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A Note on the Variational Formulation of PDEs and Solution by Finite Elements

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Abstract

Mathematical modeling of differential equations is, in some cases, complex to develop through an analytical solution. In this article a description of the variational method is presented, which is used for the qualitative study of partial differential equations: existence, uniqueness and regularity of the solution [1]. The analysis of linear second-order elliptic partial differential equations is shown as an illustration. Finally, an application of the weak formulation of the Poisson equation is shown by the finite element method.

Keywords: Variational formulation, weak formulation, finite element, regularity of the solution

1 Introduction

The variational formulation also known as weak formulation allows to find in a fast and simple way the solution to phenomena or problems modeled through PDEs, these when analyzed with the techniques or classical theory of PDE, it is very complex to find a solution that satisfies said equations. For this reason the development of methods that allow to reduce to a large extent the inconveniences that may arise has been motivated. This technique is described as an alternative in which the PDE is written in an integral form, which results in simpler equations using linear algebra methods over a vector space of infinite dimension or functional space [2].

The objective of this article is to present the variational method, which is the most widely used for its versatility, allowing to study various types of equations with slight modifications and difficulties inherent to the problem being analyzed. In addition, it is intended to solve an PDE problem starting from the variational formulation through the finite element method using the *Freefem software*.

2 Variational formulation of PDEs

Given a space of Hilbert $(H, \|\cdot\|)$, a linear and continuous form $f : H \to \mathbb{R}$, a linear form $a : H \times H \to \mathbb{R}$, by variational problem we understand the problem of finding $u \in H$, such that:

$$a\,(u, v) = \langle f, v \rangle \ \forall \ v \in H \tag{1}$$

This element $u \in H$, is called weak solution of Eq.(1). The main theorem that is presented for the variational formulation is known as the *Lax-Milgram Theorem*, which studies and guarantees the existence and uniqueness of the solution of the PDE.

2.1 Lax-Milgram Theorem

The Lax-Milgram Theorem is of great importance for both the Functional Analysis and the partial differential equations. Among its many applications, it guarantees the existence and uniqueness of solutions for elliptical equations, and to a lesser extent, parabolic ones. In combination with the Finite Element Method, we can not only ensure the existence of such a solution, but also give an explicit approximation of it [3].

If the bilinear form $a(\cdot, \cdot)$, is continuous (i.e., there exists $M > 0$ such that $|a(u, v) \leq M \parallel u \parallel \parallel v \parallel, \forall\, u, v \in H)$ and coercive (this means that there exists

$m > 0$ such that $a(v,v) \geq m\| v \|^2, \forall v, \in H$), then the variational problem has a single weak solution, in addition

$$\| u \| \leq \frac{1}{m} \| f \|,$$

where $| \| f \| | = \sup\{|\langle f, v\rangle|,\ v \in H,\ \| v \| \leq 1\ \}$.

Now, considering the following differential equation with its boundary conditions:

$$\begin{cases} D(u) \\ u|_{\Gamma} = \mu_0, \quad \Omega :\to\ \mathbb{R}^n \qquad \Gamma \subset \partial\Omega, \end{cases} \tag{2}$$

where D is a differential operator, u is an unknown function or solution sought and $f$is a known mathematical function (defines the problem).

For the variational or weak formulation, certain reasonable conditions must be assumed about the solution:

$$\left.\begin{array}{r} f\ \in V_f \\ u \in V_u \end{array}\right\} \to (V_f, V_u)$$

Here, the space of functions (V_f, V_u) has a structure of reflective Banach spaces.

$$V_u = V_u'' \wedge\ V_f = V_f''$$

2.2 Reflective Banach Spaces

A Banach space is typically known as a space of functions of infinite dimension. It is a vector space V with a norm $\| \cdot \|$, such that every Cauchy sequence (with respect to the metric $d(x,y) = \| x - y \|$)) in V has a limit in V.

The common hypothesis is that the Banach space to which the unknown function belongs is a subspace of the dual space of V_f $\left(V_u \subset V_f'\right)$, defined these precisions the problem can be formulated as:

$$A_D u = f \in\ V_u', \tag{3}$$

where $A_D : V_u \to\ V_u'$ and $f \in\ V_u'$ is dual space of V_u. When formulated in this way, equations 2 and 3 are essentially equivalent and equally difficult. The weak form of the problem is obtained from the calculation of variations that tells us if u is a solution of the first equation then it is also a solution of the second.

$$[A_D u]V = f(v),\ \forall v \in V_u \tag{4}$$

The v functions are called test functions and the set of all of them generates the Banach space V_u. When the operator AD is linear, then problem 4 can be written using a bilinear form $a(\cdot,\cdot) : V_u \times V_u \to \mathbb{R}^n$ as:

$$a(u,v) := [A_D u]v \tag{5}$$

3 Poisson's equation

The Poisson's equation [4] is a typical example of an PDE with wide use in applications in various areas of engineering. The general form of this equation is presented below:

$$\Delta v = f,$$

where Δ is the Laplace operator. The aim is to analyze the problem of the Poisson's equation by obtaining a solution in an domain $\Omega \subset \mathbb{R}^d$ taking the Dirichlet boundary condition:

$$u(x)\,h = 0, \quad \forall x \in \partial\Omega = \Gamma.$$

For the general case we take Ω

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

$$\int_\Omega fv dx = -\int_\Omega \nabla v v dx = -v\nabla v|_\Omega + \int_\Omega \nabla v \cdot \nabla \nu dx.$$

Therefore, the term $-v\nabla v|_\Omega$ is canceled because we look for $H_0^1(\Omega)$.

$$\int_\Omega fv dx = \int_\Omega \nabla v \cdot \nabla \nu dx.$$

Finally, it can be expressed in a variational way as:

