On the Existence and Non Existence of Solitons for a Generalized KP Equation of Higher Order

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Abstract
We shall establish the existence and non existence of solitons (travelling waves of finite energy) for a large number of generalized KP equations, which include models for long nonlinear waves with small amplitude with the rotation effect. Solitons are characterized via a variational approach as critical points of the action functional. Existence of solitons follows by the Concentration-Compactness principle by P.-L. Lions, applied to an appropriated minimization problem. It is also shown that solitons are analytic functions.

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1 Introduction

In this paper we are interested in establishing the existence and non existence results of solitons (travelling waves of finite energy) for a generalized KP
equation of higher order (named hereafter as the gKP equation)

\[
(M_1 u_t - M_2 u_x + (f(u, u_x, u_{xx}))_x)_x - M_3 u_{yy} + \gamma u = 0,
\]

where \( M_i \) is a differential operator of order 4 \((i = 1, 2)\) and \( M_3 \) is a differential operator of order 2 of the form

\[
M_i = \sum_{j=0}^{2} (-1)^j a_{i,j} \partial_x^{2j}, \quad M_3 = \sum_{j=0}^{1} (-1)^j a_j \partial_x^{2j},
\]

where \( a_{i,j} \) are constants and the nonlinear term \( f \) is a homogeneous function of degree \( p + 1 \) in the variable \( u, u_x, \) and \( u_{xx} \) having the form

\[
f(q, r, s) = \partial_q F(q, r) - r \partial_{rq} F(q, r) - s \partial_{rr} F(q, r),
\]

where \( F \) is a homogeneous function of degree \( p + 2 \) of the form

\[
F(q, r) = \sum_{j=1}^{k} F_j(q, r),
\]

such that for \( 1 \leq j \leq k, \)

\[
F_j(\lambda q, r) = \lambda^{p_{1,j}} F_j(q, r), \quad F_j(q, \lambda r) = \lambda^{p_{2,j}} F(q, r),
\]

with \( p + 2 = p_{1,j} + p_{2,j} \) (if \( F_j \) depends only in either \( q \) or \( r \), we assume that \( F_j \) is homogeneous of degree \( p + 2 \)). In particular, we have \( F \) is a homogeneous function of degree \( p + 2 \), since we have that

\[
F(\lambda q, \lambda r) = \lambda^{p+2} F(q, r).
\]

The generalized model (1) is associated with multiple dispersive models related for example with long wave propagation of fluid, models for the deformations of a hyperelastic compressible plate relative to a uniformly pre-stressed state, models for gravity surface waves in a shallow water channel and internal waves in the ocean, capillary surface waves or oblique magneto-acoustic waves in plasma, long internal waves in a rotating fluid propagating in one dominant direction with slow transverse effects, including the Coriolis phenomenon, among others. For instance, if \( M_1 = I_d, M_3 = \alpha I_d, M_2 = -\partial_x^2, \gamma = 0 \) (absence of rotation effect) and \( F(q, r) = \frac{1}{p+2} q^{p+2} \), we obtain the well known generalized KP model that describes for \( p = 1 \) long waves with small-amplitude in a fluid propagating in one dominant direction with slow transverse effects (see [11]):

\[
\left( u_t + u_{xxx} + (u^{p+1})_x \right)_x - \alpha u_{yy} = 0.
\]
On the other hand, for $M_1 = I_d$, $M_2 = \alpha I_d$, $M_2 = \beta \partial^2 x$, and $F(q,r) = \frac{1}{p+q}q^{p+2}$ we obtain a 2D equation that describes for $p = 1$ small-amplitude, long internal waves in a rotating fluid propagating in one dominant direction with slow transverse effects, known as the rotation KP equation (see [9], [10]):

$$
(u_t - \beta u_{xxx} + \left(u^{p+1}\right)_{x})_x - \alpha u_{yy} + \gamma u = 0.
$$

(5)

The parameter $\beta$ determines the type of dispersion; in the case $\beta < 0$ (negative dispersion), the equation models gravity surface waves in a shallow water channel and internal waves in the ocean, while in the case $\beta > 0$ (positive dispersion), it models capillary surface waves or oblique magneto-acoustic waves in plasma. The constant $\gamma$ measures the effects of rotation and is proportional to the Coriolis force (see [9], [10]).

The Ostrovsky equation is included in the equation (5) in the case $\gamma \neq 0$, (nontrivial rotation effects) and $\alpha = 0$

$$
(u_t + u_{xxx} + \left(u^{p+1}\right)_{x})_x + \gamma u = 0.
$$

(6)

Reciprocally, equation (5) may be viewed as modification of the Kadomtsev-Petviashvili equation (4) to accommodate the effects of rotation, on one hand, or an extension of the Ostrovsky equation with allowance for weak transverse effects. To our knowledge, in many of earth’s lakes, sea straits and coastal regions, the transverse scale is not negligible when compared with the Rossby radius (see [15]), indicating that the weak transverse effects may not be ignored.

For $M_2 \equiv 0$, $\gamma = 0$ and $F(q, r) = \frac{1}{2}q^3 + \frac{2}{3}qr^2$, R. M. Chen in [5] derived a model for the deformations of a hyperelastic compressible plate relative to a uniformly pre-stressed state, known as the hyperelastic dispersive equation:

$$
(u_t - u_{xxx} + \delta u_{xxxx} + \left(\frac{3}{2}u^2 - \mu \left(uu_{xx} + \frac{1}{2}(u_x)^2\right)\right)_x)_x - \alpha u_{yy} + \beta u_{xyy} = 0.
$$

(7)

In this model, the scalar $\delta$ describes the stiffness of the plate which is nonnegative. The coefficients $\alpha$ and $\beta$ are material constants that measure weak transverse effects. The material constant $\mu$ occurs as a consequence of the balance between the nonlinear and dispersive effects. Equation (7) generalizes several well-known equations including the BBM equation [1] when $\delta = \mu = \alpha = \beta = 0$, the regularized long-wave KP equation [3] (also referred as KP-BBM equation, see [16]) when $\delta = \mu = \beta = 0$, and the Camassa-Holm (CH) equation [4] when $\delta = \alpha = \beta = 0, \mu = 1$. In contrast to the derivation in [5] of nonlinear dispersive waves in a hyperelastic plate, these particular equations are usually derived as models of water waves. In equation (7), the two spatial dimensions make the analysis very different from the BBM and CH equation. J. Quintero and A. Montes established orbital stability of solitons via the variational approach for the (gKP) equation.
This paper is organized as follows. In section 2 we present some preliminary results related with properties of homogeneous functions and embedding theorems. In section 3 we establish conditions for the nonexistence of solitons. In Section 4, we show the existence of solitons (travelling waves of finite energy in $L^2$ type Sobolev spaces) for the generalized KP equation (1). Using Lions Concentration-Compactness Lemma, we prove that any minimizing sequence converges strongly, after an appropriate translation to a minimizer. In Section 5, we establish that solitons for the generalized KP equation (1) are already smooth and analytic functions.

