On the \((h, q)\)-Stirling-Like Polynomials Related to \((h, q)\)-Bernoulli Polynomials and Stirling Polynomials Associated with the \(p\)-Adic Integral on \(\mathbb{Z}_p\)

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Abstract

In this paper, we construct the \((h, q)\)-Stirling-Like numbers \(SR_{n,q}^{(h)}\) and \((h, q)\)-Stirling-Like polynomials \(SR_{n,q}^{(h)}(x)\) related to \((h, q)\)-Bernoulli polynomials and Stirling polynomials. Some interesting results and relationships are obtained.

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1 Introduction

Mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, and tangent numbers and polynomials (see [1-5]). Throughout this paper, we always make use of the following notations: \(\mathbb{N}\) denotes the set of natural numbers and \(\mathbb{Z}_+ = \mathbb{N} \cup \{0\}\), \(\mathbb{C}\) denotes the set of complex numbers, \(\mathbb{Z}_p\) denotes the ring of \(p\)-adic rational integers, \(\mathbb{Q}_p\) denotes the field of \(p\)-adic rational numbers, and \(\mathbb{C}_p\) denotes the completion of algebraic closure of \(\mathbb{Q}_p\). Let \(\nu_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-\nu_p(p)} = p^{-1}\). When one
talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the bosonic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_1(g) = \lim_{q \to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x), \quad \text{cf. [2]}. \quad (1.1)$$

By (1.1), we easily see that

$$I_1(g_1) = I_1(g) + g'(0), \quad \text{cf. [1]}, \quad (1.2)$$

where $g_1(x) = g(x+1)$ and $g'(0) = \frac{dg(x)}{dx}|_{x=0}$. For a real or complex parameter $x$, the Stirling polynomials are defined by the following generating function

$$\sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!} = \left(\frac{t}{1 - e^{-t}}\right)^{x+1} (|t| < 2\pi; 1^1 := 1), \text{ see [3]}. $$

Recently, we constructed the numbers $SR_n$ and polynomials $SR_n(x)$ related to Bernoulli polynomials and Stirling polynomials (see [5]). The polynomials $SR_n(x)$ and numbers $SR_n$ are defined by the following generating functions

$$\sum_{n=0}^{\infty} SR_n(x) \frac{t^n}{n!} = \left(\frac{t}{1 - e^{-t}}\right) e^{xt}, \quad |t| < 2\pi, \quad (1.3)$$

$$\sum_{n=0}^{\infty} SR_n \frac{t^n}{n!} = \frac{t}{1 - e^{-t}}, \quad |t| < 2\pi, \quad (1.4)$$

respectively. We studied some properties which are related to numbers $SR_n$ and polynomials $SR_n(x)$. More studies and results in this subject we may see reference [1, 5]. Our aim in this paper is to define $(h,q)$-extension of $SR_n(x)$. We investigate some properties which are related to $(h,q)$-Stirling-Like numbers $SR_{n,q}^{(h)}$ and $(h,q)$-Stirling-Like polynomials $SR_{n,q}^{(h)}(x)$. We also derive the existence of a specific interpolation function which interpolate $(h,q)$-Stirling-Like numbers $SR_{n,q}^{(h)}$ and $(h,q)$-Stirling-Like polynomials $SR_{n,q}^{(h)}(x)$ at negative integers.
2 \((h, q)\)-Stirling-Like numbers and polynomials

Our primary goal of this section is to define \((h, q)\)-Stirling-Like numbers \(SR_{n,q}^{(h)}\) and polynomials \(SR_{n,q}^{(h)}(x)\). We also find generating functions of \((h, q)\)-Stirling-Like numbers \(SR_{n,q}^{(h)}\) and polynomials \(SR_{n,q}^{(h)}(x)\) and investigate their properties. For \(h \in \mathbb{Z}\) and \(q \in \mathbb{C}_p\) with \(|1 - q|_p \leq 1\), if we take \(g(x) = q^{hx} e^{-xt}\) in (1.2), then we easily see that

\[
I_1(q^{hx} e^{-xt}) = \int_{\mathbb{Z}_p} q^{hx} e^{-xt} d\mu_1(x) = \frac{h \log q - t}{q^h e^{-t} - 1}.
\]

