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Canonical Exterior Formalism for a Production System

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Abstract

There are techniques of the fields theory that can be applied to study the dynamics of a production system described from a Lagrangian, which is a function of the variables $(q, \dot{q})$, coordinates of a given configuration space. These techniques allow a more general and abstract application than those used by other formalisms from physics, thus giving the possibility of studying the dynamics of various models. In addition, they are an effective tool to obtain more detailed information of the model. Today many of these techniques are applied in classical mechanics, quantum mechanics, particle physics, condensed matter, and in statistical physics models. In this paper they will be presented as an alternative geometric formalism for the study of the dynamics of an economic productive system.

Keywords: Field Theories, Canonical Exterior Formalism, Production System

1 Introduction

To model the dynamics of a production system, different theoretical frameworks of physics have been defined and used (M. Estola 2013 and M. Estola and A.A. Dannenberg 2016), [1, 2, 3]. In the work mentioned, the dynamic of a production is modeled from Newton's second law and the application of Lagrangian and Newtonian formalisms. In this paper, it is studied the dynamics of the neo-classical theory (static equilibrium state), and show that this theory corresponds to the particular case of the mentioned formalisms (zero force).

Indeed, since physics the static versions of the models must be a particular case of the dynamic version. On the other hand, the coherence between the Newtonian and Lagrangian formalisms is an absolutely necessary condition, since they are equivalent formalisms and the equations that they define have equal physical meaning when describing the dynamics of a system. The works of M. Estola et al., achieve this clearly. These properties, derived from physics, are critical to the consistency of the application of these formalisms in the description of any model.

It is important to mention that, in order to follow the reasoning and the way of constructing exact arguments used by physics, [4], the meanings of the co-related magnitudes between physics and economics, such as forces, inertial masses, velocity, acceleration (of accumulation of a good), and kinetic and potential energy, have a fundamental weight for the development of consistent ideas. It is possible that this set of concepts, definitions and procedures do not have the same argumentative weight as in physics, and not belonging to the field of pure mathematics, but they may have general application to a given number of different models, that is, to have a certain universe of application.

In economics it is possible to define intuitive dynamic principles, since economic units, in spite of their free will to operate, are constrained by their own will reaching the goals they have set themselves and to be consistent with their objectives. When studying the dynamics of a system, this type of reasoning allows to associate argument and tools from physics.

I. Fisher (2006, original work in 1892) in his doctoral thesis proposes a vector formulation for economics, being the first published work where the correspondences between physics and economy, [5], are explicitly defined.

In the description of the dynamics of a system there are different approaches or formalisms. Newton's second law studt the evolution of a particle in time from the knowledge of the initial conditions and forces acting on it, stating that the trajectory that describes in its movement is that which minimizes its energy. While the Lagrangian formalism (or Hamiltonian formalism) contains all the physical information of the state and the forces acting on the system from the knowledge of the kinetic and potential energies expressed in a function called Lagrangian (polynomial function). The temporal evolution in this case is obtained by means of the integral in time of the Lagrangian, called the function action (S). Action in physics is an abstract concept containing all the dynamic information and interactions of the system. The temporal evolution takes place through a trajectory in which S has an extreme (a minimum).

In this sense, and with a more general vision coming from the fields theory, it would be possible to model the dynamics of any system that can be described from a trajectory in coordinates $(q, \dot{q})$ of a given configuration space [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. By correctly defining the Lagrangian of the model, which, as already mentioned, contains the kinetic or free terms (bi-linear terms in $\dot{q}$) and the interaction terms of the system, through the application of techniques from the fields theory, it is possible to obtain the equations of motion, the Feynman rules and the diagrammatic of the model (schematization of the dynamics of the system). It is important to remark the importance of the correct definition of the concepts of kinetic (Ec) and potential (Ep) energies of the productive economic system in consistency with the conceptualizations from physics, as generators of the dynamics.

M. Estola et al. (2016) agree with I. Fisher (2006) that economic kinematics can be described as the position of an economic quantity according to the movement of a representative point in a coordinate system (economic quantity of coordinates (q, q*) in a configuration space), from:

$$Q_i(t) = Q_0(t) \ + \int \ q_i(s) \ ds \ , \tag{1}$$

M. Estola et al., add to this idea the existence of factors that resist the changes of these economic quantities, modeling these inertia according to the definitions of physics, defining forces and "inertial masses" of economic magnitudes.