2 Some Preliminary Results

Let $X$ be the completion of the space $Y = \{ \partial_x \varphi : \varphi \in C_0^\infty(\mathbb{R}^2) \}$, with respect to the the norm $\|u\| = (u, u)_X^{\frac{1}{2}}$, where the scalar product is defined by

$$(u, v)_X = \int_{\mathbb{R}^2} \left( \sum_{j=0}^{2} \partial_x^j u \partial_y^j v + \sum_{j=0}^{1} \partial_x^{j-1} (\partial_y u) \partial_y^{j-1} (\partial_y v) + (\partial_x^{-1} u)(\partial_x^{-1} v) \right) dx dy.$$ 

By a solitary wave solution of the generalized KP equation in $X$, we mean a solution $u$ of (1) of the form $u(x, y, t) = v(x - ct, y)$ of finite energy (derivatives in some $L^2$ type space). It is straightforward to see that $v$ satisfies the differential equation

$$(- (cM_1 + M_2) v + f(v, v_x, v_{xx}) - M_3 \partial_x^{-2} v_{yy} + \gamma \partial_x^{-2} v)_x = 0,$$

(8)

where the operator $\partial_x^{-1}$ is defined for $L^2(\mathbb{R}^2)$ functions with zero mean as

$$\partial_x^{-1} w(x, y) = \int_{-\infty}^{x} w(s, y) ds,$$

or defined via the Fourier symbol of $\partial_x^{-1}$ given by $\mathcal{F}(\partial_x^{-1} v)(\xi) = \frac{\hat{v}(\xi)}{\xi}$, where $\mathcal{F}(v)$ and $\hat{v}$ denote the Fourier transform of $v$.

Some embedding results. From the local version of the embedding theorem for anisotropic Sobolev space (see [2], p. 187), for any compact $\Omega \subset \mathbb{R}^2$ and $2 \leq s \leq 6$, we have that

$$\|v\|_{L^s(\Omega)}^2 \leq C \int_{\Omega} \left( v^2 + (\partial_x v)^2 + (\partial_x^{-1}(\partial_y v))^2 \right) dx dy.$$

(9)

Then it is not hard to prove the following embedding result. For the proof we see details in [8] in the case of travelling waves for the Kadomtsev-Petviashvili equation.
Lemma 2.1. The embedding $X \hookrightarrow L^s(\mathbb{R}^2)$ is continuous for $2 \leq s \leq 6$ and $X \hookrightarrow L^s_{\text{loc}}(\mathbb{R}^2)$ is compact for $2 \leq s < 6$.

Moreover, R. Chen in the work [6] (see Lemma 2.4) showed that

Lemma 2.2. There exists $C > 0$ such that if $v, v_y, v_{xx} \in L^2(\mathbb{R}^2)$, then

$$\|v\|_{L^\infty(\mathbb{R}^2)} \leq C \left( \|v\|_{L^2(\mathbb{R}^2)} + \|v_{xx}\|_{L^2(\mathbb{R}^2)} + \|v_y\|_{L^2(\mathbb{R}^2)} \right).$$

In particular, there exists $C > 0$ such that for all $v \in X$,

$$\|v\|_{L^\infty(\mathbb{R}^2)} \leq C \|v\|_X.$$  \hfill (11)

Also, we have the following result by J. Bona, Y. Liu and M. Tom in [3] (see Lemma 2.1).

Lemma 2.3. Let $2 \leq s \leq 6$. Then there exists $C > 0$ such that if $v, \partial_x^{-1}v_y, v_x \in L^2(\mathbb{R}^2)$, then

$$\|v\|_{L^s(\mathbb{R}^2)} \leq C \left( \|v\|_{L^2(\mathbb{R}^2)}^{(6-s)/2} \left( \|v\|_{L^2(\mathbb{R}^2)} + \|v_x\|_{L^2(\mathbb{R}^2)} \right)^{s-2} \|\partial_x^{-1}v_y\|_{L^2(\mathbb{R}^2)}^{(s-2)/2} \right).$$

In particular, there exists $C > 0$ such that for all $v \in X$,

$$\|v_x\|_{L^s(\mathbb{R}^2)} \leq C \|v\|_X.$$  \hfill (13)

Some Properties for homogeneous functions. Before going further, we establish some basic properties of the homogeneous functions $F = \sum_{j=1}^m F_j$ where $F_j$ satisfies (3).

Lemma 2.4. Let $F = \sum_{j=1}^m F_j$ where $F_j$ is a homogeneous function satisfying properties (3). Then we have that

$$q\partial_q F(q,r) + r\partial_r F(q,r) = (p + 2)F(q,r),$$  \hfill (14)

$$|F(q,r)| \leq M(|q|^{p+2} + |r|^{p+2}).$$  \hfill (15)

Proof. 1. We only perform the proof in the case $m = 1, p_1 = p_{1,1}, p_2 = p_{2,1}$ and $F = F_1$. From the homogeneity with respect to $r$, we have that $F(q,r) = r^{p_2}F(1,1)$, and so $\partial_r F(q,r) = p_2r^{p_2-1}F(q,1)$, which implies that

$$r\partial_r F(q,r) = p_2r^{p_2}F(q,1) = p_2F(q,r).$$

The same argument shows that

$$q\partial_q F(q,r) = p_1F(q,r).$$

So, using that $p + 2 = p_1 + p_2$, we have that

$$q\partial_q F(q,r) + r\partial_r F(q,r) = (p + 2)F(q,r).$$
2. First suppose that \(0 < |q| \leq |r|\). Then using that \(p + 2 = p_1 + p_2\),

\[
|F(q, r)| = |r|^{p_2} |F(q, 1)| = |r|^{p_2} |q|^{p_1} |F(1, 1)| \leq (|q| + |r|)^{p+2} |F(1, 1)|.
\]

The result for \(0 < |r| < |q|\) is quite similar. \(\square\)

**Lemma 2.5.** Let \(F = \sum_{j=1}^{m} F_j\) where \(F_j\) is a homogeneous function satisfying properties (3). Then the functional

\[
K(v) = \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy
\]

is locally Lipschitz in the following sense: There exists \(M > 0\) such that for \(v, u \in X\), we have that

\[
|K(v) - K(u)| \leq M \left[ L_{p+2}(v) + L_{p+2}(u) \right]^{p+1} L_{p+2}(v - u),
\]

where \(L_s\) is defined by

\[
L_s(v) = \|v\|_{L^s(\mathbb{R}^2)} + \|v_x\|_{L^s(\mathbb{R}^2)}.
\]

**Proof.** As above, we only perform the proof in the case \(m = 1, p_1 = p_{1,1}, p_2 = p_{2,1}\) and \(F = F_1\). Since \(F\) is a homogeneous function of degree \(p + 2\), hence \(K\) are homogeneous functions of degree \(p + 1\).

