x_\mu^0 \vert \leq \frac{\ell}{\vert N_{\Delta x_\chi}\vert}.
\]

Since the neighborhood of the point \(x^0\) assigned by the condition (35) is fully lying about the point specified by the condition (33), in this neighborhood the metric \(g^{\mu\nu}(x)\) is equivalent to the Minkowskian metric \(\eta^{\mu\nu}(x)\) in continuous space-time.

But, in turn, \(\eta^{\mu\nu}(x)\) can be represented, to however high accuracy, for the integer number \(N'_{\Delta x_\chi}; \vert N'_{\Delta x_\chi}\gg 1\) sufficiently high in absolute value, in the form of measurable metrics \(\eta^{\mu\nu}(x,\{N'_{\Delta x_\chi}\})\).

So, within the concept of measurability, the **Strong Principle of Equivalence (SPE)** may be formulated as follows:

**Definition 3.1. Measurable Variant of SPE at Low Energies.**

*For sufficiently small measurable neighborhood of the point \(x^0\), (the term*
"measurable neighborhood" means that all points of this neighborhood arise from \( x^0 \) by means of measurable variations), the measurable metric 
\[ g^{\mu\nu}(x, \{N_{\Delta x}\}) \]
with the integer number \( N_{\Delta x} \) sufficiently high in absolute value, is equivalent to the measurable Minkowskian metric 
\[ \eta^{\mu\nu}(x, \{N'_{\Delta x}\}) \]
with the integer \( N'_{\Delta x} \) sufficiently high in absolute value. In other words, in a sufficiently small measurable neighborhood of the point \( x^0 \) we can obtain, to however high accuracy, the equivalence of the two measurable metrics
\[ g^{\mu\nu}(x, \{N_{\Delta x}\}) \equiv \eta^{\mu\nu}(x, \{N'_{\Delta x}\}) . \tag{36} \]

It is clear that, taking maximal absolute values from both sets \( N_{\Delta x} \) and \( N'_{\Delta x} \),
\[ |N_{\Delta x}| = \max\{|N_{\Delta x}|, |N'_{\Delta x}|\}, \]
we can have for (36) the coincident sets \( \{N_{\Delta x}\} \) and \( \{N'_{\Delta x}\} \):
\[ g^{\mu\nu}(x, \{N_{\Delta x}\}) \equiv \eta^{\mu\nu}(x, \{N'_{\Delta x}\}) . \tag{37} \]

**Remark 3.2**

Again without loss of generality, we can takes as a sufficiently small measurable neighborhood of the point \( x^0 \) the neighborhood \( X^0(g^{\mu\nu}) \) specified by formula (33).

It is clear that, as the energies under study are low (\( E \ll E_p \)), we have \( a_\mu = N_{a_\mu} \ell \) and \( N_{a_\mu} \gg 1 \). Of course, the quantity \( a_\mu = N_{a_\mu} \ell \) is not necessarily primarily measurable, i.e., the number \( N_{a_\mu} \) is not necessarily integer. But we can always make it so, taking, instead of the number \( N_{a_\mu} \), its integer part \([N_{a_\mu}]\). Then the primarily measurable quantity \( a_\mu = [N_{a_\mu}] \ell \) is also satisfying the condition specified in formula (33).

The condition "sufficiently small measurable neighborhood" indicates that the numbers \( N_{a_\mu} \) should set the upper bound as follows:
\[ 1 \ll N_{a_\mu} \ll N_\mu(g^{\mu\nu}), \tag{38} \]
where the high positive number \( N_\mu(g^{\mu\nu}) \) (i.e. \( N_\mu(g^{\mu\nu}) \gg 1 \)) is dependent on the metric \( g^{\mu\nu} \).

For complete consideration of SPE at low energies \( E \ll E_p \) in terms of measurability notion, we should study the coordinate transformations of a continuous theory in terms of measurable quantities.