Today, the real economies are considered as complex dynamic systems in which they happen and accumulate small random events that select the final result. Time enters naturally here via the processes of adjustment and change: As the elements react, the aggregate changes; as the aggregate changes, elements react anew. One way to approach this is to adopt the perspective of complexity, emphasizing the formation of structures (their dynamics) rather than their existence. At the same time, when the models of prediction adopted are not obvious, and must be formulated individually by the intervening agents, who are also not aware of the expectations of other agents, the study and modeling of their dynamics uses tools from the more specific physics according to each type of system (or model). These properties have counterparts in nonlinear physics where there are similar positive feedbacks, (W.B. Arthur) [16].

Undoubtedly, this generates a complex system whose dynamics is not easy to handle if it is not possible to clearly define the variables $(q, \dot{q})$ of the configuration space where the evolution of each constituent element of the model is described. That is, it is defines properly the elements (or agents) and interactions of the economic model, as is done in physics with particles and the physical system.

This work presents and analyzes the possibility of applying a more general technique, typical of field theory such as external canonical formalism (FCE), to the study of the dynamics of a productive economic system.

The FCE plays an important role, because of its more simple and compact structure. In particular, the FCE can be used as an interesting geometrical formalism to derive and analyze the dynamic.

On the other hand, the FCE does not define a standard mechanical system in the sense that it is not proper Hamiltonian theory as others formalism, which propagates data defined on an initial surface Σ. The first question is that in the construction of the FCE, the "form" brackets are introduced and they must be related to the usual Poisson brackets defined in others formalism. Besides, it must be pointed out that the FCE takes the exterior derivative as a form-observable, which does not have direct analogue. Thus, in the FCE the first class dynamical quantity defined as the Hamiltonian density is not the Hamiltonian which generates the time evolution of generic functionals (fields). However, this can be corrected as will see in practice taking in the FCE some field equations of motion as constraints.

Other motivation to consider this kind of theories is due to the fact that by adding terms highly derived in the Lagrangian, they can be meant for regularizing divergences in the theory. On the other hand, the second order formalism is unavoidable to address the study in other framework.

2 Preliminaries and definitions

In the first order of this geometrical formalism the dynamics is described by the 1-form fields $q^A = (V^a, \omega^{ab})$, where the index $A = (a,\ ab)$. The fields V^a and ω^{ab} play the role of the coordinates of a configuration space. So, V^a represents a given quantity, and ω^{ab} is a field of geometric origin (in gravity, the dreibein and the Lorentz spin connection, respectively). The 2-forms $\dot{q}^A \equiv dq^A$ play the role of velocities. The curvature 2-forms corresponding to the above fields are called $R^A = (R^a, R^{ab})$, and are defined by:

$$R^A = dq^A - \frac{1}{2} C^A{}_{BC}\, q^C \wedge q^B \ , \tag{2}$$

where the graded structure constant C^A_{BC} and the constant symmetric Killing metric γ_{AD} are related by the equation:

$$C_{ABC} = \gamma_{AD} C^D_{BC} \ . \tag{3}$$

The explicit expressions for the curvatures are written:

$$R^a = dV^a + \omega^{ab} \wedge V_b \ , \tag{4}$$

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b \ . \tag{5}$$

3 Lagrangian density $\mathcal{L}$

The action in a three dimensional space (two spacial-one temporal) is defined by means of a Lagrangian density (3-form) given by:

$$\mathcal{L} = R^{ab} \wedge V^c \varepsilon_{abc} + d\omega^{ab} \wedge \omega_{ab} - \frac{2}{3} \omega^{ab} \wedge \omega_b{}^c \wedge \omega_{ca} , \tag{6}$$

where the usual Einstein-Hilbert term play the role of the "mass" term. The other terms are viewed as the "kinetic" terms, in higher derivative (second time derivative). An equivalent Lagrangian density is:

$$\begin{aligned} \mathcal{L} &= dV^a \wedge \omega^{bc} \varepsilon_{abc} + d\omega^{ab} \wedge \omega_{ab} \\ &\quad - \frac{2}{3} \omega^{ab} \wedge \omega_b{}^c \wedge \omega_{ca} + \omega^{ad} \wedge \omega_d{}^b \wedge V^c \varepsilon_{abc} , \end{aligned} \tag{7}$$

that differs of eq. (6) in a total derivative. The canonical momenta (1-forms) π_A conjugate to the 1-forms field variables q^A obtained by the functional variation of the Lagrangian density (7) with respect to the 2-forms velocities $dq^A \equiv \dot{q}^A$ are given by:

$$\pi_A = \frac{\partial \mathcal{L}}{\partial dq^A} . \tag{8}$$

Therefore:

$$\pi_a = \omega^{bc} \varepsilon_{abc} , \tag{9}$$

$$\pi_{ab} = \omega_{ab} . \tag{10}$$

The set of primary constraints can be obtained from the Lagrangian density and they are the relationship between the field and momentum variables not depending on the velocities:

$$\Phi_a = \pi_a - \omega^{bc} \varepsilon_{abc} \approx 0 , \tag{11}$$

$$\Phi_{ab} = \pi_{ab} - \omega_{ab} \approx 0 , \tag{12}$$

where the symbol $\approx$ implies weakly zero.

It is necessary to define a suitable operation involving forms with the help of which the Hamiltonian equation of motion may be written. As it was shown in [17, 18] the Poisson brackets yields more information than the form brackets. They can be related by means of an integral relationship.

Starting from the Lagrangian (7) the canonical Hamiltonian can be defined:

$$\mathcal{H}_{can} = dq^A \wedge \pi_A - \mathcal{L} = \\ -\omega^{ad} \wedge \omega_d{}^b \wedge V^c \varepsilon_{abc} + \frac{2}{3}\omega^{ab} \wedge \omega_b{}^c \wedge \omega_{ca} \, . \tag{13}$$

Therefore, the total Hamiltonian can be defined as follows [19]:

$$\mathcal{H}_T = \mathcal{H}_{can} + \Lambda^A \wedge \Phi_A = \\ -\omega^{ad} \wedge \omega_d{}^b \wedge V^c \varepsilon_{abc} + \frac{2}{3}\omega^{ab} \wedge \omega_b{}^c \wedge \omega_{ca} \\ +\Lambda^a \wedge \left(\pi_a - \omega^{bc} \varepsilon_{abc}\right) + \Lambda^{ab} \wedge (\pi_{ab} - \omega_{ab}) \; , \tag{14}$$

where $\Lambda^A = \left(\Lambda^a, \Lambda^{ab}\right)$ are the Lagrange multipliers.

Now, it is necessary to introduce the fundamental equation of motion in the formalims, in analogy to classical mechanics, as was mentioned in the introduction, the following equation involving the form-bracket is introduced:

$$dA = (A \; , \; \mathcal{H}_T) + \partial A \; , \tag{15}$$

where $A = (q \; , \; \pi)$ is a generic polynomial in the canonical variables q^A and π^A The operator ∂ acts nontrivially on external fields only. Therefore, for the canonical variables:

$$\partial q^A = \partial \pi A = 0 \; , \tag{16}$$

and also for constraints:

$$\partial \Phi_A = 0 \; . \tag{17}$$

Considering the equation (15) we can write the following Hamiltonian equations:

$$dq^A = \left(q^A \; , \; \mathcal{H}_T\right) \; , \tag{18}$$

and taking into account the expression (14) for $\mathcal{H}_T$, by straightforward calculation we find the following general results:

$$\Lambda_A = dq_A \; . \tag{19}$$

It is also necessary to prove whether there are secondary constraints in the theory. For this purpose, we must impose the consistency condition on the primary constraints. We must use (15) for Φ_A and impose the condition:

$$d\Phi_A = (\Phi_A \; , \; \mathcal{H}_T) \; \approx 0 \; , \tag{20}$$

where (17) was used.