Now, we known from equation (15) in Lemma 2.4 for \(\omega = q, r\), that

\[
|\partial_\omega F(q, r)| \leq M \left( |q|^{p+1} + |r|^{p+1} \right),
\]

which implies that

\[
\|\nabla F(q, r)\| \leq |\partial_q F(q, r)| + |\partial_r F(q, r)| \leq M_2 \left( |q|^{p+1} + |r|^{p+1} \right).
\]

Let \(v, u \in X\) be given. Take \(a = (v, v_x)\) and \(a_0 = (u, u_x)\). Then, using the Generalized Mean Value Theorem and estimate (17), we have that

\[
|F(a) - F(a_0)| \leq \sup_{t \in [0,1]} \|\nabla F(a_0 + t(a - a_0))\| \|a_0 - a\| \\
\leq M_3 \left( |v|^{p+1} + |v_x|^{p+1} + |v_x|^{p+1} + |u_x|^{p+1} \right) (|v - u| + |v_x - u_x|).
\]
From this after using Hölder inequality, we have that
\[ |K(v) - K(u)| \leq \int_{\mathbb{R}} |F\left(v, v_x\right) - F\left(u, u_x\right)| \, dx \, dy \]
\[ \leq M_4 \int_{\mathbb{R}} \left(|v|^{p+1} + |u|^{p+1} + |v_x|^{p+1} + |u_x|^{p+1}\right) \left(|v - u| + |v_x - u_x|\right) \, dx \]
\[ \leq M_5 \left(\left\|v\right\|_{L^{p+2}(\mathbb{R}^2)}^{p+1} + \left\|u\right\|_{L^{p+2}(\mathbb{R}^2)}^{p+1} + \left\|v_x\right\|_{L^{p+2}(\mathbb{R}^2)}^{p+1} + \left\|u_x\right\|_{L^{p+2}(\mathbb{R}^2)}^{p+1}\right) L_{p+2}(u - v) \]
\[ \leq M_5 \left(L_{p+2}(v) + L_{p+2}(u)\right)^{p+1} L_{p+2}(u - v), \]
meaning that \( K \) is locally Lipschitz in the sense described above. 

\[ \square \]

### 3 Non existence of travelling wave solutions

Our goal is to give some conditions to guarantee non existence of solitons (travelling waves of finite energy) for the generalized KP type equation
\[ (- (cM_1 + M_2) v + f(v, v_x, v_{xx}) - M_3 \partial_x^{-2} v_{yy} + \gamma \partial_x^{-2} v)_x = 0, \]
where the nonlinear terms \( f \) is a homogeneous function of degree \( p+1 \) in the variable \( u, u_x, \) and \( u_{xx} \) satisfying the property (2) and \( F \) is a homogeneous functions of degree \( p+2 \) satisfying the property (3). Hereafter, we set the notation \( C_j = ca_{1,j} + a_{2,j} \) for \( j = 0, 1, 2. \)

**Lemma 3.1.** Let \( v \) be a solution of the equation (8). Then we have that
\[ \int_{\mathbb{R}^2} \left(C_0 v^2 + C_1 v_x^2 + C_2 v_{xx}^2 + a_0 (\partial_x^{-1} v_y)^2 + a_1 v_y^2 + \gamma (\partial_x^{-1} v)^2\right) \, dxdy = \left(p + 2\right) \int_{\mathbb{R}^2} F(v, v_x) \, dxdy. \] (18)

**Proof.** if \( v \in X \) satisfies (8) in the distributional sense, then we have that \( v \) satisfies
\[ - (cM_1 + M_2) v + f(v, v_x, v_{xx}) - M_3 \partial_x^{-2} v_{yy} + \gamma \partial_x^{-2} v = 0, \quad \text{in} \ X'. \]

Note that \( \partial_x^{-1} v_y \in L^2(\mathbb{R}^2) \) and \( \partial_x^{-1} v \in L^2(\mathbb{R}^2) \), we have that \( \partial_x^{-2} v_{yy} \in X' \) and \( \partial_x^{-2} v \in X' \), and so we have taking \( X - X' \) duality product that
\[ \langle v, \partial_x^{-2} v \rangle_{X, X'} = - \int_{\mathbb{R}^2} (\partial_x^{-1} v)^2 \, dxdy, \]
\[ \langle v, \partial_x^{-2} v_{yy} \rangle_{X, X'} = \int_{\mathbb{R}^2} (\partial_x^{-1} v_y)^2 \, dxdy. \]
On the other hand, we see that
\[ f(v, v_x, v_{xx}) = \partial_q F(v, v_x) - v_x \partial_{qF} F(v, v_x) - v_{xx} \partial_{rr} F(v, v_x) = \partial_q F(v, v_x) - (\partial_r F(v, v_x))_x. \]

Then from Lemma (2.4) we have that
\[ \int_{\mathbb{R}^2} f(v, v_x, v_{xx}) v \, dx \, dy = \int_{\mathbb{R}^2} (\partial_q F(v, v_x) - (\partial_r F(v, v_x))_x) v \, dx \, dy = \int_{\mathbb{R}^2} (v F(v, v_x) + v_x \partial_r F(v, v_x)) \, dx \, dy = \int_{\mathbb{R}^2} (p_1 F(v, v_x) + p_2 F(v, v_x)) \, dx \, dy = (p + 2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy. \]

Finally, using integration by parts in the remaining terms, we see that
\[ \int_{\mathbb{R}^2} (C_0 v^2 + C_1 v_x^2 + C_2 v_{xx}^2 + a_0 (\partial_x^{-1} v_y)^2 + a_1 v_y^2 + \gamma (\partial_x^{-1} v)^2) \, dx \, dy = (p + 2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy. \] (19)

Lemma 3.2. Let \( v \) be a solution of the equation (1). Then we have that
\[ \int_{\mathbb{R}^2} (C_0 v^2 + 3C_1 v_x^2 + C_2 v_{xx}^2 - a_0 (\partial_x^{-1} v_y)^2 + a_1 v_y^2 - \gamma (\partial_x^{-1} v)^2) \, dx \, dy = 2(p + 1 + p_2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy. \] (20)