Let us consider any coordinate transformation \( x^\mu = x^\mu(\bar{x}^\nu) \) of the space–time coordinates in continuous space-time. Then we have
\[ dx^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\nu} d\bar{x}^\nu . \tag{39} \]
As mentioned at the Section 2 (formula (10)), in terms of measurable quantities we have the substitution

\[ dx^\mu \mapsto \frac{\ell}{N_{\Delta x_\mu}}, \quad d\bar{x}^\nu \mapsto \frac{\ell}{N_{\Delta \bar{x}_\nu}}, \]  

(40)

where \( N_{\Delta x_\mu}, \bar{N}_{\Delta \bar{x}_\nu} - \) integers \(|N_{\Delta x_\mu}| \gg 1, |\bar{N}_{\Delta \bar{x}_\nu}| \gg 1\) sufficiently high in absolute value, and hence in the measurable case (39) is replaced by

\[ \frac{\ell}{N_{\Delta x_\mu}} = \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \frac{\ell}{N_{\Delta \bar{x}_\nu}}. \]  

(41)

Equivalently, in terms of the primary measurable momenta we have

\[ p_{N_{\Delta x_\mu}} = \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) p_{N_{\Delta \bar{x}_\nu}}, \]  

(42)

where \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \) is the corresponding matrix represented in terms of measurable quantities.

It is clear that, in accordance with formula (40), in passage to the limit we get

\[ \lim_{|N_{\Delta x_\mu}| \to \infty} \frac{\ell}{N_{\Delta x_\mu}} = dx^\mu = \lim_{|N_{\Delta x_\mu}| \to \infty} \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \frac{\ell}{N_{\Delta \bar{x}_\nu}} \frac{\partial x^\mu}{\partial x^\nu} dx^\nu. \]  

(43)

Equivalently, passage to the limit (43) may be written in terms of the primary measurable momenta \( p_{N_{\Delta x_\mu}}, p_{N_{\Delta \bar{x}_\nu}} \) multiplied by the constant \( \ell^2/h \).

How we understand formulae (40)–(43)?

The initial (continuous) coordinate transformations \( x^\mu = x^\mu(\bar{x}^\nu) \) gives the matrix \( \partial x^\mu/\partial \bar{x}^\nu \). Then, for the integers sufficiently high in absolute value \( \bar{N}_{\Delta \bar{x}_\nu}, |\bar{N}_{\Delta \bar{x}_\nu}| \gg 1 \), we can derive

\[ \frac{\ell}{N_{\Delta x_\mu}} = \frac{\partial x^\mu}{\partial \bar{x}^\nu} \frac{\ell}{N_{\Delta \bar{x}_\nu}}, \]  

(44)

where \(|N_{\Delta x_\mu}| \gg 1\) but the numbers for \( N_{\Delta x_\mu}\) are not necessarily integer. Still, as noted above, the difference between \( \ell/N_{\Delta x_\mu}\) and \( \ell/[N_{\Delta x_\mu}] \) (and hence between \( p_{N_{\Delta x_\mu}} \) and \( p_{[N_{\Delta x_\mu}]} \)) is negligible.

Then substitution of \([N_{\Delta x_\mu}]/N_{\Delta x_\mu}\) for \( \bar{N}_{\Delta \bar{x}_\nu} \) in the left-hand side of (44) leads to replacement of the initial matrix \( \partial x^\mu/\partial \bar{x}^\nu \) by the matrix \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \) represented in terms of measurable quantities and, finally, to the formula (41). Clearly, for sufficiently high \(|N_{\Delta x_\mu}|, |\bar{N}_{\Delta \bar{x}_\nu}|\), the matrix \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \) may be selected no matter how close to \( \partial x^\mu/\partial \bar{x}^\nu \).

Similarly, in the measurable format we can get the formula

\[ d\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} dx^\nu \]  

(45)
and correspondingly the matrix \( \tilde{\Delta}_{\mu\nu}(\bar{x}^\mu, \bar{x}^\nu, 1/\bar{N}_{\Delta x_\mu}, 1/\bar{N}_{\Delta x_\nu}) \) with the property 

\[
\frac{\ell}{\bar{N}_{\Delta x_\mu}} = \tilde{\Delta}_{\mu\nu}(\bar{x}^\mu, \bar{x}^\nu, 1/\bar{N}_{\Delta x_\mu}, 1/\bar{N}_{\Delta x_\nu}) \frac{\ell}{\bar{N}_{\Delta x_\nu}},
\]

(46)

Thus, any coordinate transformation may be represented, to however high accuracy, by the measurable transformation (i.e., written in terms of measurable quantities), where the principal components are the measurable quantities \( \ell/\bar{N}_{\Delta x_\mu} \) or the primary measurable momenta \( p_{N_{\Delta x_\mu}} \).

