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The Cauchy Problem for a Symmetric System of Keyfitz-Kranzer Type with Linear Damping

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Abstract

We prove the existence of a weak solution for the Cauchy problem associated with a 2×2 symmetric system of Keyfitz-Kranzer type with linear damping.

Keywords: Symmetric system of Keyfitz-Kranzer type, linear damping, existence, weak solution, compensated compactness method

1 Introduction

The following system of partial differential equations

$$\begin{cases} u_t + \big(u\phi(r)\big)_x = 0 \\ v_t + \big(v\phi(r)\big)_x = 0\,, \end{cases} \tag{1.1}$$

where $\phi(r)$ is a nonlinear symmetric function of u and v, is a 2×2 symmetric system of Keyfitz-Kranzer type.

A system of the form (1.1) was first introduced in [6] by B. Keyfitz and H. Kranzer as a model in elasticity theory. Also, this type of system appear in magnetohydrodynamics, chromatography and enhanced oil recovery [1, 4, 7]. Symmetric systems of Keyfitz-Kranzer type have been studied by many authors [1, 2, 4, 5, 6, 7, 8, 9].

In this paper we shall apply the vanishing viscosity method together with the Murat's lemma and the div-curl lemma, to study the Cauchy problem for the 2×2 symmetric system of Keyfitz-Kranzer type with linear damping

$$\begin{cases} u_t + \big(u\phi(r)\big)_x = -au \\ v_t + \big(v\phi(r)\big)_x = -bv, \end{cases} \tag{1.2}$$

with bounded measurable initial data

$$\big(u(x,0), v(x,0)\big) = \big(u_0(x), v_0(x)\big), \tag{1.3}$$

where $\phi(r) \in C^2(\mathbb{R}^+)$ and $\phi(r)$ is strictly increasing or decreasing for positive r, a, b are constants such that $b \geq a > 0$ and

$$r = |u|^\alpha + |v|^\alpha, \tag{1.4}$$

for any $\alpha > 1$ fixed.

From [3], we see that the system (1.2) has eigenvalues

$$\lambda_1 = \phi(r) + \alpha r \phi'(r), \qquad \lambda_2 = \phi(r),$$

and Riemann invariants

$$z(u,v) = \frac{v}{u}, \qquad w(u,v) = \phi(r). \tag{1.5}$$

2 Existence of Weak Solution

We consider the Cauchy problem for the system

$$\begin{cases} u_t^\epsilon + \big(u^\epsilon \phi(r^\epsilon)\big)_x = -au^\epsilon + \epsilon u_{xx}^\epsilon \\ v_t^\epsilon + \big(v^\epsilon \phi(r^\epsilon)\big)_x = -bv^\epsilon + \epsilon v_{xx}^\epsilon\,, \end{cases} \tag{2.1}$$

with initial data (1.3).

Lemma 2.1. *For any $\epsilon > 0$ and any $T > 0$, we have the a-priori bounds for the Cauchy problem (2.1)-(1.3)*

$$|u^\epsilon(x,t)| \leq M(T), \qquad |v^\epsilon(x,t)| \leq M(T), \qquad (x,t) \in \mathbb{R} \times [0,T], \tag{2.2}$$

for a positive constant $M(T)$ independent of ϵ.

Proof. We multiply the first and second equations of system (2.1) respectively by $\alpha|u|^{\alpha-2}u$ and $\alpha|v|^{\alpha-2}v$, adding the results, we obtain

$$r_t + \lambda_1 r_x + a\alpha|u|^\alpha + b\alpha|v|^\alpha = \epsilon r_{xx} - \epsilon\alpha(\alpha-1)\Big(|u|^{\alpha-2}u_x^2 + |v|^{\alpha-2}v_x^2\Big). \tag{2.3}$$

We have from (2.3) the inequality

$$r_t + \lambda_1 r_x + a\alpha r \le \epsilon r_{xx}. \tag{2.4}$$

Applying the maximum principle to (2.4) we get the estimate $r^\epsilon \le N(T)$, where $N(T)$ is a positive constant, being independent of ϵ. Estimate from which we obtain the a-priori bounds in (2.2). □

A consequence of the previous lemma is the following.

