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On the Boundary Layer Flow of a Shear Thinning Liquid over a 2-Dimensional Stretching Surface

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Abstract

Shear thinning phenomenon in inelastic non-Newtonian fluids is widely encountered in several engineering and industrial applications. In this paper, we have considered steady boundary layer flow of a special type of pseudoplastic fluid over a 2-dimensional stretching sheet. Using the Williamson fluid model, we have carried out an analytico-numerical study to investigate the effect of higher order terms in the perturbation analysis involving a non-Newtonian parameter. A number of illustrative plots of velocity components have been presented to showcase several features of this flow. It has been shown that higher order relative effects for flow over a stretching surface do not strictly exhibit the non-monotonic features observed in stagnation point flow of pseudoplastic as well as dilatant fluids, over a stationary surface. Furthermore, the magnitude of the wall shear stress at the stretching sheet is quite sensitive to non-Newtonian parameter as well as relative order effects in the perturbation expansion. In both cases, the wall shear stress increases with increasing non-Newtonian effects of the pseudoplastic fluid.

Subject Classification: 76A05, 76D10

Keywords: Non-Newtonian flow, pseudoplastic fluid, boundary layer, stretching sheet, higher order effects

1 Introduction

Rheological flows involving inelastic fluids are known to have a shear dependent viscosity, and thus possess nonlinear relationship between the stress tensor and the rate of deformation tensor. A number of fluid models has been reported in the vast literature on non-Newtonian flows exhibiting this shear dependent phenomenon. In general, such models are known to have two to three fluid parameters [1–3, 9, 11–14]. The extent to which the non-Newtonian properties of the fluids influence the flow features in various applications in engineering and industry varies from one application to another. While dealing with inelastic fluid models, it is known that the non-Newtonian features depend on the way the apparent viscosity of the fluid varies with the shear rate. For instance, a number of fluids are encountered in real life applications for which the apparent viscosity may decrease or increase with increasing shear rate. The inelastic fluids exhibiting increase in apparent viscosity with shear rate have been termed as shear thickening (dilatant) fluids, while those showing the opposite behavior are known as shear thinning (pseudoplastic) fluids. It is worth remarking here that majority of non-Newtonian inelastic fluids encountered in the real life applications (e.g., food processing, polymer, cosmetics, cement, biomedical, etc.) show pseudoplastic behavior.

In this work, we have analyzed a special type of pseudoplastic fluid governed by Williamson model. Although the Williamson constitutive equation is a three-parameter model, in several applications, one can use a special case of this by assuming the infinite shear viscosity parameter to be negligible [4, 6, 13]. Using this special case, we have carried out a semi-analytical study of the boundary layer flow over a 2-dimensional impermeable stretching sheet. Flows near stretching boundary sheets have been studied widely in fluid dynamics literature of real fluids due to their applications in technology, e.g., processing and polymer industries. Using the 2-dimensional boundary layer equations

[13], we have transformed [4, 7, 13] the partial differential equations together with the accompanying boundary conditions to a boundary value problem involving nonlinear ODE, using some similarity functions. This ODE contains two non-dimensional parameters indicating spatial and non-Newtonian characteristics. Our main aim in this work is to bring out the following in the boundary layer flow:
(i) Rheological effects, and (ii) Relative higher order effects vis-à-vis zeroth order Newtonian flow in the perturbation analysis to be employed for the solution.

The boundary value problems arising from the perturbation expansion have been solved numerically for a range of parameter values. The results have been presented through a host of plots and tables for the similarity functions representing longitudinal and transverse velocities. We have also shown how the shear stress at the bounding wall varies with the pseudoplastic parameter.

2 Boundary layer analysis

Consider a 2-dimensional rigid stretching surface along the $x-$axis and the $y-$axis perpendicular to it into the fluid. The boundary layer equations for the 2-parameter Williamson fluid can be derived following the same procedure as given in [4, 13]. However, for completeness and ready reference, we give below the main steps relevant to the constitutive equation and the boundary layer equations. The constitutive equation of the Williamson model has been employed in the literature by several researchers (see, e.g., [5]–[7], [13], [15]). The expression for the Cauchy stress tensor for the Williamson fluid is given by

$$S = -pI + T \tag{1}$$

where p, I, T denote the pressure, unit tensor and the deviatoric stress tensor, respectively. For the two-dimensional flow considered in this study, the Williamson fluid is a three-parameter model in which T is given by

$$T = \left[\mu_\infty + \frac{\sqrt{2}(\mu_0 - \mu_\infty)}{\sqrt{2} - \Gamma\sqrt{I_2}}\right] A_1 \tag{2}$$

In the above, μ_0 and μ_∞ are the zero shear viscosity and infinite shear viscosity, respectively; Γ is a time constant representing non-Newtonian effects (assumed small), I_2 is the second invariant of the rate of deformation tensor, and A_1 is the first Rivlin-Erickson tensor [16].