From this it follows that all the components necessary for the formulation of a measurable variant of SPE at low energies \( E \ll E_p \) are available – all of them are represented in terms of the measurability notion, making the above definition of a measurable variant of SPE at low energies \( E \ll E_p \) correct.

4 Measurability in Gravity and Strong Principle of Equivalence at All Energy Scales

In this section, based on the results from [10], within the scope of the space-time foam notion we perform an initial analysis of the possibility for generalization of SPE in a measurable analog of gravity to the ultraviolet (Planck) energy region.

As directly follows from the first part of Section 3, specifically from formulae (25)–(27), the principal components involved in gravitational equations of General Relativity have measurable analogs [10].

In particular, the Christoffel symbols [25],[26]

\[
\Gamma^\alpha_{\mu\nu}(x) = \frac{1}{2} g^{\alpha\beta}(x) \left( \partial_{\nu} g_{\beta\mu}(x) + \partial_{\mu} g_{\nu\beta}(x) - \partial_{\beta} g_{\nu\mu}(x) \right)
\]

(47)

have the measurable analog [10]

\[
\Gamma^\alpha_{\mu\nu}(x, N_{x_\chi}) = \frac{1}{2} g^{\alpha\beta}(x, N_{x_\chi}) \left( \Delta_{\nu} g_{\beta\mu}(x, N_{x_\chi}) + \Delta_{\mu} g_{\nu\beta}(x, N_{x_\chi}) - \Delta_{\beta} g_{\mu\nu}(x, N_{x_\chi}) \right).
\]

(48)

Similarly, for the Riemann tensor in a continuous theory we have [25],[26]:

\[
R^\mu_{\nu\rho\beta}(x) \equiv \partial_{\alpha} \Gamma^\mu_{\nu\beta}(x) - \partial_{\beta} \Gamma^\mu_{\nu\alpha}(x) + \Gamma^\mu_{\gamma\alpha}(x) \Gamma^\gamma_{\nu\beta}(x) - \Gamma^\mu_{\gamma\beta}(x) \Gamma^\gamma_{\nu\alpha}(x).
\]

(49)

With the use of formula (48), we can get the corresponding measurable analog, i.e. the quantity \( R^\mu_{\nu\alpha\beta}(x, N_{x_\chi}) \) [10].

In a similar way we can obtain the measurable variant of Ricci tensor, \( R_{\mu\nu}(x, N_{x_\chi}) \equiv R^\alpha_{\mu\alpha\nu}(x, N_{x_\chi}) \), and the measurable variant of Ricci scalar.
\[ R(x, N_{x_{\chi}}) \equiv R_{\mu\nu}(x, N_{x_{\chi}}) g^{\mu\nu}(x, N_{x_{\chi}}) \] [10].

So, for the Einstein Equations (EU) in a continuous theory [25],[26]

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} \Lambda g_{\mu\nu} = 8 \pi G T_{\mu\nu} \] (50)

we can derive their **measurable** analog, for short denoted as (EUM)\textbf{Einstein Equations Measurable} [10]:

\[
R_{\mu\nu}(x, N_{x_{\chi}}) - \frac{1}{2} R(x, N_{x_{\chi}}) g^{\mu\nu}(x, N_{x_{\chi}}) - \frac{1}{2} \Lambda(x, N_{x_{\chi}}) g^{\mu\nu}(x, N_{x_{\chi}}) = \\
= 8 \pi G T_{\mu\nu}(x, N_{x_{\chi}}),
\] (51)

where \( G \) – Newton’s gravitational constant.

For correspondence with a continuous theory, the following passage to the limit must take place for all the points \( x \):

\[
\lim_{|N_{x_{\chi}}| \to \infty} \Lambda(x, N_{x_{\chi}}) = \Lambda,
\] (52)

where the cosmological constant \( \Lambda \) is taken from formula (50).