Proof. As done by A. de Bouard and J. C. Saut in [8], if we take a \( \chi \in C_0^\infty \) such that \( \chi(t) = 0 \) for \( |t| > 2 \) and \( \chi(t) = 1 \) for \( |t| \leq 1 \), then we have for \( \chi_j(z) = \chi (\frac{|z|}{2^j}) \) with \( z \in \mathbb{R}^2 \) and \( j \in \mathbb{N} \) that as \( j \to \infty \),
\[ \int_{\mathbb{R}^2} (- (cM_1 + M_2) v_x - M_3 \partial_x^{-1} v_{yy} + \gamma \partial_x^{-1} v) \, dx \chi_j(x, y) v \, dx \, dy \to \]
\[ \frac{1}{2} \int_{\mathbb{R}^2} (C_0 v^2 + 3C_1 v_x^2 + C_2 v_{xx}^2 - a_0 (\partial_x^{-1} v_y)^2 + a_1 v_y^2 - \gamma (\partial_x^{-1} v)^2) \, dx \, dy. \] (21)

On the other hand,
\[ \int_{\mathbb{R}^2} \left( f(v, v_x, v_{xx}) \right)_x \, dx \chi_j v \, dz = - \int_{\mathbb{R}^2} (x \chi_j(x, y) v)_x f(v, v_x, v_{xx}) \, dz = - \int_{\mathbb{R}^2} (\chi_j(x, y) v + 2x^2 \chi_j v + x \chi_j(x, y) v_x) f(v, v_x, v_{xx}) \, dx \, dy. \]
Now, we have for \( \bar{v} = (v, v_x, v_{xx}) \) that
\[
\int_{\mathbb{R}^2} \chi_j(x, y) v f(\bar{v}) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} \chi_j(x, y) v (\partial_q F(v, v_x) - (\partial_{\bar{v}} F(v, v_x))_x) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} (\chi_j v \partial_q F(v, v_x) + (\chi_j v_x) \partial_{\bar{v}} F(v, v_x)) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} (\chi_j (v \partial_q F(v, v_x) + v_x \partial_{\bar{v}} F(v, v_x)) + \frac{2x}{j^2} \chi_j v \partial_{v_x} F(v, v_x)) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} ((p + 2) \chi_j F(v, v_x) + \frac{2x}{j^2} \chi_j v \partial_{v_x} F(v, v_x)) \, dx \, dy
\]
\[
\to (p + 2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy, \quad \text{as} \quad j \to \infty.
\]
From similar arguments we obtain that
\[
\int_{\mathbb{R}^2} x \chi_j(x, y) v_x f(\bar{v}) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} (x \chi_j v_x \partial_q F(v, v_x) + (x \chi_j v_x) \partial_{\bar{v}} F(v, v_x)) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} (x \chi_j (v_x \partial_q F(v, v_x) + v_{xx} \partial_{\bar{v}} F(v, v_x)) + (x \chi_j)_{xx} v_x \partial_{v_x} F(v, v_x)) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} \left( x \chi_j (F(v, v_x))_x + \chi_j v_x + \frac{2x^2}{j^2} \chi_j v_{xx} \partial_{v_x} F(v, v_x) \right) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} \left( -x \chi_j F(v, v_x) + \chi_j v_x + \frac{2x^2}{j^2} \chi_j v_{xx} \partial_{v_x} F(v, v_x) \right) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} \chi_j (-F(v, v_x) + v_x \partial_{v_x} F(v, v_x)) \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^2} \frac{2x^2}{j^2} \chi_j (-F(v, v_x) + v_x \partial_{v_x} F(v, v_x)) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} \chi_j (-F(v, v_x) + p_2 F(v, v_x)) \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^2} \frac{2x^2}{j^2} \chi_j (-F(v, v_x) + v_x \partial_{v_x} F(v, v_x)) \, dx \, dy
\]
\[
\to (-1 + p_2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy, \quad \text{as} \quad j \to \infty.
\]
Putting together previous estimates, we have that as \( j \to \infty \),
\[
\int_{\mathbb{R}^2} (f(v, v_x, v_{xx}))_x x \chi_j v \, dz \to -(p + 1 + p_2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy.
\]
From previous estimate and estimate (5), we conclude that
\[
\int_{\mathbb{R}^2} (C_0 v^2 + 3 C_1 v_x^2 + C_2 v_{xx}^2 - a_0 (\partial_x^{-1} v_y)^2 + a_1 v_y^2 - \gamma (\partial_x^{-1} v)^2) \, dx \, dy
\]
\[
= 2(p + 1 + p_2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy.
\]

**Lemma 3.3.** Let \( v \) be a solution of the equation (1). Then we have that
\[
\int_{\mathbb{R}^2} (C_0 v^2 + C_1 v_x^2 + C_2 v_{xx}^2 - a_0 (\partial_x^{-1} v_y)^2 - a_1 v_y^2 + \gamma (\partial_x^{-1} v)^2) \, dx \, dy
\]
\[
= 2 \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy. \quad (22)
\]

**Proof.** Recall that \( v \) satisfies
\[
-(cM_1 + M_2) v_x + (f(v, v_x, v_{xx}))_x - M_3 \partial_x^{-1} v_{yy} + \gamma \partial_x^{-1} v = 0.
\]
Then, multiplying equation (8) by \( y \partial_x^{-1} v_y \) and integrating over \( \mathbb{R}^2 \), we have that for \( j = 0, 1, 2 \)
\[
\int_{\mathbb{R}^2} y \partial_x^{-1} v_y \partial_x^{2j+1} v \, dx \, dy = (-1)^{j+1} \int_{\mathbb{R}^2} y \partial_x^j v_y \partial_x^j v \, dx \, dy = \frac{(-1)^j}{2} \int_{\mathbb{R}^2} \partial_x^j v^2 \, dx \, dy,
\]
\[
\int_{\mathbb{R}^2} y \partial_x^{-1} v_y \partial_x^{-1} v_{yy} \, dx \, dy = -\frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^{-1} v_y)^2 \, dx \, dy,
\]
\[
\int_{\mathbb{R}^2} y \partial_x^{-1} v_y v_{xyy} \, dx \, dy = \frac{1}{2} \int_{\mathbb{R}^2} (v_y)^2 \, dx \, dy,
\]
\[
\int_{\mathbb{R}^2} y \partial_x^{-1} v_y \partial_x^{-1} v \, dx \, dy = -\frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^{-1} v)^2 \, dx \, dy.
\]
On the other hand, from Lemma (2.4) we have that
\[
\int_{\mathbb{R}^2} y \partial_x^{-1} v_y (f(v, v_x, v_{xx}))_x \, dx \, dy = -\int_{\mathbb{R}^2} y v_y f(v, v_x, v_{xx}) \, dx \, dy
\]
\[
= -\int_{\mathbb{R}^2} y v_y (\partial_q F(v, v_x))_x - (\partial_q F(v, v_x))_x \, dx \, dy
\]
\[
= -\int_{\mathbb{R}^2} y v_y (\partial_q F(v, v_x) + v_{xy} \partial_q F(v, v_x)) \, dx \, dy
\]
\[
= -\int_{\mathbb{R}^2} y (F(v, v_x))_y \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy.
\]
Finally, putting together previous computations, we have that

\[
\int_{\mathbb{R}^2} \left( C_0 v^2 + C_1 v_x^2 + C_2 v_{xx}^2 - a_0 (\partial_x^{-1} v_y)^2 - a_1 v_y^2 + \gamma (\partial_x^{-1} v)^2 \right) \, dx \, dy = 2 \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy,
\]

as claimed.