Let us define the \((h, q)\)-Stirling-Like numbers \(SR_{n,q}^{(h)}\) and \((h, q)\)-Stirling-Like polynomials \(SR_{n,q}^{(h)}(x)\) as follows:

\[
I_1(q^{hy} e^{-yt}) = \int_{\mathbb{Z}_p} q^{hy} e^{-yt} d\mu_1(y) = \sum_{n=0}^{\infty} SR_{n,q}^{(h)} \frac{t^n}{n!}, \quad (2.1)
\]

\[
I_1(q^{hy} e^{(x-y)t}) = \int_{\mathbb{Z}_p} q^{hy} e^{(x-y)t} d\mu_1(y) = \sum_{n=0}^{\infty} SR_{n,q}^{(h)}(x) \frac{t^n}{n!}. \quad (2.2)
\]

By (2.1) and (2.2), we obtain the following Witt’s formula.

**Theorem 2.1** For \(h \in \mathbb{Z}\) and \(q \in \mathbb{C}_p\) with \(|1 - q|_p \leq 1\), we have

\[
\int_{\mathbb{Z}_p} q^{hx} (-x)^n d\mu_1(x) = SR_{n,q}^{(h)}, \quad \int_{\mathbb{Z}_p} q^{hy} (x-y)^n d\mu_1(y) = SR_{n,q}^{(h)}(x).
\]

Note that \(SR_{n,q}^{(h)} = (-1)^n B_{n,q}^{(h)}\) and \(SR_{n,q}^{(h)}(-x) = (-1)^n B_{n,q}^{(h)}(x)\), where \(B_{n,q}^{(h)}\) and \(B_{n,q}^{(h)}(x)\) denote the \((h, q)\)-Bernoulli numbers and the \((h, q)\)-Bernoulli polynomials, respectively (see [6]). If \(q \to 1\), then \(SR_{n,q}^{(h)} \to SR_n\) and \(SR_{n,q}^{(h)}(x) \to SR_n(x)\). By using \(p\)-adic integral on \(\mathbb{Z}_p\), we obtain,

\[
\int_{\mathbb{Z}_p} q^{hx} e^{-xt} d\mu_1(x) = (h \log q - t) \sum_{m=0}^{\infty} q^{-h(m+1)} e^{(m+1)t}. \quad (2.3)
\]

Thus \((h, q)\)-Stirling-Like numbers \(SR_{n,q}^{(h)}\) are defined by means of the generating function

\[
F_q^{(h)}(t) = \sum_{n=0}^{\infty} SR_{n,q}^{(h)} \frac{t^n}{n!} = (h \log q - t) \sum_{m=0}^{\infty} q^{-h(m+1)} e^{(m+1)t}. \quad (2.4)
\]
Using similar method as above, by using $p$-adic integral on $\mathbb{Z}_p$, we have

$$\sum_{n=0}^{\infty} SP_{n,q}(h) x^n n! = \left( \frac{h \log q - t}{q^h e^{-t} - 1} \right) e^{xt}. \quad (2.5)$$

By using (2.2) and (2.5), we obtain

$$F_q^{(h)}(t, x) = \sum_{n=0}^{\infty} SR_{n,q}(h) \frac{t^n}{n!} = (h \log q - t) \sum_{m=0}^{\infty} (-1)^m q^{-h(m+1)} e^{(m+1+x)t}. \quad (2.6)$$

By Theorem 2.1, we easily obtain that

$$SR_{n,q}(h) = \int_{\mathbb{Z}_p} q^{hy} (x - y)^n d\mu_1(y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} SR_{k,q}^{(h)}$$

$$= (x + SR_q^{(h)})^n$$

$$= (h \log q - t) \sum_{m=0}^{\infty} q^{-h(m+1)} (m + 1 + x)^m. \quad (2.7)$$

The following elementary properties of $(h, q)$-Stirling-Like polynomials $SR_{n,q}^{(h)}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [4]-[6].