In the present study, we consider a special pseudoplastic fluid for which the infinite shear viscosity μ_∞ is negligible, and $\Gamma\sqrt{I_2} < 1$. Thus, we can expand the right side of Eq (2) in a binomial series to get

$$T = \mu_0 \left[1 + \Gamma\left(0.5I_2\right)^{1/2} + \Gamma^2\left(0.5I_2\right) + \Gamma^3\left(0.5I_2\right)^{3/2} + \cdots\right] A_1 \tag{3}$$

Assuming $\Gamma\left(0.5 I_2\right)^{1/2} \ll 1$, we can finally approximate T by

$$T = \mu_0 \left[1 + \Gamma\left(0.5 I_2\right)^{1/2}\right] A_1 \tag{4}$$

In the present study, we shall use Eq (4) as the first order approximation for T. In order to obtain the boundary layer equations for the two-dimensional steady flow, we consider the equation of continuity and the momentum equations, given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} \tag{6}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} \tag{7}$$

where ρ is the fluid density, and u, v are the velocity components in the x and y directions, respectively. Using the expressions for the stress components T_{xx}, T_{xy}, T_{yx} and T_{yy} as given in [13], and employing the usual boundary layer approximations, the boundary layer equations for the two-dimensional steady flow are: Eq (5), the equation $\partial p/\partial y = 0$, and the modified u-component of the momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu_0\frac{\partial^2 u}{\partial y^2} + \sqrt{2}\,\Gamma\nu_0\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \tag{8}$$

where ν_0 is the zero shear kinematic viscosity. We shall use Eq (8) in the next section for the analysis of the two-dimensional flow over a linearly stretching sheet.

3 Flow over a stretching sheet

The physical situation corresponds to the flow of a viscous incompressible pseudoplastic fluid modelled by the linearised Williamson constitutive equation, over a linearly stretching plane surface. Fluid motions over stretching surfaces have been extensively studied in the literature because of their applications in a number of industrial and engineering areas. For the flow of the Williamson fluid considered here, the boundary conditions are

$$u = U(x), \;\; v = 0 \;\text{ at }\; y = 0, \;\text{ and }\; u \to 0 \;\text{ as }\; y \to \infty \tag{9}$$

where $U = U(x) = ax$ and a is a constant. Furthermore, the pressure gradient term in the boundary layer equation (8) is zero. We thus re-write Eq (8) as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_0\frac{\partial^2 u}{\partial y^2} + \sqrt{2}\,\Gamma\nu_0\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \tag{10}$$

We now transform Eq (10) and the boundary conditions (9) using the similarity transformations

$$\eta = \sqrt{a/\nu_0}\, y, \quad u = Uf'(\eta), \quad v = -\sqrt{a\nu_0}\, f(\eta) \tag{11}$$

The above transformations automatically satisfy the continuity equation (5). Furthermore, using Eq (11) in Eq (10), we obtain a third order nonlinear ODE governing the similarity function f, in the form

$$f''' + ff'' - (f')^2 + hNf''f''' = 0 \tag{12}$$

The boundary conditions satisfied by f are

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{13}$$

In Eq (12), primes denote differentiation with respect to $\eta = \sqrt{a/\nu_0}\, y$, and the non-dimensional quantities h and N are defined by

$$h = \sqrt{a/\nu_0}\, x, \quad N = \sqrt{2}\, a\Gamma \tag{14}$$

Equations (12) and (13) describe the boundary value problem of the stretching flow of the fluid considered. It is clear that the features of the ensuing flow depend on the flow configuration as well as the fluid characteristics represented by the pseudoplastic parameter N. In the following, our main aim thus is to analyze the effect of the parameter N on the fluid velocity, particularly with regard to its higher order effects. We investigate this feature through a perturbation analysis in terms of the parameter N, assumed small. We write

$$f(\eta) = f_0(\eta) + Nf_1(\eta) + N^2 f_2(\eta) + N^3 f_3(\eta) + \cdots \tag{15}$$