Moreover, for high \(|N_{x_{\chi}}|\), the quantity \( \Lambda(x, N_{x_{\chi}}) \) should be practically independent of the point \( x \), and we have

\[
\Lambda(x, N_{x_{\chi}}) \approx \Lambda(x', N'_{x_{\chi}}) \approx \Lambda,
\] (53)

where \( x \neq x' \) and \(|N_{x_{\chi}}| \gg 1, |N'_{x_{\chi}}| \gg 1\).

Actually, it is clear that formula (52) reflects the fact that (EUM) given by formula (51) represents deformation of the Einstein equations (EU) (50) in the sense of the Definition given in [28] with the deformation parameter \( N_{x_{\chi}} \), and we have

\[
\lim_{|N_{x_{\chi}}| \to \infty} (EUM) = (EU).
\] (54)

We denote this deformation as \((EUM)[N_{x_{\chi}}]\). Since at low energies \( E \ll E_P \) and to within the known constants we have \( \ell/N_{x_{\chi}} = p_{N_{x_{\chi}}} \), the following deformations of (EU) are equivalent to

\[
(EUM)[N_{x_{\chi}}] \equiv (EUM)[p_{N_{x_{\chi}}}].
\] (55)

So, on passage from (EU) to the **measurable** deformation of \((EUM)[N_{x_{\chi}}]\) (or identically \((EUM)[p_{N_{x_{\chi}}}]\)) we can find solutions for the gravitational equations on the metric bundle \( \hat{B}_{g,N_{x_{\chi}}} = g_{\mu\nu}(x, \{N_{x_{\chi}}\}) \) (formula (28)) given by formula (25) [10].
What are the advantages of this approach?

4.1. First, as $|N_{x\chi}| \gg 1$, from the above formulae it follows that the metric $g_{\mu\nu}(x, \{N_{x\chi}\})$ belonging to $B_{g,N_{x\chi}}$ and representing a solution for $(EUM)[N_{x\chi}]$, to a high accuracy, is a solution for the Einstein equations (EU) in a continuous theory. Besides, formula (54) shows that at sufficiently high $|N_{x\chi}|$ this accuracy may be however high. In this way the Principle of Correspondence to Continuous Theory (PCCT) (Section 2) to a continuous theory takes place.

4.2. We replace the abstract infinitesimal quantities $dx_\mu$, incomparable with each other, by the specific small quantities $\ell/N_{x\mu}$ which may be made however small at sufficiently high $|N_{x\chi}|$, still being ordered and comparable. Because of this, we can compare small values of the squared intervals $\Delta s^2_{\{N_{x\chi}\}}(x)$ from formula (25). Possibly, this will help to recover the causality property for all solutions in $(EUM)[N_{x\chi}]$ without pathological solutions in the form of the Closed Time-like Curves (CTC), involved in some models of General Relativity [29]–[32].

4.3. Finally, this approach from the start is quantum in character due to the fact that the fundamental length $\ell$ is proportional to the Planck length $\ell \propto l_P$ and includes the whole three fundamental constants, the Planck constant $\hbar$ as well. Besides, it is naturally dependent on the energy scale: sets of the metrics $g_{\mu\nu}(x, \{N_{x\chi}\})$ with the lowest value $|N_{x\chi}|$ correspond to higher energies as they correspond to the momenta $\{p_{N_{x\chi}}\}$ which are higher in absolute value. This is the case for all the energies $E$.

However, minimal measurable increments for the energies $E \approx E_P$ are not of the form $\ell/N_{x\mu}$ because the corresponding momenta $\{p_{N_{x\chi}}\}$ are no longer primary measurable, as indicated by the results in Section 2.

So, in the proposed paradigm the problem of the ultraviolet generalization of the low-energy measurable gravity $(EUM)[N_{x\chi}]$ (formula (51)) is actually reduced to the problem: what becomes with the primary measurable momenta $\{p_{N_{x\chi}}\}, |N_{x\chi}| \gg 1$ at high Planck’s energies.