**Theorem 2.2** For any positive integer $n$, we have

$$SR_{n,q}^{(h)}(x) = (-1)^n q^{-h} SR_{n,q}^{(h)}(-1 - x).$$

**Theorem 2.3** For any positive integer $m (= odd)$, we have

$$SR_{n,q}^{(h)}(x) = n^m \sum_{a=0}^{m-1} (-1)^a q^{ha} SR_{n,q}^{(h)} \left( \frac{x - a}{m} \right), \quad n \in \mathbb{Z}_+. $$

By (1.3), (2.1), and (2.2), we easily see that

$$h \log q \sum_{l=0}^{n-1} q^{hl} (-l)^m - \sum_{l=0}^{n-1} mq^{hl} (-l)^{m-1} = q^{hn} SR_{n,q}^{(h)}(-n) - SR_{n,q}^{(h)}.$$ 

Hence, we have the following theorem.
**Theorem 2.4** For $m \in \mathbb{Z}_+$, we have

$$q^h \text{SR}^{(h)}_{m,q}(-n) - \text{SR}^{(h)}_{m,q} = h \log q \sum_{l=0}^{n-1} q^h (-l)^m - m \sum_{l=0}^{n-1} q^h (-l)^{m-1}.$$ 

From (1.3), we note that

$$h \log q - t = q^h \int_{\mathbb{Z}_p} q^h e^{-x-1} t d\mu_1(x) - \int_{\mathbb{Z}_p} q^h e^{-xt} d\mu_1(x)$$

$$= \sum_{n=0}^\infty \left( q^h \int_{\mathbb{Z}_p} q^h (-x-1)^n d\mu_1(x) - \int_{\mathbb{Z}_p} q^h (-x)^n d\mu_1(x) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^\infty (q^h \text{SR}^{(h)}_{n,q}(-1) - \text{SR}^{(h)}_{n,q}) \frac{t^n}{n!}.$$ 

Therefore, we have the following theorem.

**Theorem 2.5** For $n \in \mathbb{Z}_+$, we have

$$q^h \text{SR}^{(h)}_{n,q}(-1) - \text{SR}^{(h)}_{n,q} = \begin{cases} 
 h \log q, & \text{if } n = 0, \\
 -1, & \text{if } n = 1 \\
 0, & \text{if } n \neq 1.
\end{cases}$$

By (2.7) and Theorem 2.5, we have the following corollary.

**Corollary 2.6** For $n \in \mathbb{Z}_+$, we have

$$q^h (\text{SR}^{(h)}_q - 1)^n + \text{SR}^{(h)}_{n,q} = \begin{cases} 
 h \log q, & \text{if } n = 0, \\
 -1, & \text{if } n = 1 \\
 0, & \text{if } n \neq 1.
\end{cases}$$

with the usual convention of replacing $(\text{SR}^{(h)}_q)^n$ by $\text{SR}^{(h)}_{n,q}$.

**Theorem 2.7** For $n \in \mathbb{Z}_+$, we have

$$\text{SR}^{(h)}_{n,q}(x + y) = \sum_{k=0}^{n} \binom{n}{k} \text{SR}^{(h)}_{k,q}(x) y^{n-k}.$$ 

By Theorem 2.1, we easily get

$$\text{SR}^{(h)}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} q^h (-y)^l d\mu_1(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \text{SR}^{(h)}_{l,q}.$$ 

Therefore, we obtain the following theorem.