Computing explicitly the form-bracket appearing in (20), we arrive to the general equation:

$$d\Phi_A = -\,[equation\ of\ motion] \;+\; \Big(\Phi_A\ ,\ \Lambda^B\Big) \wedge \Lambda_B\ . \tag{21}$$

As $\Big(\Phi_A\ ,\ \Lambda^B\Big)\ \wedge\ \Phi_B$ is a weakly zero trem, (21) implies the lack of secondary constraints in the FCE. Moreover, the equation (21) guarantees that the Hamiltonian defined in (14) is a first class dynamical quantity. On the other hand, by using (20) and after lengthy algebraic manipulations, we find:

$$d\Phi_a = -R^{bc}\,\varepsilon_{abc} + \Big(\Phi_a\ ,\ \Lambda^A\Big) \wedge \Lambda_A\ , \tag{22}$$

$$d\Phi_{ab} = -\,2\,R_{ab} - R^c\,\varepsilon_{abc} + \Big(\Phi_{ab}\ ,\ \Lambda^A\Big) \wedge \Lambda_A\ . \tag{23}$$

These results and properties can be obtained from the FCE in a general form [17, 18].

4 Space-time decomposition

To obtain the proper Hamiltonian $\tilde{\mathcal{H}}$ of the theory, generator of the time evolution of generic functionals, we must consider q^A (1-form gauge fields) written in the holonomic basis $q^A = q^A_\mu\, dx^\mu$, with $\mu = 0, 1, 2$ and $A = a$, ab. Thus, the proper Hamiltonian $\tilde{\mathcal{H}}$ is defined by (see eq. 14):

$$\int \mathcal{H}_{\mathcal{T}} = \int dx^0\ \wedge\ \tilde{\mathcal{H}}\ , \tag{24}$$

where $\tilde{\mathcal{H}}$ can be written:

$$\tilde{\mathcal{H}} = \int q^A_0\, \mathcal{H}_{\mathcal{A}}(x)\, dx^3 = \int \Big(L^a_0\, \mathcal{H}_a(x)\ +\ \frac{1}{2}\,\omega^{ab}_0\, \mathcal{H}_{ab}(x)\Big)\ dx^3\ , \tag{25}$$

where q^A_0 are the temporal components of the 1-form fields q^A. We assume that the primary constraints (11) and (12) in the FCE remain at least weakly zero in the canonical component formalism [17, 18, 19]. On the other hand, in an usual canonical component formalism, the components π^0_A of the momenta vanish and define primary constraints [15]. In the FCE these vanishing components do not even appear among the components of the momenta. The choice of the time variable determines which components vanish. Consequently, when in the FCE the forms are restricted to a $t = x^0 =$ constant surface, the time components of the momenta do not appear. Therefore, we put $\pi^0_A = 0$ in the primary constraints and R^A_{0i} with $i = 1, 2$. Considering the restriction of

(11) and (12) to the surface Σ and according to what was assumed above, the following prescription is made:

$$\Phi_a|_\Sigma = \psi_a \approx 0 \,, \tag{26}$$

$$\Phi_{ab}|_\Sigma = \psi_{ab} \approx 0 \,. \tag{27}$$

Then, the explicit expressions for $\tilde{\mathcal{H}}(x) = \mathcal{H}_a(x)\, d^2x$ are:

$$\mathcal{H}_a(x) = -R^{bc}\,\varepsilon_{abc} - \omega^b_a \wedge \psi_b \approx 0 \,, \tag{28}$$

$$\mathcal{H}_{ab}(x) = -4\,R_{ab} - 2\,R^c\varepsilon_{abc} + (V_b \wedge \psi_a - V_a \wedge \psi_b) \approx 0 \,, \tag{29}$$

where the antisymmetric and weakly zero quantities,

$$M_{ab}\, d^3x \equiv (V_b \wedge \psi_a - V_a \wedge \psi_b) \approx 0 \,, \tag{30}$$

appear directly as primary constraints because they are functionals of the constraints ψ_A, which are the restriction to Σ of the primary constraint Φ_A, and they are the generators of the local Lorentz group of the theory. Note that, in the context of this formalism, the appearance of the generators M_{ab} is absolutely natural. By computing the form brackets between the constraints (28) and (29) we find:

$$\left(\tilde{\mathcal{H}}_A\,,\,\tilde{\mathcal{H}}_B\right) = C^D_{AB}\,\tilde{\mathcal{H}}_D + W_{AB}\,(\mu\,,\,\psi) \,, \tag{31}$$

where W_{AB} are weakly zero 2-forms, functionals of the primary second-class constraints ψ_A. The equation (31) is very important because it shows that the quantities defined in (28) and (29) are first class constraints in the Dirac sense. So, we conclude that the proper Hamiltonian $\tilde{\mathcal{H}}$ defined in (25) can be written as a linear combination of the set $\tilde{\mathcal{H}}_A$ of first class constraints, corresponding to invariances of the theory under local gauge transformations, which are the two degrees of freedom of this theory.

5 Second-order Hamiltonian formalism

The higher derivative character of the theory, that makes it possible regularize mathematical divergences, do not allow go over to the second order formalism directly from the FCE. This is due to the presence of the second time derivatives. We consider all the forms written the dreibein V^a in the holonomic basis as $V^a = L^a_\mu\, dx^\mu$. The indices $\mu, \nu = 0, 1, 2$ for space-time, and the indices $i, j = 1, 2$ to label spatial components only. We define the metric tensor $g_{\mu\nu}$ split in the shift function $N^\perp$ and lapse function N_i.

The Lagrangian density we are able to construct the second-order formalism, which is obtained by considering the following equation of motion.

$$R^a = 0\ . \tag{32}$$

From the equation for this curvature, it can be obtained:

$$\omega_\mu^{ab} = \omega_\mu^{ab}\left(L\right)\ , \tag{33}$$

$$\begin{aligned}\omega_\mu^{ab}(L) &= \frac{1}{2}L^{a\nu}\left(\partial_\mu L_\nu^b - \partial_\nu L_\mu^b\right) - \frac{1}{2}L^{b\nu}\left(\partial_\mu L_\nu^a - \partial_\nu L_\mu^a\right)\\ &\quad -\frac{1}{2}L^{a\rho}L^{b\sigma}\left(\partial_\rho L_{\sigma c} - \partial_\sigma L_{\rho c}\right)L_\mu^c\ .\end{aligned} \tag{34}$$

Therefore, the Lagrangian density only depends on the field V, once the equation is used to eliminate ω_{ab} as independent dynamical variable.

The Lagrangian density contains second times derivatives on the dreibein components and because of the form of the term of the Lorentz-Chern-Simons expression it is not possible to eliminate it by partial integration. Consequently, we are in the presence of a constrained Hamiltonian system with a singular higher-order Lagrangian, in the framework of the Dirac formalism. Therefore, we consider the Ostrogradski transformation to introduce canonical momenta in this higher derivative theory [20, 21, 22, 23, 24]. In those papers, we are going to apply the same ideas to construct the canonical formalism in our case. Moreover, we will work as close as possible to the Dirac conjectures [17, 18]. In this paper we will give only the constructive method. We will not explicitly write all the results because the computations, even proved straightforwardly, involve tedious algebraic manipulations. We start by defining the following independent dynamical field variables: $L_{a\mu} = (L_{a0}\,,\, L_{ai})$, with $L_{ai} = n_a\, N^\perp + N^i$, and $B_{a\mu} = \partial_0\; L_{a\mu}$.

The Ostrogradski transformation respectively introduces the following canonical momenta:

$$\overset{(1)}{\Pi}{}_\mu^d = \frac{\partial\mathcal{L}}{\partial\; \dot{L}_\mu^d} - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\;\partial_\nu\; \dot{L}_\mu^d\;)}\ , \tag{35}$$

$$\overset{(2)}{\Pi}{}_\mu^d = \frac{\partial\mathcal{L}}{\partial(\partial_0\; \dot{L}_\mu^d)}\ . \tag{36}$$