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega fv\ d(x).$$

Making use of the Cauchy-Schwarz inequality we note that

$$\int_\Omega |\nabla u \cdot \nabla v| dx \leq \left(\int_\Omega |\nabla u(x)|\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v(x)|\right)^{\frac{1}{2}} = \| \nabla u \|_{L_2(\Omega)} \| \nabla v \|_{L_2(\Omega)} \leq M \| u \| \| v \| .$$

This indicates that the function is bounded. Now using the *Poincaré-Friedrichs inequality*, it is verified that the function is coercive.

$$\int_{\Omega} |\nabla \upsilon|^2 \, dx = \| \nabla \upsilon \|^2_{L_2(\Omega)} \geq \delta \| \nabla \upsilon \|^2_{H^1_0(\Omega)}.$$

Finally, we have that

$$|F(\upsilon)| = |\int_{\Omega} f \upsilon dx| \leq \| f \|^2_{L_2(\Omega)} \| \upsilon \|^2_{L_2(\Omega)} \leq \| f \|^2_{L_2(\Omega)} \| \upsilon \|^2_{H^1_0(\Omega)}$$

This shows that F is bounded, and with this the conditions of the Lax-Milgram Theorem are met, which ensures that the problem has a solution and is also unique.

4 Numerical result

As mentioned at the beginning of the paper, the main objective was to enunciate the variational formulation of partial differential equations. By having a simpler way to work through linear algebra, it is also easier to use computational techniques to find an approximate solution [5]. Through the use of *freefem software* it is intended to solve an equation in its variational form. Next, the code with which the problem is formulated is presented, which corresponds to the Poisson equation previously analyzed and is solved for the mesh presented in Figure 2.

```
//Construccion de la malla
border a(t=0,5){x=t;y=0;label=1;};
border b(t=0,3){x=5;y=t;label=2;};
border c(t=0,5){x=5-t;y=3;label=3;};
border d(t=0,3){x=0;y=3-t;label=4;};
border e(t=1,2){x=t;y=1;label=5;};
border f(t=1,2){x=2;y=t;label=6;};
border g(t=2,1){x=t;y=2;label=7;};
border h(t=2,1){x=1;y=t;label=8;};
border j(t=3,4){x=t;y=1;label=9;};
border k(t=1,2){x=4;y=t;label=10;};
border l(t=4,3){x=t;y=2;label=11;};
border m(t=2,1){x=3;y=t;label=12;};
int n=5;

mesh th=buildmesh(a(n)+b(n)+c(n)+d(n)+e(-n)+f(-n)+g(-n)+h(-n)+j(-n)+k(-n)+l(-n)+m(-n));
plot(th);

//Define espacio V
fespace Vh(th,P2);
Vh u,v;

//Forma variacional
func F=sin(pi*x)*cos(pi*y);
problem poisson(u,v)=int2d(th)(dx(u)*dx(v)+dy(u)*dy(v)) // Parte bilineal
                    +int2d(th)(-g*v) // Parte lineal
       +on(1,2,3,4,u=0)+on(5,7,8,9,10,11,12,u=20)+on(6,u=20);// Condiciones de frontera

poisson; // Solucion de la ecuacion
plot(u,wait=1,ps="Poisson-escalon.eps");
```

Figure 1: Code implemented in freefem.

In the code described above, the PDE entered in variational form can be seen. Then the mesh in which you want to solve the proposed problem is presented.

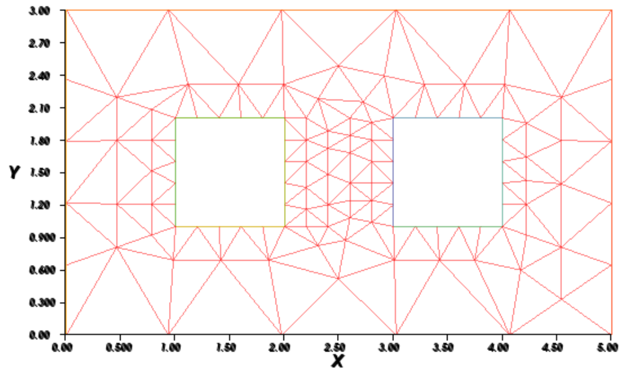

Figure 2: Generated mesh.

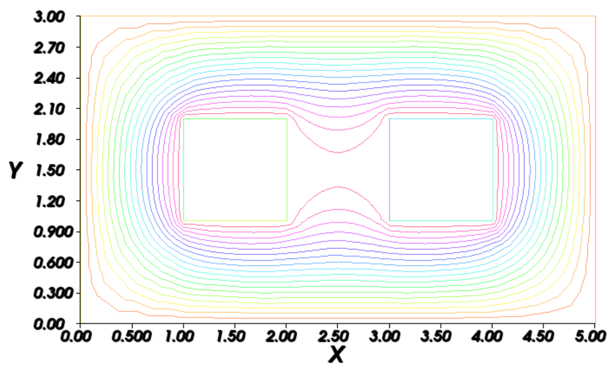

Figure 3: Solution using freefem.

5 Conclusion

The variational method or approach is a very useful tool for the qualitative study of partial differential equations for allowing to study the solutions in a very general environment, and thus overcome the problems presented by classical methods. In addition, its easy adaptability to various situations, exposed in a partial way in the present work, has allowed it to be the predominant technique for the analysis of problems of partial differential equations.

From a variational formulation, the problems that involve PDE can be developed in a much simpler way since one can work with linear algebra methods, such is the case of the finite element method. To solve a problem of this type, freefem software was used, which requires only the equation expressed in a variational form and boundary conditions to solve any problem in a previously defined space and mesh.

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