**Theorem 2.8** For $n \in \mathbb{Z}_+$, we have

$$\text{SR}^{(h)}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \text{SR}^{(h)}_{l,q} x^{n-l}.$$
3. \((h,q)\)-Hurwitz-type zeta function and \((h,q)\)-type zeta function

Our primary aim in this section is to define \((h,q)\)-Hurwitz-type zeta functions. We give the relation between generating function in (2.6) and \((h,q)\)-Hurwitz-type zeta functions. These functions interpolate the \((h,q)\)-Stirling-Like numbers \(SR_{n,q}^{(h)}\) and polynomials \(SR_{n,q}^{(h)}(x)\), respectively. We assume that \(q \in \mathbb{C}\) with \(|q| < 1\). Let \(\Gamma(s)\) be the gamma function. The gamma function is defined for all complex numbers except the non-positive integers. For complex numbers with a positive real part, it is defined via a convergent improper integral:

\[
\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.
\]

For \(s \in \mathbb{C}\), by applying the Mellin transformation to (2.6), we obtain

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2}F_q^{(h)}(-t,x)dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \frac{q^{-h}e^{-t}}{1-q^{-h}e^{-t}} \right) e^{-xt}dt = \frac{h \log q}{(s-1)} \sum_{m=0}^\infty \frac{(q^{-h})^{m+1}}{(x+m+1)^{s-1}} + \sum_{m=0}^\infty \frac{1}{(x+1+m)^s}.
\]

By using the above equation, we obtain integral representations of the now \((h,q)\)-Hurwitz-type zeta function.

Definition 3.1 Let \(s \in \mathbb{C}\) and \(x \in \mathbb{R}^+\). We define

\[
\zeta_q^{(h)}(s,x) = \frac{h \log q}{(s-1)} \sum_{m=0}^\infty \frac{(q^{-h})^{m+1}}{(x+m+1)^{s-1}} + \sum_{m=0}^\infty \frac{1}{(x+1+m)^s}.
\]

Relation between \(\zeta_q^{(h)}(s,x)\) and \(SR_{k,q}^{(h)}(x)\) is given by the following theorem. For \(s = 1 - n, n \in \mathbb{N}\) and by using Cauchy Residue Theorem in (3.1) we arrive at the desired result, so we choose to omit the details involved.

Theorem 3.2 For \(k \in \mathbb{N}\), we have

\[
\zeta_q^{(h)}(1-k,x) = -\frac{SR_{k,q}^{(h)}(x)}{k}.
\]

By (2.4), we can readily see that

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2}F_q^{(h)}(-t)dt = \frac{h \log q}{(s-1)} \sum_{m=0}^\infty \frac{(q^{-h})^{m+1}}{(m+1)^{s-1}} + \sum_{m=0}^\infty \frac{1}{(m+1)^s}.
\]
**Definition 3.3** Let $s \in \mathbb{C}$ with $Re(s) > 1$. We define

$$
\zeta_q^{(h)}(s) = \frac{h \log q}{(s-1)} \sum_{m=0}^{\infty} \frac{(q^{-h})^{m+1}}{(m+1)^{s-1}} + \sum_{m=0}^{\infty} \frac{1}{(m+1)^s}.
$$

(3.5)

Observe that $x = 0$ in (3.2), we easily see that $\zeta_q^{(h)}(s,0) = \zeta_q^{(h)}(s)$. Using (3.5) and (2.1), relation between $\zeta_q^{(h)}(s)$ and $SR_{k,q}^{(h)}$ is given by the following theorem.

**Theorem 3.4** For $k \in \mathbb{N}$, we have

$$
\zeta_q^{(h)}(1-k) = -\frac{SR_{k,q}^{(h)}}{k}.
$$

**References**

https://doi.org/10.12988/ams.2014.410846


https://doi.org/10.1016/j.aml.2005.11.019

https://doi.org/10.12988/ams.2014.49727

https://doi.org/10.12988/astp.2013.3886

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