Solution of the Huppert Equation Using
Lattice-Boltzmann and a Solitary Wave Method

F. Fonseca

Universidad Nacional de Colombia
Grupo de Ciencia de Materiales y Superficies
Departamento de Física
Bogotá, Colombia

Copyright © 2018 F. Fonseca. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we use a lattice Boltzmann model, along with a $d1q3$ velocity scheme, for the solution of the nonlinear one-dimensional Huppert equation (HEq). Also, applying a field transformation, we get the right balance in order to find solitary wave methods. We use the Tanh and Riccati solutions, finding several families of solutions.

Keywords: Huppert equation, lattice-Boltzmann, Tanh method, Riccati equation

1 Introduction

The Huppert equation plays a prominent role in the physical description of gravity currents phenomena [1]-[2], and in geological fluid mechanics [3]. On the other hand, the Lattice-Boltzmann technique [4]-[5], has generated important results in fields like engineering and science [6]-[9]. In the same way, the solitary wave solution method called tanh [10], has become a very useful analytical tool, in the achievement of solutions of nonlinear differential equations. In this paper we solve the Huppert equation using LB the solutions provided by Tanh and the solutions of the Riccati equation [11].
2 The lattice Boltzmann model

The lattice Boltzmann equation [5] in the B.G.K. approximation [6], is:

\[ f_i(x + e_i \epsilon, t + \epsilon) - f_i(x, t) = -\frac{1}{\tau} (f_i(x, t) - f_i^{eq}(x, t)) \]  

(1)

Expanding in a Taylor series, the distribution function, up to order fourth, we have:

\[ f_i(x + e_i \epsilon, t + \epsilon) - f_i(x, t) = \epsilon \left( \frac{\partial}{\partial t} + e_i \frac{\partial}{\partial x} \right) f_i + \frac{\epsilon^2}{2} \left( \frac{\partial}{\partial t} + e_i \frac{\partial}{\partial x} \right)^2 f_i \]  

(2)

\[ + \frac{\epsilon^3}{6} \left( \frac{\partial}{\partial t} + e_i \frac{\partial}{\partial x} \right)^3 f_i + \frac{\epsilon^4}{24} \left( \frac{\partial}{\partial t} + e_i \frac{\partial}{\partial x} \right)^4 f_i + O(\epsilon^5) \]

Doing a perturbative expansion of the derivatives in time in powers of \( \epsilon \), we get:

\[ f_i = f_i^{(0)} + \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \epsilon^3 f_i^{(3)} + \epsilon^4 f_i^{(4)} \]  

(3)

And assuming:

\[ f_i^{(0)} = f_i^{(eq)} \]  

(4)

Where the temporal scales are defined as:

\[ t_0 = t \quad t_1 = \epsilon t \quad t_2 = \epsilon t^2 \quad t_3 = \epsilon t^3 \quad t_4 = \epsilon t^4 \]  

(5)

And the perturbative expansion in parameter \( \epsilon \) of the temporal derivative operator

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \epsilon^3 \frac{\partial}{\partial t_3} + \epsilon^4 \frac{\partial}{\partial t_4} \]  

(6)

Replacing eqs. (2-6) in eq. (1), we get at first and second order in \( \epsilon \), respectively, the next set of equations:

\[ \frac{\partial f_i^0}{\partial t_0} + e_i \frac{\partial f_i^0}{\partial x} = -\frac{1}{\tau} f_i^1 \]  

(7)

\[ \frac{\partial f_i^0}{\partial t_1} - \tau (1 - \frac{1}{\tau}) \left( \frac{\partial}{\partial t_0} + e_i \frac{\partial}{\partial x} \right)^2 f_i = -\frac{1}{\tau} f_i^2 \]  

(8)
3 The moments of the distribution

The moments of the distribution are:

\[ \sum_i f_i^{(0)} = \phi = \sum_i f_i^{(eq)} \]  
(9)

\[ \sum_i e_i f_i^{(0)} = 0 \]  
(10)

\[ \sum_i e_{i,i} f_i^{(0)} = \lambda \frac{1}{4} \phi^4 \delta_{ij} \]  
(11)

Where \( \delta_{ij} \) is Kronecker’s delta. Assuming the superior orders of the moments of the distribution as:

\[ \sum_i f_i^{(k)} = 0, \quad \text{with} \quad k > 0 \]  
(12)