Corollary 2.2. *For $\epsilon > 0$ and $T > 0$ the viscosity solution $\big(u^\epsilon(x,t), v^\epsilon(x,t)\big)$ for the Cauchy problem (2.1)-(1.3) exists on $\mathbb{R} \times [0,T]$.*

Lemma 2.3. *If $u_0(x) \ge c_1$ for a positive constant c_1, then*

$$u^\epsilon(x,t) \ge c(t,\epsilon,c_1) > 0, \tag{2.5}$$

where $c(t,\epsilon,c_1)$ could tend to 0 as $t \to +\infty$ or $\epsilon \to 0$.

Proof. We set $\nu = -\ln u$ and deduce from the first equation of the system (2.1) that

$$\nu_t - \epsilon\nu_{xx} \le \frac{\big(\phi(r)\big)^2}{\epsilon} + \phi(r)_x + a.$$

Then using the previous inequality, we obtain

$$\nu(x,t) \le \nu_0(x) * \frac{1}{\sqrt{4\epsilon\pi t}} e^{-\frac{x^2}{4\epsilon t}} + \int_0^t \Big(\tfrac{1}{\epsilon}\big(\phi(r)\big)^2 + \phi(r)_x + a\Big) *_x \frac{1}{\sqrt{4\epsilon\pi(t-s)}} e^{-\frac{x^2}{4\epsilon(t-s)}}\, ds\,,$$

where $\nu_0(x) = -\ln u_0^\epsilon(x)$. Hence

$$\nu(x,t) \le -\ln c_1 + \frac{N_1}{\epsilon} t + N_2\sqrt{\frac{t}{\epsilon}}\,.$$

It follows that

$$u(x,t) \ge c_1 \exp -\left(\frac{N_1}{\epsilon} t + N_2\sqrt{\frac{t}{\epsilon}}\right) \ge c(t,\epsilon,c_1) > 0. \qquad \square$$

Lemma 2.4. *Let z be the Riemann invariant given in (1.5). If in addition to the assumption of lemma 2.3, $z_0(x) = z(x,0) \in L^\infty(\mathbb{R})$ and $z_0'(x) \in L^1(\mathbb{R})$, then $\big(\frac{v^\epsilon}{u^\epsilon}\big)(x,t) \in L^\infty(\mathbb{R} \times [0,T])$, $\big(\frac{v^\epsilon}{u^\epsilon}\big)_x(\cdot,t) \in L^1(\mathbb{R})$. Moreover*

$$TV\left(\Big(\tfrac{v^\epsilon}{u^\epsilon}\Big)(\cdot,t)\right) = \int_{-\infty}^{+\infty} \Big|\Big(\tfrac{v^\epsilon}{u^\epsilon}\Big)_x(x,t)\Big|\, dx \le \int_{-\infty}^{+\infty} \Big|\Big(\tfrac{v_0}{u_0}\Big)'(x)\Big|\, dx = TV\left(\Big(\tfrac{v_0}{u_0}\Big)(x)\right), \tag{2.6}$$

where TV is the total variation.

Proof. Multiplying the first equation of (2.1) by $-\frac{v}{u^2}$ and the second equation by $\frac{1}{u}$ and summing them up, one obtains

$$\left(\frac{v}{u}\right)_t + \phi(r)\left(\frac{v}{u}\right)_x + (b-a)\frac{v}{u} = \epsilon\left(\frac{v}{u}\right)_{xx} + 2\epsilon\frac{u_x}{u}\left(\frac{v}{u}\right)_x, \tag{2.7}$$