\[\square\]

We also have that

**Theorem 3.1.** Let \(C_j = ca_{1,j} + a_{2,j} > 0\) for \(1 \leq j \leq 3\), and \(\gamma \geq 0\). If either

\[a_0 \leq 0, \quad a_1 \leq 0, \quad p \geq 1, \quad \text{or} \quad a_1 \leq 0, \quad p \geq 4,\]

then equation (1) has non trivial travelling wave solutions.

**Proof.** Multiplying equation (18) by 2 and equation (22) by \(-(p + 2)\), and adding them, we have that

\[
p \int_{\mathbb{R}^2} \left( C_0 v^2 + C_1 v_x^2 + C_2 v_{xx}^2 + \gamma (\partial_x^{-1} v)^2 \right) \, dx \, dy - (p + 4) \int_{\mathbb{R}^2} (a_0 (\partial_x^{-1} v_y)^2 + a_1 v_y^2) \, dx \, dy = 0.
\]

So, if \(a_0 \leq 0\) and \(a_1 \leq 0\), we have that \(v = 0\). Now assume that \(p \geq 4\) and \(a_1 \leq 0\). Then adding (18) and (22), we have that

\[
\int_{\mathbb{R}^2} \left( C_0 v^2 + C_1 v_x^2 + C_2 v_{xx}^2 + \gamma (\partial_x^{-1} v)^2 \right) \, dx \, dy = \frac{p + 4}{2} \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy > 0.
\]

On the other hand, subtracting (20) and (22), we have that

\[
\int_{\mathbb{R}^2} \left( C_1 v_x^2 + a_1 (\partial_y v)^2 - \gamma (\partial_x^{-1} v)^2 \right) \, dx \, dy = (p + p_2) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy.
\]

If we subtract from equation (23) the last equation, we see that

\[
\int_{\mathbb{R}^2} \left( C_0 v^2 + C_2 v_{xx}^2 - a_1 (\partial_y v)^2 + 2\gamma (\partial_x^{-1} v)^2 \right) \, dx \, dy = \left( \frac{4 - p}{2} - p_2 \right) \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy.
\]

So, if we have that \(a_1 \leq 0\) and \(p \geq 4\), we conclude that \(v\) must be trivial.

\[\square\]
4 Existence of travelling wave solutions

Existence of solitons (travelling wave of finite energy) will follow as a consequence of the Concentration-Compactness principle by P. L. Lions in [12]-[13] by considering an appropriate minimization problem with restrictions (see for example among many [7], [11], [14], [18]). As in many cases, solitary wave solutions for the generalized KP model (8) corresponds to critical points for the functional \( J : X \to \mathbb{R} \) defined by

\[
J(v) = I_c(v) - K(v),
\]

where functionals \( I_c \) and \( K \) are defined by

\[
I_c(v) = \frac{1}{2} \int_{\mathbb{R}^2} \left[ \sum_{j=0}^{2} (ca_{1,j} + a_{2,j})(\partial_j^2 v)^2 + \sum_{j=0}^{1} a_j(\partial_j^{-1}(\partial_y v))^2 + \gamma (\partial_x^{-1} v)^2 \right] \, dx \, dy
\]

\[
K(v) = \int_{\mathbb{R}^2} F(v, v_x) \, dx \, dy.
\]

It is straightforward to verify that \( J \in C^2(X, \mathbb{R}) \). For instance, we see directly that

\[
\langle I'_c(v), w \rangle = \int_{\mathbb{R}^2} \left[ \sum_{j=0}^{2} (ca_{1,j} + a_{2,j})\partial_j^2 v \partial_j^2 w + \sum_{j=0}^{1} a_j\partial_j^{-1}(\partial_y v)\partial_j^{-1}(\partial_y w) \right] \, dx \, dy \\
+ \gamma \int_{\mathbb{R}^2} (\partial_x^{-1} v)(\partial_x^{-1} w) \, dx \, dy,
\]

\[
\langle K'(v), w \rangle = \int_{\mathbb{R}^2} (\partial_q F(v, v_x)w + \partial_r F(v, v_x)w_x) \, dx \, dy \\
= \int_{\mathbb{R}^2} (\partial_q F(v, v_x) - (\partial_r F(v, v_x))_x) w \, dx \, dy \\
= \int_{\mathbb{R}^2} f(v, v_x, v_{xx})w \, dx \, dy.
\]

Moreover, we see that

\[
J'(v) = (cM_1 + M_2)v - f(v, v_x, v_{xx}) + M_3\partial_x^{-2} v_{yy} - \gamma \partial_x^{-2} v,
\]

meaning that weak solutions of the travelling wave equation (8) are the critical points of \( J \). Note that by taking \( v = w \), we also have that

\[
I'_c(v)(v) = 2I_c(v)
\]

\[
K'(v)(v) = (p + 2)K(v)
\]

\[
J'(v)(v) = 2I_c(v) - (p + 2)K(v)
= 2J(v) - pK(v).
\]
The first observation is that $I_c(v) \sim \|v\|^2_X$. In fact, if we set

$$C_1 = \frac{1}{2} \min_{0 \leq j \leq 2} \{ca_{1,j} + a_{2,j}, a_0, a_1, \gamma\} \quad \text{and} \quad C_2 = \frac{1}{2} \max_{0 \leq j \leq 2} \{ca_{1,j} + a_{2,j}, a_0, a_1, \gamma\},$$

then we have

**Lemma 4.1.** Assume that $c > 0$, $a_{i,j} > 0$, $a_i > 0$, for $i = 0, 1, j = 0, 1, 2$, and $\gamma > 0$, then $\sqrt{I_c}$ is like a norm in $X$. More exactly, there are constants $C_1, C_2 > 0$ such that for $v \in X$,

$$C_1 \|v\|^2_X \leq I_c(v) \leq C_2 \|v\|^2_X.$$

Now, we use that the functional $\sqrt{I_c}$ is a norm in $X$ to consider the following minimization problem with restrictions

$$\mathcal{I}_c = \inf_{v \in X} \{I_c(v) : K(v) = (-1)^{p+1}\}. \quad (30)$$

The importance of this approach can be observed in the following result,

**Lemma 4.2.** Assume that $c > 0$, $a_{i,j} > 0$, $a_i > 0$, for $i = 0, 1, j = 0, 1, 2$, and $\gamma > 0$ and that $\mathcal{I}_c > 0$. If $v_0 \in X$ is a minimizer for the minimization problem (30), then there exists a nontrivial weak solution $w_0 \in X$ of the travelling wave equation $\mathcal{L}_c = 0$.