In a relatively simple case of GUP in Section 2 we have the answer. And, using the fact that $(EUM)[N_{x\chi}] \equiv (EUM)[p_{N_{x\chi}}]$ (55), based on the results of Section 2, we can construct a correct high-energy passage to the Planck energies $E \approx E_p$ [10]

$$g_{\mu\nu}(x, \{N_{x\chi}\}) 
\Rightarrow (EUM)[p_{N_{x\chi}}(GUP), |N_{x\chi}| \approx 1], \quad (56)$$

where $p_{N_{x\chi}}(GUP) = \Delta p(\Delta x_{\chi}, GUP)$ according to formula (15) of Section 2. In this specific case, we can construct the natural ultraviolet generalization $(EUM)[p_{N_{x\chi}}, |N_{x\chi}| \gg 1] \equiv (EUM)[p_{N_{x\chi}}]$. The theoretical calculations $(EUM)[p_{N_{x\chi}}(GUP), |N_{x\chi}| \approx 1]$ derived at Planck’s energies are obviously dis-
crete, measurable, and represent a high-energy deformation in the sense of the \[28\] measurable gravitational theory \((EUM)[p_{N_{x\chi}}, |N_{x\chi}| \gg 1]\).

**Strong Principle of Equivalence in Measurable Variant at All Energy Scales**

The Equivalence Principle (weak or strong) in its initial form has been formulated for a low-energy gravitational theory, i.e. for the energies \(E \ll E_p\) in continuous space-time \([27]\). There is nothing similar for the energies \(E \approx E_p\).

However, in the proposed approach (or in the present paradigm) we go from continuous space-time to the measurable discrete space-time but in such a way that at low energies \(E \ll E_p\) the introduced measurable discrete space-time is close to the continuous space-time, enabling the author to form a measurable analog of the Strong Principle of Equivalence at Low Energies in Section 3.

The basic parameters used to form the measurability notion for all the energy scales are the integer numbers \(N_{x\mu}, \mu = 0, ..., 3\) (or identically \(N_{x\mu}\)). At low energies \(E \ll E_p\) these numbers satisfy the condition \(|N_{x\mu}| \gg 1\). As it has been demonstrated above, the corresponding primarily measurable momenta \(p_{N_{x\mu}}\) (and space-time variations \(\ell/N_{\Delta x}\)) are adequate to form a measurable variant of gravity at these energy scales.

At high energies \(E \approx E_p\) (same \(E \approx E_{\ell}\) (case B) from Section 2), due to the fact that for \(|N_{x\mu}| \approx 1\) a theory in terms of measurable quantities becomes really discrete, the primarily measurable momenta \(p_{N_{\Delta x}\mu}\), in line with formula (15), are inadequate for the correct examination of this case.

In the general case the transition from high \(E \approx E_p\) to low energies for a measurable variant of gravity is given by reversal of the arrow from formula (56):

\[
(EUM)[p_{N_{x\chi}}, |N_{x\chi}| \approx 1] \leftrightarrow (EUM)[p_{N_{x\chi}}, |N_{x\chi}| \gg 1],
\]

(57)

where \(p_{N_{x\chi}}\) for \(|N_{x\chi}| \approx 1\) the generalized measurable (or simply measurable) momenta are so that we have

\[
p_{N_{x\chi}}; (|N_{x\chi}| \approx 1) \Rightarrow |N_{x\chi}| \gg 1 \Rightarrow p_{N_{x\chi}}; (|N_{x\chi}| \gg 1).
\]

(58)

The momenta in the right-hand part of formula (58), i.e. \(p_{N_{x\chi}}, (|N_{x\chi}| \gg 1)\), are the primary measurable momenta at low energies \(E \ll E_p\).

In Section 2 it is shown that the momenta \(p_{N_{x\chi}}(GUP), |N_{x\chi}| \approx 1\) specified by formula (15) just satisfy the conditions of (57),(58). But it is obvious that in the general case at the energies \(E \approx E_p\) the momenta \(p_{N_{x\chi}}, (|N_{x\chi}| \approx 1, meeting the conditions (57),(58), may be of a more complex form. For example, the form of GUP may be more complex than that considered in the survey work
In this case on passage to quantum gravity the formulas (56)–(58) are still valid.