Appliying the maximum principle to (2.7), we thus find that $\left(\frac{v^\epsilon}{u^\epsilon}\right)(x,t) \in L^\infty(\mathbb{R}\times[0,T])$. Now we differentiate (2.7) with respect to x and then we do $\theta = \left(\frac{v}{u}\right)_x$ to get

$$\theta_t + \big(\phi(r)\theta\big)_x + (b-a)\theta = \epsilon\theta_{xx} + \big(2\epsilon u^{-1}u_x\theta\big)_x,$$

multiplying this equation by the sequence of smooth functions $g'(\theta,\alpha)$, where α is a parameter, we obtain

$$g(\theta,\alpha)_t + \big(\phi(r)g(\theta,\alpha)\big)_x + \phi(r)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big) + (b-a)g'(\theta,\alpha)\theta_x = \epsilon g(\theta,\alpha)_{xx}$$
$$- \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon u^{-1}u_x g(\theta,\alpha)\big)_x + \big(2\epsilon u^{-1}u_x\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big). \tag{2.8}$$

We choose $g(\theta,\alpha)$ such that $g''(\theta,\alpha)\geq 0$, $g'(\theta,\alpha)\to \operatorname{sign}\theta$ and $g(\theta,\alpha)\to|\theta|$ as $\alpha\to 0$, we have from (2.8)

$$|\theta|_t + \big(\phi(r)|\theta|\big)_x \leq \epsilon|\theta|_{xx} + \big(2\epsilon u^{-1}u_x|\theta|\big)_x. \tag{2.9}$$

Integrating (2.9) in $\mathbb{R}\times[0,t]$, we obtain (2.6). □

We establish the results related to compactness in H^{-1}_{loc} that allow us to apply the div-curl lemma.

Lemma 2.5. *We assume the same conditions given in the Lemma 2.1. Then*

$$r^\epsilon_t + \left(\int^{r^\epsilon}\big(\phi(s) + \alpha s\phi'(s)\big)\,ds\right)_x \tag{2.10}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. We rewrite (2.3) as

$$r_t + \left(\int^{r}\big(\phi(s) + \alpha s\phi'(s)\big)\,ds\right)_x = \epsilon r_{xx} - \epsilon\alpha(\alpha-1)\Big(|u|^{\alpha-2}u_x^2 + |v|^{\alpha-2}v_x^2\Big)$$
$$- \alpha\big(a|u|^\alpha + b|v|^\alpha\big). \tag{2.11}$$

Noting that the last term in the right-hand side of (2.11) is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and using the same type of argument given in Lemma 5 of [3], we obtain the conclusion of the lemma. □

Lemma 2.6. *Under the assumptions of Lemma 2.1, it follows that*

$$\left(\int^{r^\epsilon}\big(\phi(s)+\alpha s\phi'(s)\big)\,ds\right)_t+\left(\int^{r^\epsilon}\big(\phi(s)+\alpha s\phi'(s)\big)^2\,ds\right)_x \tag{2.12}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Multiplying the equation (2.3) by $\phi(r)+\alpha r\phi'(r)$, we obtain

$$\begin{aligned}\left(\int^{r^\epsilon}\big(\phi(s)+\alpha s\phi'(s)\big)\,ds\right)_t+\left(\int^{r^\epsilon}\big(\phi(s)+\alpha s\phi'(s)\big)^2\,ds\right)_x &= \epsilon r_{xx}\big(\phi(r)\\ +\alpha r\phi'(r)\big)-\epsilon\alpha(\alpha-1)\Big(|u|^{\alpha-2}u_x^2+|v|^{\alpha-2}v_x^2\Big)&\big(\phi(r)+\alpha r\phi'(r)\big)\\ -\alpha\big(a|u|^\alpha+b|v|^\alpha\big)\big(\phi(r)+\alpha r\phi'(r)\big),&\end{aligned} \tag{2.13}$$

as the last term is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, we conclude that (2.12) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ by following the proof given in Lemma 6 of [3]. □