The relationship between field and momentum independent of the velocities gives rise to the following primary constraints:

$$\overset{(1)}{\Phi}{}^0_d = \overset{(1)}{\Pi}{}^0_d - \partial_i \left(\omega^0_{jk}\, L^k_d\right) - \partial_i \left(\frac{1}{N^\perp}\right) N^\perp \omega^0_{jk}\, L^k_d - \partial^k \left(\omega^0_{kj}\, L_{id}\right)$$
$$+ B_{ia}\, \omega^a_{jd} + L_{ib}\, B^b_k \left(L^k_a + \frac{N^k}{N^\perp}\, n_a\right) \omega^a_{jd} \approx 0 \,, \tag{37}$$

$$\overset{(2)}{\Phi}{}^0_d = \overset{(2)}{\Pi}{}^0_d \approx 0 \,, \tag{38}$$

$$\overset{(2)}{\Phi}{}^i_d = \overset{(2)}{\Pi}{}^i_d + N^\perp g^{1/2} \left(\omega^{j0}_k\, \varepsilon^{ik} + \omega^{i0}_k\, \varepsilon^{jk}\right) L_{jd} \approx 0 \,. \tag{39}$$

The remaining momentum which depends on the velocities is $\overset{(1)}{\Pi}{}^i_c$.

By means of these momenta, the canonical Hamiltonian remains defined by:

$$\mathcal{H}_{can} = B^d_\mu\, \overset{(1)}{\Pi}{}^\mu_d + \dot{B}^d_\mu\, \overset{(2)}{\Pi}{}^\mu_d - \mathcal{L} \,, \tag{40}$$

where $\dot{L}^a_\mu$ by B^a_μ was replaced. We note that the canonical Hamiltonian is formed by eliminating only the velocity $\dot{B}^a_\mu$. The field B^a_μ cannot be eliminated from the formalism when we treat with higher derivative Lagrangians. Once the Lagrangian is used and the velocities $\dot{B}^a_\mu$ is eliminated.

Finally, we can write the extended Hamiltonian (first class dynamical quantity):

$$H_T = \int d^2x\, \mathcal{H}_T \,, \tag{41}$$

which is the generator of time evolutions of generic functionals. The Hamiltonian density $\mathcal{H}_T$ remains defined by:

$$\mathcal{H}_T = \mathcal{H}_{can} + \overset{(1)}{\lambda}{}^d_0\, \overset{(1)}{\Phi}{}^0_d + \overset{(2)}{\lambda}{}^d_\mu\, \overset{(2)}{\Phi}{}^\mu_d \,, \tag{42}$$

where $\overset{(1)}{\lambda}{}^d_0$ and $\overset{(2)}{\lambda}{}^d_\mu$ are arbitrary Lagrange multipliers.

Now, we must go on with the Dirac' s algorithm and impose the consistency conditions on the constraints according to:

$$\Omega^{(k)} = \dot{\Omega}^{(k-1)} = \left[\Omega^{(k-1)}\ H_T\right] \approx 0 \,. \tag{43}$$

Hence, for the constraint $\overset{(2)}{\Phi}{}^0_d$ we find the following secondary constraint:

$$\Omega^{(1)} = \dot{\overset{(2)}{\Phi}}{}^0_d = \left[\overset{(2)}{\Phi}{}^0_d, H_T\right]_{PB} = -\overset{(1)}{\Pi}{}^0_d + \partial_i \overset{(2)}{\Pi}{}^i_d \approx 0 \ . \tag{44}$$

From now on, following the Dirac's prescriptions, the procedure can be continued for each one of the constraints.

6 Conclusions

In this work we have applied techniques of field theory to a productive economic system following the approach taken by other authors in this area. The formalism thus obtained corresponds to a constrained Hamiltonian system containing primary and secondary constraints and they are of first and second class. To analyze this system, we have worked as closed as possible to the Dirac prescriptions [19]. The second order formalism is obtained, starting from the Lagrangian density, by considering strongly equal to zero constraint on curvature and by eliminating ω_μ^{ab} as independent dynamical variable. In order to analyze this singular system the Ostrogradsky transformation is considered. The formalism that we present is from the point of view of the broader and more general analytical mathematics. It includes the law of Euler Lagrange, the Lagrangian and Newtonian formalisms studied by others. This formalism, coming from the field theory would give the possibility to study the model dynamics more general and complex.

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