**Proof.** Suppose that there is $v_0 \in X$ such that $\mathcal{I}_c = I_c(v_0)$ with $K(v_0) = (-1)^{p+1}$. Using Lagrange Multiplier Theorem, we know that there is $\beta \neq 0$ such that for any $u \in X$

$$I'_c(v_0)(u) = \beta K'(v_0)(u). \quad (31)$$

Now, by previous computations, we have that

$$(cM_1 + M_2)v_0 + M_3 \partial_x^{-2}[(v_0)_{yy}] - \gamma \partial_x^{-2}v_0 - \beta f(v_0, (v_0)_x, (v_0)_{xx}).$$

Now, from (31), we have that

$$2I_c(v_0) = \beta (p + 2) K(v_0).$$

We know that $K(v_0) = (-1)^{p+1}$ and $I_c(v_0) = \mathcal{I}_c$. Then we conclude that

$$\beta = (-1)^{p+1} \frac{2}{p + 2} \mathcal{I}_c.$$

Now, we see that for an appropriated value of $\rho$, $w_0 = \rho v_0$ is critical point for the functional $J$. In fact, using that $f$ is homogeneous of degree $p + 1$, we have that

$$(cM_1 + M_2)w_0 + M_3 \partial_x^{-2}[(w_0)_{yy}] - \gamma \partial_x^{-2}w_0 - \frac{\beta}{\rho^p} f(w_0, (w_0)_x, (w_0)_{xx}) = 0.$$
So, taking $\frac{\beta}{\rho} = 1$, \( \rho = \left( \frac{2}{p+2} \mathcal{I}_c \right)^{1/p} \), we see that \( w_0 \) satisfies the equation

\[
(cM_1 + M_2)w_0 + M_3\partial_x^{-2}[(w_0)_{yy}] - \gamma \partial_x^{-2}w_0 - f(w_0, (w_0)_x, (w_0)_{xx}) = 0,
\]
as desired. In other words, there exists a nontrivial weak solution \( w_0 \in X \) of the travelling wave equation (8).

In order to prove the existence of a travelling wave solution for the gKP equation via the variational approach, we will use Concentration-Compactness Principle applied to the measure with density \( \rho \) of the form

\[
\rho(\psi) = \sum_{j=0}^{2} (ca_{1,j} + a_{2,j})(\partial_x^j \psi)^2 + \sum_{j=0}^{1} a_j(\partial_x^{j-1}(\partial_y \psi))^2 + \gamma(\partial_x^{-1} \psi)^2.
\]

In other words, if we have a minimizing sequence \( (\psi_n)_n \subset X \) with \( K(\psi_n) = (-1)^{p+1} \) and \( I_c(\psi_n) \to \mathcal{I}_c \), we consider the sequence of measures \( (\nu_n)_n \) given by

\[
\nu_n(A) = \int_A \rho(\psi_n) \, dx dy.
\]

By Lions’ Concentration-compactness principle, there exist a subsequence of \( (\nu_n)_n \) (denoted the same) such that one of the following three condition holds:

1. **Compactness**: There exists a sequence \( (x_n, y_n)_n \subset \mathbb{R}^2 \) such that for any \( \epsilon > 0 \) there exists a radius \( R > 0 \) such that for all \( n \)

\[
\int_{B_R(x_n, y_n)} \, d\nu_n \geq \mathcal{I}_c - \epsilon.
\]

2. **Vanishing**: For all \( R > 0 \) there holds:

\[
\lim_{n \to \infty} \left( \sup_{(x, y) \in \mathbb{R}^2} \int_{B_R(x, y)} d\nu_n \right) = 0.
\]

3. **Dichotomy**: There exists \( \theta \in (0, \mathcal{I}_c) \) such that

\[
\lim_{t \to \infty} Q(t) = \theta,
\]

where \( Q \) is defined as

\[
Q(t) = \lim_{n \to \infty} \sup_{(x, y) \in \mathbb{R}^2} \int_{B_t(x, y)} \rho(\psi_n) \, dz dw
\]

Before going further, we will rule out vanishing.
Lemma 4.3. For $1 \leq p < 4$, Vanishing is impossible.

Proof. Assume that vanishing is true. Thus, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have that

$$\sup_{(x,y) \in \mathbb{R}^2} \int_{B_1(x,y)} \rho(\psi_n) \, dx \, dy < \epsilon.$$  \hfill (35)

For $j = 0, 1, 2 \leq q < 6$ and $\psi \in X$, we have from the embedding of $X$ in $L^q(\mathbb{R}^2)$ (see Lemma (2.1)) that,

$$\int_{B_1(x,y)} |\partial^j_x \psi|^q \, dx \, dy \leq \left( \int_{B_1(x,y)} \left[ \sum_{j=0}^2 (ca_{1,j} + a_{2,j})(\partial^j_x \psi)^2 + \sum_{j=0}^1 a_j(\partial_x^{-1}(\partial_y \psi))^2 + \gamma(\partial_x^{-1}\psi)^2 \right] \, dx \, dy \right)^{\frac{q}{2}}.$$

which implies that

$$\int_{B_1(x,y)} |\partial^j_x \psi|^q \, dx \, dy \leq \left( \Gamma(\psi) \right)^{\frac{q-2}{2}} \left( \int_{B_1(x,y)} \left[ \sum_{j=0}^2 (ca_{1,j} + a_{2,j})(\partial^j_x \psi)^2 + \sum_{j=0}^1 a_j(\partial_x^{-1}(\partial_y \psi))^2 + \gamma(\partial_x^{-1}\psi)^2 \right] \, dx \, dy \right)^{\frac{q}{2}},$$

where $\Gamma(\psi)$ is defined by

$$\Gamma(\psi) = \sup_{(x,y) \in \mathbb{R}^2} \int_{B_1(x,y)} \left( \sum_{j=0}^2 (ca_{1,j} + a_{2,j})(\partial^j_x \psi)^2 + \sum_{j=0}^1 a_j(\partial_x^{-1}(\partial_y \psi))^2 + \gamma(\partial_x^{-1}\psi)^2 \right) \, dx \, dy.$$