Lemma 2.7. *Suppose the conditions of Lemma 2.4 holds. Then*

$$\Big((u^\epsilon)^\alpha\Big)_t+\left(\frac{(u^\epsilon)^\alpha}{r^\epsilon}\int^{r^\epsilon}\big(\phi(s)+\alpha s\phi'(s)\big)\,ds\right)_x \tag{2.14}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. The first equation of the system (2.1) can be written as

$$u_t+u_x\Big(\phi\big(|u|^\alpha\varphi\big)+\alpha|u|^\alpha\varphi\phi'\big(|u|^\alpha\varphi\big)\Big)=\epsilon u_{xx}-|u|^\alpha u\varphi_x\phi'\big(|u|^\alpha\varphi\big)-au, \tag{2.15}$$

where the auxiliary function $\varphi(x,t)$ is defined by

$$\varphi=1+\left|\frac{v}{u}\right|^\alpha, \tag{2.16}$$

by Lemma 2.4 $\varphi(\cdot,t)_x$ is bounded in $L^1(\mathbb{R})$. Multiplying the above equation by $\alpha|u|^{\alpha-2}u$, we obtain

$$\begin{aligned}\big(u^\alpha\big)_t+\left(\frac{u^\alpha}{r}\int^r\big(\phi(s)+\alpha s\phi'(s)\big)\,ds\right)_x &= \epsilon\big(u^\alpha\big)_{xx}-\epsilon\alpha(\alpha-1)u^{\alpha-2}u_x^2\\ &-\alpha\big(u^\alpha\big)^2\varphi_x\phi'\big(u^\alpha\varphi\big)-a\alpha|u|^\alpha,\end{aligned} \tag{2.17}$$

the term $a\alpha|u|^\alpha$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. We skip the rest of the proof since the result is derived by following exactly the same proof of Lemma 7 in [3]. □

Lemma 2.8. *Let the assumptions in Lemma 2.4 hold. Then*

$$u_t^\epsilon + \Big(u^\epsilon \phi\big(r^\epsilon\big)\Big)_x \tag{2.18}$$

is compact in $H_{loc}^{-1}(\mathbb{R}\times\mathbb{R}^+)$.

We skip the proof of this lemma since it is similar to the one exposed to establish Lemma 8 in [3].

Corollary 2.9. *Suppose the conditions of Lemma 2.4. Then we have*

$$u_t^\epsilon + \left(u^\epsilon \phi\big(r^\epsilon\big) + \frac{v^\epsilon}{u^\epsilon}\right)_x \tag{2.19}$$

is compact in $H_{loc}^{-1}(\mathbb{R}\times\mathbb{R}^+)$.

The proof of the following result is an easy adaptation of the proof given in [3], Lemma 10.

Lemma 2.10. *Assuming the hypotheses as in lemma 2.4, then*

$$v_t^\epsilon + \Big(v^\epsilon \phi\big(r^\epsilon\big)\Big)_x \tag{2.20}$$

is compact in $H_{loc}^{-1}(\mathbb{R}\times\mathbb{R}^+)$.

Corollary 2.11. *Suppose the conditions of Lemma 2.4. Then we have*

$$v_t^\epsilon + \left(v^\epsilon \phi(r^\epsilon) + \left(\frac{v^\epsilon}{u^\epsilon}\right)^2\right)_x \tag{2.21}$$

is compact in $H_{loc}^{-1}(\mathbb{R}\times\mathbb{R}^+)$.

Lemma 2.12. *When the assumptions of Lemma 2.1 are satisfied and*

$$meas\{r : (2n+1)\phi'(r) + 2nr\phi''(r) = 0\} = 0, \tag{2.22}$$

then there exists a subsequence of $\{r^\epsilon(x,t)\}$ *which converges pointwisely.*

Proof. We use the div-curl lemma, which can be applied to the functions (2.10) and (2.12) (for more details see [3], Lemma 12). □

Lemma 2.13. *Assume the hypotheses of the lemmas 2.4 and 2.12, then there is a subsequence of* $\{u^\epsilon(x,t)\}$ *which converges pointwisely.*

Proof. The proof is the same to that of Lemma 13, [3]. □

Lemma 2.14. *Under the assumptions of lemma 2.13, there is a subsequence of* $\{v^\epsilon\}$ *such that it converges pointwise.*