4 The Huppert equation

Summing on \( i \) in eq. (7) and using eq. (9) and (12) we get:

\[ \frac{\partial \phi}{\partial t_0} + 0 = -\frac{1}{\tau} \sum_i f_i^1 = 0 \rightarrow \frac{\partial \phi}{\partial t_0} = 0 \]  
(13)

An now summing on \( i \) in eq. (8) and multiplying by \( \epsilon \)

\[ \epsilon \frac{\partial \sum_i f_i^0}{\partial t_1} - \epsilon \tau (1 - \frac{1}{\tau}) \sum_i \left( \frac{\partial}{\partial t_0} + e_i \frac{\partial}{\partial x} \right) f_i = -\frac{1}{\tau} \sum_i f_i^{(2)} \]  
(14)

And using eqs. (9-10) and (12), we have:

\[ \frac{\epsilon}{\partial t_1} - \epsilon \tau (1 - \frac{1}{\tau}) \left( \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \sum_i f_i e_{i,k} e_{i,j} \right) = 0 \]  
(15)

Summing eq. (15) to eq. (13) and using eq. (11):

\[ \frac{\partial \phi}{\partial t_0} + \epsilon \frac{\partial \phi}{\partial t_1} - \epsilon \tau (1 - \frac{1}{\tau}) \frac{\partial^2}{\partial x^2} \left( \frac{\lambda}{4} \phi^4 \right) = 0 \]  
(16)

Also, using eq. (6), at first order:

\[ \frac{\partial \phi}{\partial t} - \lambda \left( \frac{1}{\tau} - 1 \right) \frac{\partial^2 \phi^4}{\partial x^2} = 0 \]  
(17)

Defining \( D = (\lambda \epsilon (\frac{1}{\tau} - 1)) \), we have the third order diffusion equation

\[ \frac{\partial \phi}{\partial t} = D \frac{\partial}{\partial x} \left( \phi^3 \frac{\partial \phi}{\partial x} \right) \]  
(18)
5 The equilibrium distribution function

In figure (1) we show a d1q3 one-dimensional velocity scheme with $e_\alpha = \{0, c, -c\}$ [4]. So, the one particle equilibrium equilibrium distribution function is:

$$f^{(eq)}_i = \begin{cases} \phi - \frac{\lambda}{4c^4} \phi^4 & \rightarrow i = 0 \\ \frac{\lambda}{8c^4} \phi^4 & \rightarrow i = 1 \\ \frac{\lambda}{8c^4} \phi^4 & \rightarrow i = 2 \end{cases}$$ (19)

6 Solitary wave solution 1

using the transformation

$$\zeta = x - ct$$ (20)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta}, \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \zeta^2}, \frac{\partial}{\partial t} = -c \frac{\partial}{\partial \zeta}$$ (21)

Setting $D = 1$, and replacing in eq. (18):

$$-c \frac{d\phi}{d\zeta} + \frac{3}{4} \phi^2 \left(\frac{d\phi}{d\zeta}\right)^2 + \frac{1}{4} \phi^3 \frac{d^2\phi}{d\zeta^2} = 0$$ (22)

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (22), [4]. We get:

$$\frac{d\phi}{d\zeta} \rightarrow \phi^3 \frac{d^2\phi}{d\zeta^2} \rightarrow m + 1 = 3m + m + 2 \rightarrow m = -1/3$$ (23)

Then, we do the next transformation

$$\phi = h^{-1/3} \rightarrow \frac{d\phi}{d\zeta} = \frac{d\phi}{dh} \frac{dh}{d\zeta} = -\frac{1}{3} h^{-4/3} \frac{dh}{d\zeta}$$ (24)

$$\frac{d^2\phi}{d\zeta^2} = -\frac{d}{d\zeta} \left(\frac{1}{3} h^{-4/3} \frac{dh}{d\zeta}\right) = \frac{4}{9} h^{-7/3} \left(\frac{dh}{d\zeta}\right)^2 - \frac{1}{3} h^{-4/3} \frac{d^2h}{d\zeta^2}$$ (25)
Then, eq. (22)

$$c \frac{1}{3} h^{-4/3} \frac{dh}{d\zeta} + \frac{1}{12} h^{-10/3} \left( \frac{dh}{d\zeta} \right)^2 + \frac{1}{9} h^{-10/3} \left( \frac{dh}{d\zeta} \right)^2 - \frac{1}{12} h^{-7/3} \frac{d^2h}{d\zeta^2} = 0$$  \hspace{1cm} (26)