If we cover $\mathbb{R}^2$ with balls of radius 1 such that any point of $\mathbb{R}^2$ is contained in at most 3 balls, then we have that

$$\int_{\mathbb{R}^2} |\partial^j_x \psi|^q \, dx \, dy \leq 3\Gamma(\psi)\|\psi\|_X^2.$$

From this fact and the hypothesis, we conclude that $\psi_n$ and $\partial_x \psi_n$ tend to zero in $L^{p+2}(\mathbb{R}^2)$ for $1 \leq p < 4$, meaning from the estimate (15) that

$$1 = |K(\psi_n)| \leq \int_{\mathbb{R}^2} |F(\psi_n, \partial_x \psi_n)| \, dx \, dy \leq [L_{p+2}(\psi_n)]^{p+2} \rightarrow 0, \ n \rightarrow \infty,$$

which gives us a contradiction as $n \rightarrow \infty.$  \qed
We will see that if we assume that the Dichotomy holds, then we reach a contradiction provided that it leads to the splitting of the sequence \((\psi_n)_n\) in two sequences \((\psi_{n,1})_n\) and \((\psi_{n,2})_n\) having disjoint supports. The contradiction is reached by using a standard sub-additivity argument, after proving that the sequence of measures \((\nu_n)_n\) associated with a minimizer sequence \((\psi_n)_n\) splits appropriately into two sequences of measures \((\nu_{n,1})_n\) and \((\nu_{n,2})_n\) having disjoint supports.

The first remark is that given \(\epsilon > 0\), there are \(R_0 > 0\) and \(R_n > 0\) with \(R_n \nearrow \infty\), and \((x_n, y_n) \in \mathbb{R}^2\) such that

\[
\theta - \epsilon < \int_{B_{R_0}(x_n, y_n)} \rho(\psi_n) \, dx \, dy < \theta, \quad Q_n(2R_n) \leq \theta + \epsilon,
\]

for \(n\) large enough, where \(Q_n\) is defined as

\[
Q_n(t) = \sup_{(x_0, y_0) \in \mathbb{R}^2} \int_{B_t(x_0, y_0)} \rho(\psi_n) \, dx \, dy.
\]

Then we conclude for \(n\) large enough that

\[
\int_{A(R_0, 2R_n, (x_n, y_n))} \rho(\psi_n) \, dx \, dy \leq 2\epsilon,
\]

where \(A(R, R', z_0) = \{(z \in \mathbb{R}^2 : R \leq ||z - z_0|| \leq R'\}\). We use \(A(R, 2R, z_0) = A(R, z_0)\).

Due to the nature of this problem, we use the fact that there is \(\varphi_n\) such that \(\psi_n = \partial_z \varphi_n\) to build \(\psi_{n,i} \in X\). So, we proceed to split the \(\varphi_n\), but to prove this splitting property we use the following Sobolev inequality.

**Proposition 4.1.** (A Sobolev inequality) There exists a positive constant \(C_1\) such that for all \(u \in C^\infty(\mathbb{R}^2)\) and \(q \geq 2\)

\[
\left( \int_{A(z_0, R)} |u - a_{R, z_0}(u)|^q \, dx \, dy \right)^{\frac{1}{q}} \leq C_1 R^{\frac{1}{q} + \frac{1}{2}} \left( \int_{A(z_0, R)} |\nabla u|^2 \, dx \, dy \right)^{\frac{1}{2}},
\]

where \(a(R, z_0)\) is the average of \(u\) on the annulus \(A(R, z_0)\) of radius \(R\) and \(2R\),

\[
a_{R, z_0}(u) = \frac{1}{3\pi R^2} \left( \int_{A(R, z_0)} u(x) \, dx \, dy \right).
\]

Now, we want to have a split for \(\psi_n\) in two functions \(\psi_{n,1}\) and \(\psi_{n,2}\) where \(\psi_{n,1}\) is supported in \(B_{2R_1}(x_n, y_n)\) and \(\psi_{n,2}\) is supported in \(\mathbb{R}^2 \setminus B_{R_n}(x_n, y_n)\). To do this, we consider \(\zeta, \eta \in C^\infty_0(\mathbb{R}^2, \mathbb{R}^+)\) such that \(0 \leq \zeta, \eta \leq 1\), \(\text{supp}(\eta) \subset B_2(0, 0)\),
\( \zeta \equiv 1 \) in \( B_1(0,0) \), \( \text{supp}(\eta) \subset \mathbb{R}^2 \setminus B_1(0,0) \) and \( \eta \equiv 1 \) in \( \mathbb{R}^2 \setminus B_2(0,0) \). So, we define \( \zeta_n \) and \( \eta_n \) by

\[
\zeta_n(x,y) = \zeta \left( \frac{x-x_n}{R_1}, \frac{y-y_n}{R_1} \right), \quad \eta_n(x,y) = \eta \left( \frac{x-x_n}{R_n}, \frac{y-y_n}{R_n} \right).
\]

Now, we know that there exists \( \varphi_n \) such that \( \psi_n = \partial_x \varphi_n \). Then we consider

\[
\psi_{n,1} = \partial_x((\varphi_n - a_n)\zeta_n) \quad \text{and} \quad \psi_{n,2} = \partial_x(\varphi_n \eta_n + b_n(1 - \eta_n)),
\]

where \( a_n \) and \( b_n \) are given by

\[
a_n = \frac{1}{3\pi R_1^2} \int_{A(R_1,(x_n,y_n))} \varphi_n(x,y)dydx, \quad b_n = \frac{1}{3\pi R_n^2} \int_{A(R_n,(x_n,y_n))} \varphi_n(x,y)dydx
\]

We also set \( \phi_n = \partial_x^{-1}\partial_y \psi_n \) and

\[
\phi_{n,1} = \partial_x^{-1}\partial_y \psi_{n,1} = \partial_y((\varphi_n - a_n)\zeta_n), \quad \phi_{n,2} = \partial_x^{-1}\partial_y \psi_{n,2} = \partial_y((\varphi_n - b_n)\eta_n),
\]

Before we go forward, from Hölder inequality and estimate (38) we have that

\[
|a_n| \leq \frac{1}{3\pi R_1^2} \int_{A(R_1,(x_n,y_n))} |\varphi_n(x,y)|dydx
\]

\[
\leq \frac{1}{\sqrt{3\pi} R_1} \left( \int_{A(R_1,(x_n,y_n))} |\varphi_n|^2 dydx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{\sqrt{3\pi} R_1} \left( \int_{A(R_1,2R_n,(x_n,y_n))} \rho_n(\psi_n) dydx \right)^{\frac{1}{2}}
\]

\[
\leq C\sqrt{\epsilon}.
\]