Proof. The proof is the same as for Lemma 14 in [3]. □

Theorem 2.15. *Suppose that* $\left(\frac{v_0}{u_0}\right)(x) \in L^\infty(\mathbb{R})$, $\left(\frac{v_0}{u_0}\right)'(x) \in L^1(\mathbb{R})$, $\phi(r)$ *is strictly increasing or decreasing for positive* r, $\phi(r) \in C^2(\mathbb{R}^+)$ *and* $meas\{r : (2n+1)\phi'(r) + 2nr\phi''(r)\} = 0$. *Then there exist a subsequence of* (u^ϵ, v^ϵ) *which converges pointwisely and the limit is a weak solution of the Cauchy problem (1.2)-(1.3).*

Proof. We consider the sequence of viscosity solutions (u^ϵ, v^ϵ) of the system (2.1). Let us consider $\varphi, \psi \in C_0^\infty\big(\mathbb{R}\times[0,\infty)\big)$. By multiplying the first equation of the system (2.1) by φ, the second by ψ, adding the resulting equations and integrating by parts in $\mathbb{R} \times [0,\infty)$, we obtain that u^ϵ and v^ϵ satisfy the weak formulation of the Cauchy problem (2.1)-(1.3),

$$\begin{aligned}\int_{\mathbb{R}}\int_0^{+\infty} \big(u^\epsilon\varphi_t + u^\epsilon\phi(r^\epsilon)\varphi_x - au^\epsilon\varphi\big)\,dt\,dx + \int_{\mathbb{R}} u_0\varphi(x,0)\,dx \\ + \int_{\mathbb{R}}\int_0^{+\infty} \big(v^\epsilon\psi_t + v^\epsilon\phi(r^\epsilon)\psi_x - bv^\epsilon\psi\big)\,dt\,dx + \int_{\mathbb{R}} v_0\psi(x,0)\,dx = \\ -\epsilon\int_{\mathbb{R}}\int_0^{+\infty}\big(u^\epsilon\varphi_{xx} + v^\epsilon\psi_{xx}\big)\,dt\,dx\,. \quad (2.23)\end{aligned}$$

By Lemmas 2.12, 2.13 and 2.14, we can find a subsequence of (u^ϵ, v^ϵ) (no relabeled), which converges pointwise, a. e. $(x,t) \in \mathbb{R}\times[0,T]$, to (u,v) and it is such that $r^\epsilon \to |u|^\alpha + |v|^\alpha$, a. e. $(x,t) \in \mathbb{R}\times[0,T]$. Since ϕ is continuous, $\phi(r^\epsilon) \to \phi(|u|^\alpha + |v|^\alpha)$, a. e. $(x,t) \in \mathbb{R}\times[0,T]$.
Note that by (2.2),

$$\begin{aligned}\left|\epsilon\int_{\mathbb{R}}\int_0^{+\infty} u^\epsilon\varphi_{xx}\,dt\,dx\right| &\le \epsilon||\varphi_{xx}||_{L^\infty}\int_{\text{supp}(\varphi)}|u^\epsilon|\,dt\,dx \\ &\le \epsilon N(T)\Big(meas\big(\text{supp}(\varphi)\big)\Big),\end{aligned}$$

thus we obtain

$$\lim_{\epsilon\to 0}\epsilon\int_{\mathbb{R}}\int_0^{+\infty} u^\epsilon\varphi_{xx}\,dt\,dx = 0. \quad (2.24)$$

Using the above argument we also have

$$\lim_{\epsilon\to 0}\epsilon\int_{\mathbb{R}}\int_0^{+\infty} v^\epsilon\psi_{xx}\,dt\,dx = 0. \quad (2.25)$$

We want to pass to the limit the weak formulation (2.23) to complete the proof. From (2.24) and (2.25), it follows immediately that the integral on the right-hand side of (2.23) converges to 0 as $\epsilon \to 0$. Due to the convergence almost everywhere, we can apply the Lebesgue dominated convergence theorem to (2.23) to obtain that (u, v) is a weak solution of the Cauchy problem (1.2)-(1.3). □

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