Multiplying by \(h^{4/3}\) in eq. (26)

$$12c h^2 \frac{dh}{d\zeta} + 7 \left( \frac{dh}{d\zeta} \right)^2 - 3h \frac{d^2h}{d\zeta^2} = 0$$  \hspace{1cm} (27)

Again, balancing

$$h^2 \frac{dh}{d\zeta} \rightarrow h \frac{d^2h}{d\zeta^2} \rightarrow 2m + m + 1 = m + m + 2 \rightarrow m = 1$$  \hspace{1cm} (28)

Now, we introduce a new independent variable \([4]\):

$$Y = \tanh (\zeta)$$  \hspace{1cm} (29)

Then, the first and second derivatives of \(\zeta\), are:

$$\frac{d}{d\zeta} = (1 - Y^2) \frac{d}{dY}; \quad \frac{d^2}{d\zeta^2} = -2Y(1 - Y^2) \frac{d}{dY} + (1 - Y^2)^2 \frac{d^2}{dY^2}$$  \hspace{1cm} (30)
The solutions are postulated as:

\[ h = \sum_{i=1}^{m} a_i Y^i \rightarrow h = a_0 + a_1 Y \rightarrow \frac{dh}{dY} = a_1 \rightarrow \frac{d^2 h}{dY^2} = 0 \] (31)

Then, replacing in eq. (27) and doing some algebra

\[ a_{11} = -1/4c; \quad a_{12} = 1/12c; \quad a_{0,1,2,3,4} = \pm \sqrt{-7a_1/12c} \] (32)

\[ h_1 \rightarrow (a_0, a_{11}), \quad h_2 \rightarrow (a_0, a_{11}), \quad h_3 \rightarrow (a_0, a_{12}), \quad h_4 \rightarrow (a_0, a_{12}) \] (33)

Then, we get 4 solutions using Tanh method.

### 7 Solitary wave solution 2

Also, we apply \( \phi(\xi) \) to find solutions, [11]

\[ h(\xi) = \sum_{i=1}^{m} a_i F^i \] (34)

where \( F \) solves, table (1), the Riccati equation, i.e.

\[ F' = CF^2 + A, \quad \rightarrow F'' = CFF' = 2CF(CF^2 + A) = 2C^2 F^3 + 2ACF \] (35)

here A, C are constants, table (1). Replacing in eq. (27), and taking \( m = 1 \) in eq. (34) and doing the algebra, we get, \( h(\xi) = a_0 + a_1 F \). So:

\[ h' = a_1 F' = a_1 CF^2 + a_1 A \rightarrow h'' = a_1 F'' = 2a_1 C^2 F^3 + 2a_1 ACF \] (36)
\[12c(a_0 + a_1 F)^2(a_1 CF^2 + a_1 A) + 7(a_1 CF^2 + a_1 A)^2 - 3(a_0 + a_1 F)(2a_1 C^2 F^3 + 2a_1 AC F) = 0\]

\[a_{11} = C/4c^2; \quad a_{12} = -C/12c^2\]  

\[a_{01,2,3,4} = \pm \sqrt{-7Aa_{11,2}/12c^2}\]  

\[a_{05,6,7,8} = \pm \sqrt{(-14a_{1,2} AC + 6a_{1,2} AC - 12a_{1,2} AC^2)/(12c^2 C)}\]  

\[h_5 \to (a_{01}, a_{11}), \quad h_6 \to (a_{02}, a_{11}), \quad h_7 \to (a_{03}, a_{12}), \quad h_8 \to (a_{04}, a_{12}), \quad h_9 \to (a_{05}, a_{11}), \quad h_{10} \to (a_{06}, a_{11}), \quad h_{11} \to (a_{07}, a_{12}), \quad h_{12} \to (a_{08}, a_{12})\]  

Then, we get 48 solutions using the Riccati method.

### 8 Conclusions

We solved the Huppert equation using the lattice-Boltzmann technique and the tanh and Riccati solitary wave methods. We get 52 families of solutions. As a future work, we can extend the investigation of solutions in higher dimensions.

\[\phi_i = (a_{0i} + a_{1i} \tanh(x - ct))^{-1/3}\]  

\[\phi_j = (a_{0j} + a_{1j} F(x - ct))^{-1/3}\]  

**Acknowledgements.** This research was supported by Universidad Nacional de Colombia in Hermes project (32501).

### References


**Received: December 21, 2017; Published: January 5, 2018**