Using Eq (15) in Eqs (12) and (13), and equating coefficients of different powers of N, we obtain sets of boundary value problems corresponding to the various order terms. Restricting ourselves up to and including terms of order 3, we obtain the following systems of boundary value problems:

$$f_0''' + f_0 f_0'' - (f_0')^2 = 0 \tag{16}$$

$$f_0(0) = 0, \quad f_0'(0) = 1, \quad f_0'(\infty) = 0$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = -hf_0'' f_0''' \tag{17}$$

$$f_1(0) = 0, \quad f_1'(0) = 0, \quad f_1'(\infty) = 0$$

$$f_2''' + f_0 f_2'' - 2f_0' f_2' + f_0'' f_2 = (f_1')^2 - f_1 f_1'' - h\left[f_0''' f_1'' + f_0'' f_1'''\right] \tag{18}$$

$$f_2(0) = 0, \quad f_2'(0) = 0, \quad f_2'(\infty) = 0$$

$$\begin{aligned} f_3''' + f_0 f_3'' - 2f_0' f_3' + f_0'' f_3 &= 2f_1' f_2' - f_1 f_2'' - f_1'' f_2 \\ &\quad - h\left[f_0''' f_2'' + f_1'' f_1''' + f_0'' f_2'''\right] \end{aligned} \tag{19}$$

$$f_3(0) = 0, \quad f_3'(0) = 0, \quad f_3'(\infty) = 0$$

Equations (16)–(19) subject to the applicable boundary conditions, have been solved numerically by a suitable shooting method. The results are discussed in the next section.

4 Discussion

In this section, we shall analyze the effects of the rheological parameter as well the relative order effects in the perturbation expansion. In addition, we shall also bring out the effect of the parameter h. These effects have been shown in Figs 1–6 and Tables 1 and 2.

In Figs 1 and 2, we have shown the plots of f and f', representing the transverse and longitudinal velocity components, respectively, in the boundary layer region, at the cross-section $h = 0.5$. Both velocity profiles follow expected physical features. In particular, the longitudinal velocity function decreases steadily from unity at the boundary and approaches zero near $\eta \approx 4$ (see Fig 2). It is apparent that both velocity components show enhancement with the decrease of pseudoplasticity parameter N. Within the boundary layer region, this enhancement is more pronounced away from the bounding stretched surface for the transverse velocity as compared to the longitudinal one. In Figs 3 and 4, we have shown the relative effects of retaining higher

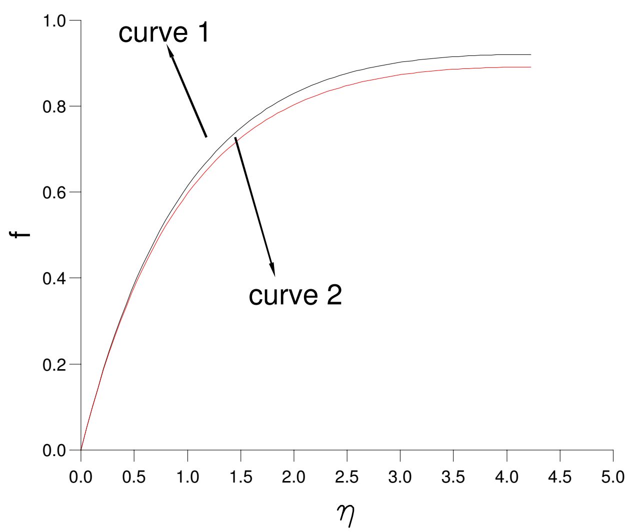

Fig 1. Variation of f: $N = 0.3$ (Curve 1), 0.9 (Curve 2)

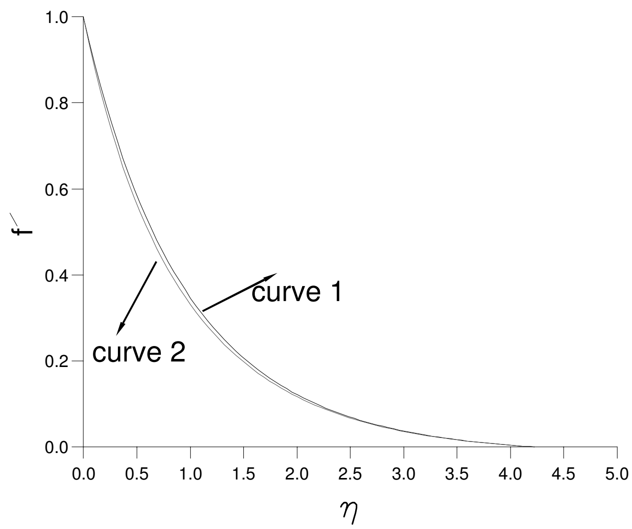

Fig 2. Variation of f': $N = 0.3$ (Curve 1), 0.9 (Curve 2)

order terms in comparison to the corresponding Newtonian flow, for the perturbation solutions near the stretching surface. The curves in these two figures correspond to the relative percentage effect of retaining up to first order (curve 1), second order (curve 2) and third order (curve 3) terms in Eq (15). Thus the curves 1, 2 and 3 represent the following quantities:

Curve 1: $\dfrac{Nf_1}{f_0} \times 100$, Curve 2: $\dfrac{Nf_1 + N^2 f_2}{f_0} \times 100$,

Curve 3: $\dfrac{Nf_1 + N^2 f_2 + N^3 f_3}{f_0} \times 100$

The curves in Fig 3 correspond to $N = 0.6$, while those in Fig 4 are for $N = 0.9$. These figures clearly indicate that higher order effects do influence the transverse velocity of the flow, and the this velocity decreases with the higher order terms. However, the effect of the pseudoplastic parameter is more discernible as one goes from the first to second order effect, while the second to third order effect is much less than the previous one. It is worth noting here that this additive increasing influence of the pseudoplastic parameter on the magnitude of the velocity for the stretching surface flow may be contrasted with the stagnation point flows of non-Newtonian fluids, e. g., viscoelastic fluid [17] and dilatant fluid [13], in which different order effects involving the non-Newtonian parameter, have been shown not to yield a monotonic behavior.

For the longitudinal velocity function f' also, we have computed the relative percentage variations of first, second and third order effects in relation to the Newtonian flow using similar expressions as for f. The curves of f' are

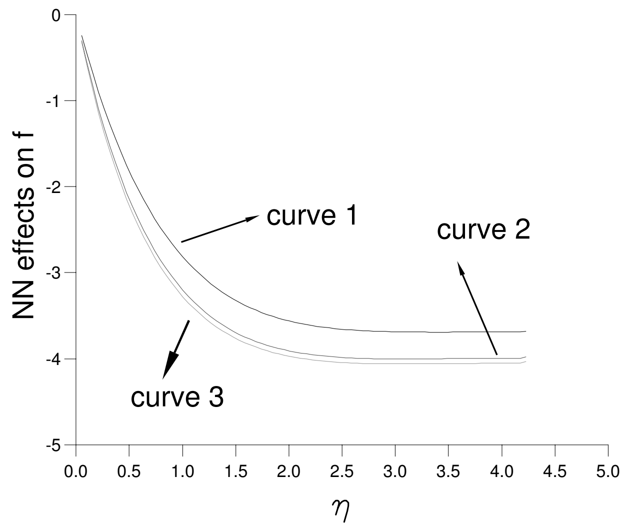

Fig 3. Higher order NN effects on relative percentage changes in f. $(N = 0.6)$

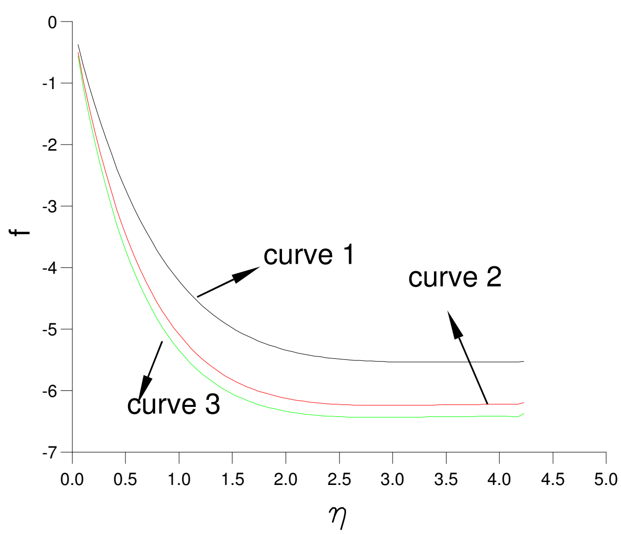

Fig 4. Higher order NN effects on relative percentage changes in f. $(N = 0.9)$

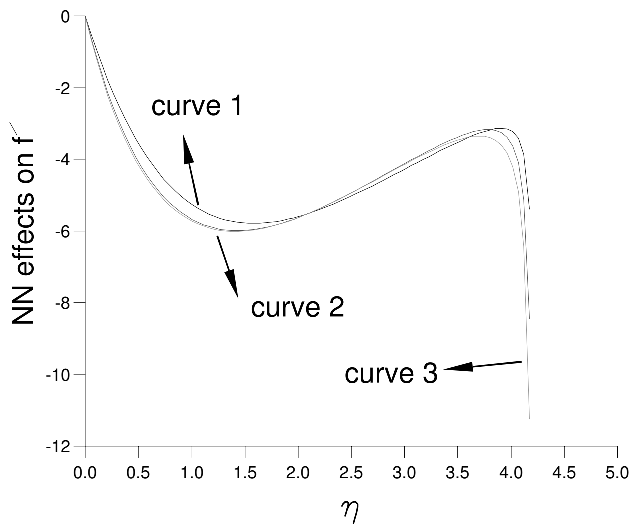

Fig 5. Higher order NN effects on relative percentage changes in f'.
$(N = 0.6)$

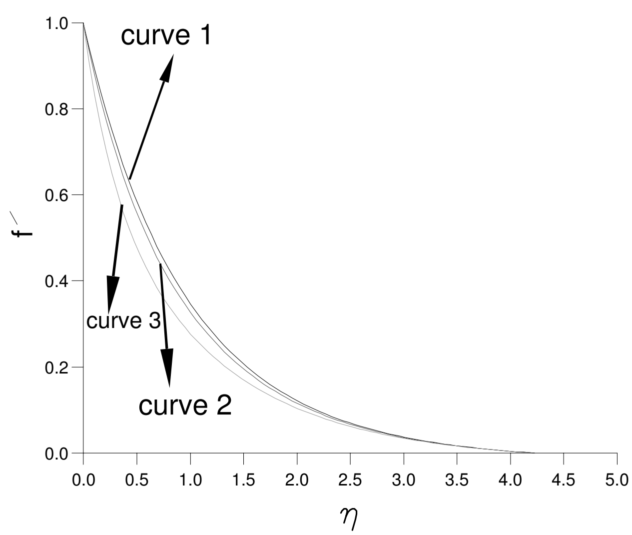

Fig 6. Variation f' with h. $h = 0.5,\ 1.0,\ 2.0$ (Curves 1, 2, 3)

Table 1: Values of f'. $(N = h = 0.5)$

y	S_1	S_2	S_3
0.00	1.0000	1.0000	1.0000
0.26	0.7504	0.7473	0.7466
0.53	0.5660	0.5630	0.5625
0.79	0.4286	0.4264	0.4261
1.00	0.3437	0.3422	0.3421
1.27	0.2610	0.2602	0.2601
1.80	0.1498	0.1497	0.1497
2.06	0.1127	0.1127	0.1127
2.32	0.0840	0.0841	0.0841
2.85	0.0445	0.0446	0.0446
3.65	0.0123	0.0123	0.0123
4.23	0.0000	0.0000	0.0000

Table 2: Values of $|f''(0)|$. $(h = 0.5)$

