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Solution of the Huppert Equation Using Lattice-Boltzmann and a Solitary Wave Method

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Abstract

In this paper, we use a lattice Boltzmann model, along with a $d1q3$ velocity scheme, for the solution of the nonlinear one-dimensional Huppert equation (HEq). Also, applying a field transformation, we get the right balance in order to find solitary wave methods. We use the Tanh and Riccati solutions, finding several families of solutions.

Keywords: Huppert equation, lattice-Boltzmann, Tanh method, Riccati equation

1 Introduction

The Huppert equation plays a prominent role in the physical description of gravity currents phenomena [1]-[2], and in geological fluid mechanics [3]. On the other hand, the Lattice-Boltzmann technique [4] -[5], has generated important results in fields like engineering and science [6]-[9]. In the same way, the solitary wave solution method called tanh [10], has become a very useful analytical tool, in the achievement of solutions of nonlinear differential equations. In this paper we solve the Huppert equation using LB the solutions provided by Tanh and the solutions of the Riccati equation [11].

2 The lattice Boltzmann model

The lattice Boltzmann equation [5] in the B.G.K. approximation [6], is:

$$f_i(x+e_i\epsilon, t+\epsilon) - f_i(x,t) = -\frac{1}{\tau}\left(f_i(x,t) - f_i^{eq}(x,t)\right) \tag{1}$$

Expanding in a Taylor series, the distribution function, up to order fourth, we have:

$$\begin{aligned} f_i(x+e_i\epsilon, t+\epsilon) \quad - \quad & f_i(x,t) = \epsilon\left(\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x}\right) f_i + \frac{\epsilon^2}{2}\left(\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x}\right)^2 f_i \quad (2)\\ + \quad & \frac{\epsilon^3}{6}\left(\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x}\right)^3 f_i + \frac{\epsilon^4}{24}\left(\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x}\right)^4 f_i + O(\epsilon^5) \end{aligned}$$

Doing a perturbative expansion of the derivatives in time in powers of ϵ, we get:

$$f_i = f_i^{(0)} + \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \epsilon^3 f_i^{(3)} + \epsilon^4 f_i^{(4)} \tag{3}$$

And assuming:

$$f_i^{(0)} = f_i^{(eq)} \tag{4}$$

Where the temporal scales are defined as:

$$t_0 = t \quad t_1 = \epsilon t \quad t_2 = \epsilon t^2 \quad t_3 = \epsilon t^3 \quad t_4 = \epsilon t^4 \tag{5}$$

And the perturbative expansion in parameter ϵ of the temporal derivative operator

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon^1\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2} + \epsilon^3\frac{\partial}{\partial t_3} + \epsilon^4\frac{\partial}{\partial t_4} \tag{6}$$

Replacing eqs. (2-6) in eq. (1), we get at first and second order in ϵ, respectively, the next set of equations:

$$\frac{\partial f_i^0}{\partial t_0} + e_i\frac{\partial f_i^0}{\partial x} = -\frac{1}{\tau} f_i^1 \tag{7}$$

$$\frac{\partial f_i^0}{\partial t_1} - \tau(1-\frac{1}{\tau})\left(\frac{\partial}{\partial t_0} + e_i\frac{\partial}{\partial x}\right)^2 f_i = -\frac{1}{\tau} f_i^2 \tag{8}$$

3 The moments of the distribution

The moments of the distribution are:

$$\sum_i f_i^{(0)} = \phi = \sum_i f_i^{(eq)} \tag{9}$$

$$\sum_i e_i f_i^{(0)} = 0 \tag{10}$$

$$\sum_l e_{l,i} e_{l,j} f_l^{(0)} = \lambda \frac{1}{4} \phi^4 \delta_{ij} \tag{11}$$

Where δ_{ij} is Kronecker's delta. Assuming the superior orders of the moments of the distribution as:

$$\sum_i f_i^{(k)} = 0, \quad \textit{with} \quad k > 0 \tag{12}$$

4 The Huppert equation

Summing on i in eq. (7) and using eq. (9) and (12) we get:

$$\frac{\partial \phi}{\partial t_0} + 0 = -\frac{1}{\tau} \sum_i f_i^1 = 0 \rightarrow \frac{\partial \phi}{\partial t_0} = 0 \tag{13}$$

An now summing on i in eq. (8) and multiplying by ϵ

$$\epsilon \frac{\partial \sum_i f_i^0}{\partial t_1} - \epsilon\tau(1 - \frac{1}{\tau}) \sum_i \left(\frac{\partial}{\partial t_0} + e_i \frac{\partial}{\partial x} \right)^2 f_i = -\epsilon \frac{1}{\tau} \sum_i f_i^{(2)} \tag{14}$$

And using eqs. (9-10) and (12), we have:

$$\epsilon \frac{\partial \phi}{\partial t_1} - \epsilon\tau(1 - \frac{1}{\tau}) \left(\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \sum_i f_i e_{i,k} e_{i,j} \right) = 0 \tag{15}$$