In all the cases for a measurable variant of gravity the transition to the ultraviolet (i.e. quantum) region may be realized by substitution of \( \ell^2 \bar{h} p N_{\Delta x^\mu} \), \(|N_{\Delta x^\mu}| \approx 1\) in Section 3 for the quantities \( \ell/N_{\Delta x^\mu} = \ell^2 \bar{h} p N_{\Delta x^\mu} \), \(|N_{\Delta x^\mu}| \gg 1\); by the corresponding corrections of formulae (25)–(31) from Section 3, of all the components necessary for derivation of gravitational equations in a measurable variant \( \Gamma_{\mu\nu}(x,N_{x_\chi}), R_{\mu\nu\alpha\beta}(x,N_{x_\chi}), \ldots \), and of formulae (48),(50),... from this Section.

It is clear that, provided at high energies \( E \approx E_p \) in the measurable case some analog of the Strong Principle of Equivalence (SPE) is involved, its formulation should be radically different from (SPE) in the measurable case at low energies \( E \ll E_p \) considered in Section 3 for the two main reasons given below.

**4.4A.** As at high energies \( E \approx E_p \) (and hence at \(|N_{\Delta x^\mu}| \approx 1\)) a measurable variant of gravity represents a discrete theory, where the notion of locality is senseless, we should involve the minimal primarily measurable spatial neighborhood and the minimal generalized measurable spatial variations \( \ell^2 \bar{h} p N_{\Delta x^\mu} \), \(|N_{\Delta x^\mu}| \approx 1\) for the arbitrary point \( x = \{x^\mu\} \) (with the naturally selected finite bounds of the numbers \( N_{\Delta x^\mu} \)).

**4.4B.** Besides, it is obvious that at high energies \( E \approx E_p \) the space curvature becomes great and this space in any measurable neighborhood of the random point \( x \) is far from the flat space with the Minkowskian metric \( \eta^{\mu\nu}(x) \).

As follows from remarks 4.4A. and 4.4B., when for a measurable variant of gravity there is some form of an analog of the Strong Principle of Equivalence (SPE) at high energies \( E \approx E_p \), its correct formulation should be completely coordinated with the transitions from high to low energies given in formulae (57), (58). In other words, on going from high to low energies, this high-energy analog of SPE should conform to SPE at low energies \( E \ll E_p \) for a measurable variant of gravity considered in Section 3.

In accordance with the modern understanding of the problem, at high energies \( E \approx E_p \) the space geometry, due to high Space-Time Quantum Fluctuations (STQF), represents the “space-time foam” (stf) \([33],[34]\). The notion of “space-time foam” was introduced by J. A. Wheeler about 60 years ago for the description and investigation of physics at Planck’s scales (Early Universe). Actually, because of high quantum fluctuations of the metric \( g_{\mu\nu} \), the space has a quantity of geometries. Despite the fact that in the last time numerous works have been devoted to physics at Planck’s scales within the scope of this notion, by this time still their no clear understanding of stf as it is.
Still, some models based on micro-black holes are very interesting and fairly promising. In particular, the models studied in [35]–[37] and based on micro-black holes, i.e. black holes with a Schwarzschild radius of several Planck’s units of length.

Without loss of generality, it may be considered that all the micro-black holes considered as “constituent parts” of stf are Schwarzschild’s black holes.

It should be noted that the case of micro-black holes with the Schwarzschild metric in terms of measurable quantities has been already studied by the author in his papers [7], [9]. In these papers, within the scope of validity of the Generalized Uncertainty Principle (GUP) of Section 2, in terms of the measurability notion the gravitational equations at the event horizon surface of these holes have been derived and their basic thermodynamic characteristics (temperature, entropy) have been obtained.

It is obvious that these holes form a discrete finite set, provided their Schwarzschild radii \( r_{mbh} \) are considered primarily measurable quantities:

\[
r_{mbh} = N_{r_{mbh}} \ell, N_{r_{mbh}} \approx 1,
\]

where \( N_{r_{mbh}} \) is an integer number.