N	$\lvert(f''(0))_1\rvert$	$\lvert(f''(0))_2\rvert$	$\lvert(f''(0))_3\rvert$
0.0	1.0033	1.0033	1.0033
0.3	1.3043	1.3946	1.4217
0.6	1.6053	1.9664	2.1832
0.9	1.9062	2.7189	3.4503

shown in Fig 5, for $N = 0.6$. We note that for the longitudinal velocity, the effects of different order terms cannot be exhibited as clearly as for f due to oscillatory nature of the variations away from the boundary $\eta = 0$. Hence we have included a table to showcase this effect on f'. In Table 1, the quantities S_1, S_2 and S_3 denote the direct contributions resulting from the inclusion of up to first, second and third order effects, respectively. It is observed that for the longitudinal velocity, retention of terms of third order and above in the perturbation expansion, are insignificant. Nevertheless, the decrement in the longitudinal velocity with increasing N, observed in Fig 2, is yet again illustrated in this table.

In the final figure (Fig 6), we have shown the effect of the spatial parameter h on the velocity function f', keeping the parameter N fixed ($N = 0.5$). The curves labelled 1, 2 and 3 in this figure correspond to the values $h = 0.5$, 1.0 and 2.0, respectively. It can be noted that the effect of increasing the parameter h is monotonic on the longitudinal velocity. This velocity component u decreases at cross-sections away from the bounding surface. Although not included here for brevity, similar behavior has been observed for the transverse velocity v as well, as one would expect on the physical grounds. However, the extent of this effect is not much appreciable.

In the Table 2, the absolute values of $f''(0)$, which are directly related to wall skin friction, are given. The values are for $N = 0$ (Newtonian), $N = 0.3$, 0.6, and 0.9. Also, the columns 2, 3, and 4 in this table correspond to first order, second order and third order effects, respectively. Both non-Newtonian parameter as well as higher order terms tend to increase the magnitude of the skin friction at the stretching sheet. This further corroborates the velocity behavior with respect to non-Newtonian as well as higher order effects, observed earlier.

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