Summing eq. (15) to eq. (13) and using eq. (11):

$$\frac{\partial \phi}{\partial t_0} + \epsilon \frac{\partial \phi}{\partial t_1} - \epsilon\tau(1 - \frac{1}{\tau}) \frac{\partial^2}{\partial x^2} (\frac{\lambda}{4} \phi^4) = 0 \tag{16}$$

Also, using eq. (6), at first order:

$$\frac{\partial \phi}{\partial t} - \frac{\lambda}{4} \epsilon\tau(1 - \frac{1}{\tau}) \frac{\partial^2 \phi^4}{\partial x^2} = 0 \tag{17}$$

Defining $D = (\lambda\epsilon\tau(\frac{1}{\tau} - 1))$, we have the third order diffusion equation

$$\frac{\partial \phi}{\partial t} = D \frac{\partial}{\partial x} (\phi^3 \frac{\partial \phi}{\partial x}) \tag{18}$$

Figure 1: Lattice-Boltzmann velocity scheme $d1q3$.

5 The equilibrium distribution function

In figure (1) we show a $d1q3$ one-dimensional velocity scheme with $e_\alpha = \{0, c, -c\}$ [4]. So, the one particle equilibrium equilibrium distribution function is:

$$f_i^{(eq)} = \left\{ \begin{array}{ccc} \phi - \frac{\lambda}{4c^2}\phi^4 & \rightarrow & i=0 \\ \frac{\lambda}{8c^2}\phi^4 & \rightarrow & i=1 \\ \frac{\lambda}{8c^2}\phi^4 & \rightarrow & i=2 \end{array} \right\} \tag{19}$$

6 Solitary wave solution 1

using the transformation

$$\zeta = x - ct \tag{20}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \zeta^2}, \quad \frac{\partial}{\partial t} = -c\frac{\partial}{\partial \zeta} \tag{21}$$

Setting $D = 1$, and replacing in eq. (18):

$$-c\frac{d\phi}{d\zeta} + \frac{3}{4}\phi^2(\frac{d\phi}{d\zeta})^2 + \frac{1}{4}\phi^3\frac{d^2\phi}{d\zeta^2} = 0 \tag{22}$$

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (22), [4]. We get:

$$\frac{d\phi}{d\zeta} \rightarrow \phi^3\frac{d^2\phi}{d\zeta^2} \rightarrow m + 1 = 3m + m + 2 \rightarrow m = -1/3 \tag{23}$$

Then, we do the next transformation

$$\phi = h^{-1/3} \rightarrow \frac{d\phi}{d\zeta} = \frac{d\phi}{dh}\frac{dh}{d\zeta} = -\frac{1}{3}h^{-4/3}\frac{dh}{d\zeta} \tag{24}$$

$$\frac{d^2\phi}{d\zeta^2} = -\frac{d}{d\zeta}(\frac{1}{3}h^{-4/3}\frac{dh}{d\zeta}) = \frac{4}{9}h^{-7/3}(\frac{dh}{d\zeta})^2 - \frac{1}{3}h^{-4/3}\frac{d^2h}{d\zeta^2} \tag{25}$$

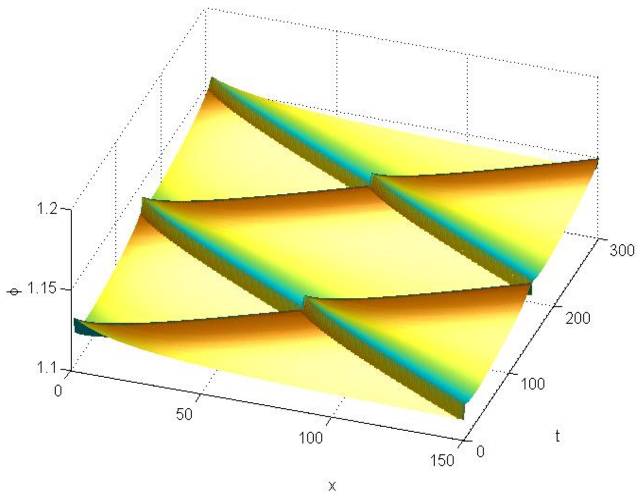

Figure 2: The spatiotemporal Lattice-Boltzmann for $\phi(x,t)$ using a $d1q3$ lattice velocity, for two initial profiles given by eq. (26).