Proceeding from all the above, a measurable variant of the Strong Principle of Equivalence at high energies \( E \approx E_p \) for stf based on the geometry of Schwarzschild’s micro-black holes may be formulated as follows.

\[\text{In a sufficiently small primarily measurable neighborhood of any spatial point } x \text{ at the Planck scale the geometry of stf is equivalent to the geometry of some micro-black hole with the Schwarzschild metric and with the corresponding Schwarzschild radius } r_{mbh} \text{ satisfying formula (59).}\]

As, in accordance with GUP of Section 2, we have

\[
p(N_{\Delta x_i}, GUP) = \frac{\hbar}{1/2(N_{\Delta x_i} + \sqrt{N_{\Delta x_i}^2 - 1})\ell}, i = 1, \ldots, 3,
\]

on passage from high energies \( E \approx E_p \) to low energies \( E \ll E_p \), formula (58) is apparently valid and we can, to a high accuracy, obtain at low energies the primarily measurable momenta \( p(N_{\Delta x_i}), |N_{\Delta x_i}| \gg 1 \) and a measurable variant of the Strong Principle of Equivalence at Low Energies from Section 3.

In the process it is assumed that formula (57) is valid by default, i.e. passage from stf at high energies \( E \approx E_p \) to low energies \( E \ll E_p \) leads to the large-scale space-time structure and to Einstein Equations.

As noted in point 4.3., in a simple case of GUP considered in Section 2 passage to quantum gravity in a measurable variant of General Relativity is
represented by formula (56). However, GUP may be of a more complex as compared to the considered in the survey work [38]. In this case on passage to quantum gravity the formula (56) is still valid.

5 Conclusion

Thus, in this paper it has been demonstrated that the Strong Principle of Equivalence (SPE) may be correctly formulated in terms of measurable quantities, i.e. for a measurable analog of gravity (or same measurable variant of gravity) at low energies $E \ll E_p$. Besides, it has been shown that, within the scope of the specific models for Space-Time Foam, SPE may be also valid for a measurable variant of gravity and at the Planck scales, or at high energies $E \approx E_p$.

Since at the present time no direct or indirect experiments at the scales on the order of Planck’s scales (i.e. at the energies associated with the quantum gravity scales) are known, all theoretical studies in this field are to some or other extent speculative. Nevertheless, considering that gravity should be formulated with the use of the same terms at all the energy scales, it must be governed by the particular unified principles the formulation of which varies depending on the available energies. Because of this, the results from Section 4 seem to be important. Of course, these results are tentative and may be corrected during further studies of gravity in terms of the measurability notion. But they give the main idea and define the trend towards the derivation of a measurable variant of gravity: framing of a correct gravitational theory at all the energy scales, with the use of a set of discrete parameters $p(N_{\Delta x})$ for all nonzero integer values of $N_{\Delta x}$, that is close to the General Relativity at low energies $E \ll E_p$ and is a new (discrete) theory at high energies $E \approx E_p$.

As noted in Section 4 (formula (54)) and in the earlier papers of the author, the above derivation of a measurable variant of gravity may be realized proceeding from the notion of the deformation of a physical theory introduced in [28]:

*Deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition.*

Denoting a measurable variant of gravity at low energies $E \ll E_p$ (that is yet incompletely derived) by $Grav[LE,meas]^\ell$, we obtain that the abovementioned deformation is nothing else but the following mapping:

$$Grav[LE,meas]^\ell \xrightarrow[\ell \rightarrow 0]{} GR,$$  \hspace{1cm} (61)
where the deformation parameters are primarily measurable momenta $p(N_{\Delta x_{\mu}}), |N_{\Delta x_{\mu}}| \gg 1$ (or the corresponding space-time variations $\ell/N_{\Delta x_{\mu}}$).

Then Einstein Equations Measurable (EUM) at low energies $E \ll E_p$ in Section 3 (formula (51)) is a low-energy deformation deformation of Einstein Equations (EU) in General Relativity (GR) as indicated by formula (54).

Considering that $\text{Grav}[\text{LE, meas}]^\ell$ and GR are very close but not identical, the author’s hypothesis is as follows:

we can frame a measurable variant of gravity $\text{Grav}[\text{LE, meas}]^\ell$, within the scope of which there is possibility for the effective solution of several problems at the joint of General Relativity and Quantum Theory: the above-mentioned Closed Time-like Curves (CTC) problem [29]–[32], black hole radiation problem, Hawking’s Information Paradox [39]–[41], etc.

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References


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