Then, eq. (22)

$$c\frac{1}{3}h^{-4/3}\frac{dh}{d\zeta}+\frac{1}{12}h^{-10/3}(\frac{dh}{d\zeta})^2+\frac{1}{9}h^{-10/3}(\frac{dh}{d\zeta})^2-\frac{1}{12}h^{-7/3}\frac{d^2h}{d\zeta^2}=0 \tag{26}$$

Multiplying by $h^{4/3}$ in eq. (26)

$$12ch^2\frac{dh}{d\zeta}+7(\frac{dh}{d\zeta})^2-3h\frac{d^2h}{d\zeta^2}=0 \tag{27}$$

Again, balancing

$$h^2\frac{dh}{d\zeta}\rightarrow h\frac{d^2h}{d\zeta^2}\rightarrow 2m+m+1=m+m+2\rightarrow m=1 \tag{28}$$

Now, we introduce a new independent variable [4]:

$$Y=\tanh(\zeta) \tag{29}$$

Then, the first and second derivatives of ζ, are:

$$\frac{d}{d\zeta}=(1-Y^2)\frac{d}{dY};\quad \frac{d^2}{d\zeta^2}=-2Y(1-Y^2)\frac{d}{dY}+(1-Y^2)^2\frac{d^2}{dY^2} \tag{30}$$

	A	C	F
1	1/2	-1/2	$coth(\xi) \pm cosh(\xi)$, $tanh(\xi) \pm isech(\xi)$
2	1/2	1/2	$sec(\xi) \pm itan(\xi)$
3	-1/2	- 1/2	$csc(\xi) \pm icot(\xi)$
4	1	- 1	$tanh(\xi)$, $coth(\xi)$
5	1	1	$tan(\xi)$
6	-1	-1	$cot(\xi)$

Table 1: Solutions for eq. (34)[11] .

The solutions are postulated as:

$$h = \sum_{i=1}^{m} a_i Y^i \rightarrow h = a_0 + a_1 Y^1 \rightarrow \frac{dh}{dY} = a_1 \rightarrow \frac{d^2h}{dY^2} = 0 \tag{31}$$

Then, replacing in eq. (27) and doing some algebra

$$a_{1_1} = -1/4c; \quad a_{1_2} = 1/12c; \quad a_{0_{1,2,3,4}} = \pm\sqrt{-7a_1/12c} \tag{32}$$

$$h_1 \rightarrow (a_{0_1}, a_{1_1}), h_2 \rightarrow (a_{0_2}, a_{1_1}), h_3 \rightarrow (a_{0_3}, a_{1_2}), h_4 \rightarrow (a_{0_4}, a_{1_2}) \tag{33}$$

Then, we get 4 solutions using Tanh method.

7 Solitary wave solution 2

Also, we apply $\phi(\xi)$ to find solutions, [11]

$$h(\xi) = \sum_{i=1}^{m} a_i F^i \tag{34}$$

where F solves, table (1), the Riccati equation, i.e.

$$F' = CF^2 + A. \rightarrow F'' = C2FF' = 2CF(CF^2 + A) = 2C^2F^3 + 2ACF \tag{35}$$

here A, C are constants, table (1). Replacing in eq. (27), and taking $m = 1$ in eq. (34) and doing the algebra, we get, $h(\xi) = a_0 + a_1 F$. So:
then

$$h' = a_1 F' = a_1 CF^2 + a_1 A \rightarrow h'' = a_1 F'' = 2a_1 C^2 F^3 + 2a_1 ACF \tag{36}$$

$$12c(a_0+a_1F)^2(a_1CF^2+a_1A)+7(a_1CF^2+a_1A)^2$$
$$-3(a_0+a_1F)(2a_1C^2F^3+2a_1ACF)=0 \quad (37)$$

$$a_{1_1}=C/4c^2; \quad a_{1_2}=-C/12c^2 \quad (38)$$

$$a_{0_{1,2,3,4}}=\pm\sqrt{-7Aa_{1_{1,2}}/12c^2} \quad (39)$$

$$a_{0_{5,6,7,8}}=\pm\sqrt{(-14a_{1_{1,2}}AC+6a_{1_{1,2}}AC-12a_{1_{1,2}}^2Ac^2)/(12c^2C)} \quad (40)$$

$$h_5\to(a_{0_1},a_{1_1}),h_6\to(a_{0_2},a_{1_1}),h_7\to(a_{0_3},a_{1_2}),h_8\to(a_{0_4},a_{1_2}) \quad (41)$$
$$h_9\to(a_{0_5},a_{1_1}),h_{10}\to(a_{0_6},a_{1_1}),h_{11}\to(a_{0_7},a_{1_2}),h_{12}\to(a_{0_8},a_{1_2})$$

Then, we get 48 solutions using Ricatti method.

8 Conclusions

We solved the Huppert equation using the lattice-Boltzmann technique and the tanh and Riccati solitary wave methods. We get 52 families of solutions. As a future work, we can extend the investigation of solutions in higher dimensions.

$$\phi_i=(a_{0_i}+a_{1_i}\tanh(x-ct))^{-1/3} \quad (42)$$

$$\phi_j=(a_{0_j}+a_{1_j}F(x-ct))^{-1/3} \quad (43)$$

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