Moreover, by the same type of arguments, we have that

\[
|b_n| \leq \frac{C\sqrt{\epsilon}}{R_n}.
\]

Now, we will prove

**Lemma 4.4.** (Splitting of a minimizing sequence). Let \( \psi_{n,1} \) and \( \psi_{n,2} \) be as above. Then for \( \epsilon > 0 \), there exists \( \delta(\epsilon) \) with \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \), such that for \( n \) large enough we have that:

\[
\|\psi_n - \psi_{n,1} - \psi_{n,2}\|_X \leq \delta(\epsilon)
\]

\[
\|\|\psi_{n,1}\|_X - \theta\| \leq \delta(\epsilon)
\]

\[
|I_c(\psi_n) - I_c(\psi_{n,1}) - I_c(\psi_{n,2})| \leq \delta(\epsilon)
\]

\[
|K(\psi_n) - K(\psi_{n,1}) - K(\psi_{n,2})| \leq \delta(\epsilon),
\]

and we also have that

\[
\lim_{n \to \infty} \text{dist} (\text{supp } \psi_{n,1}, \text{supp } \psi_{n,2}) = \infty.
\]
Proof. First note for $-1 \leq j \leq 2$ and $0 \leq k \leq 1$ that
\[
\|\partial_x^j \psi_n - \partial_x^j \psi_{n,1} - \partial_x^j \psi_{n,2}\|_{L^2} = \|\partial_x^{j+1} \varphi_n - \partial_x^{j+1} \varphi_{n,1} - \partial_x^{j+1} \varphi_{n,2}\|_{L^2}
\]
\[
\|\partial_x^{k-1} \partial_y \psi_n - \partial_x^{k-1} \partial_y \psi_{n,1} - \partial_x^{k-1} \partial_y \psi_{n,2}\|_{L^2} = \|\partial_x^k \partial_y \varphi_n - \partial_x^k \partial_y \varphi_{n,1} - \partial_x^k \partial_y \varphi_{n,2}\|_{L^2}.
\]
Now, for $j = -1$ we have that
\[
\int_{\mathbb{R}^2} (\varphi_n - \varphi_{n,1} - \varphi_{n,2})^2 \, dx
dy
= \int_{\mathbb{R}^2} (\varphi_n (1 - \zeta_n - \eta_n) + a_n \zeta_n + b_n (1 - \eta_n))^2 \, dx
dy
\leq 3 \int_{\mathbb{R}^2} \varphi_n^2 (1 - \zeta_n - \eta_n)^2 \, dx
dy + 3a_n^2 \int_{B_{2R_1}(x_n,y_n)} \zeta_n^2 \, dx
dy + 3b_n^2 \int_{B_{2R_n}(x_n,y_n)} (1 - \eta_n)^2 \, dx
dy.
\]
Now, we note from estimate (38), (39), and (40) that
\[
\int_{\mathbb{R}^2} \varphi_n^2 (1 - \zeta_n - \eta_n)^2 \, dx \leq \int_{A(R_1,2R_n,(x_n,y_n))} \rho(\psi_n) \, dx \leq C \epsilon,
\]
\[
a_n^2 \int_{B_{2R_1}(x_n,y_n)} \zeta_n^2 \, dx \leq C \epsilon,
\]
\[
b_n^2 \int_{B_{2R_n}(x_n,y_n)} (1 - \eta_n)^2 \, dx \leq CR_n^2 b_n^2 \leq C \epsilon.
\]
So, we conclude from previous estimates that
\[
\|\partial_x^{-1} \psi_n - \partial_x^{-1} \psi_{n,1} - \partial_x^{-1} \psi_{n,2}\|_{L^2} \leq C \sqrt{\epsilon}.
\]
Using the same arguments as those used by J. C. Saut et al. in [8] for the KP equation, we have for $j = 0, 1$ that
\[
\|\partial_x^j \psi_n - \partial_x^j \psi_{n,1} - \partial_x^j \psi_{n,2}\|_{L^2} + \|\partial_x^{-1} \partial_y \psi_n - \partial_x^{-1} \partial_y \psi_{n,1} - \partial_x^{-1} \partial_y \psi_{n,2}\|_{L^2} \leq C \sqrt{\epsilon}.
\]
Now, we have that
\[
\|\partial_x^2 \psi_n - \partial_x^2 \psi_{n,1} - \partial_x^2 \psi_{n,2}\|_{L^2} \leq \|(1 - \zeta_n - \eta_n) \partial_x^2 \psi_n\|_{L^2} + \|\varphi_n - a_n\|_{L^2} \|\partial_x \zeta_n\|_{L^2} + \|\varphi_n - b_n\|_{L^2} \|\partial_x \eta_n\|_{L^2} + 3\|\varphi_n \partial_x^2 \zeta_n\|_{L^2} + 3\|\varphi_n \partial_x^2 \eta_n\|_{L^2} + 3\|\partial_x \psi_n \partial_x \zeta_n\|_{L^2} + 3\|\partial_x \psi_n \partial_x \eta_n\|_{L^2}.
\]
The three first terms in the right hand side of the above inequality are bounded as the preceding ones. We will illustrate how to handle the four last terms by doing a specific estimate:
\[
\|\partial_x \psi_n \partial_x \zeta_n\|_{L^2} \leq \|\partial_x \zeta_n\|_{L^\infty} \left(\int_{A(R_1,(x_n,y_n))} \|\partial_x \psi_n\|^2 \, dx \, dy\right)^{\frac{1}{2}} \leq C \sqrt{\epsilon}.
\]
Putting these computations together, we conclude
\[ ||\psi_n - \psi_{n,1} - \psi_{n,2}||_X \leq \delta(\epsilon), \]
as desired. On the other hand, similar arguments show that
\[ |I_c(\psi_n) - I_c(\psi_{n,1}) - I_c(\psi_{n,2})| \leq \delta(\epsilon). \]

Now, we will establish the four part of the Lemma. We must recall that
\[ K(u) = \int_{\mathbb{R}^2} F(u, u_x) \, dx \, dy, \]
where \( F \) is a homogeneous function of degree \( p + 2 \) satisfying (3),
\[ F(\lambda q, r) = \lambda^p F(q, r), \quad F(q, \lambda r) = \lambda^p F(q, r), \]
(we consider \( m = 1, p_1 = p_{1,1}, p_2 = p_{2,1} \) and \( F = F_1 \)). Thus, we have that
\[ K(\psi_n) - K(\psi_{n,1}) - K(\psi_{n,2}) = C \int_{\mathbb{R}^2} [\psi_n^{p_1}(\partial_x \psi_n)^{p_2} - \psi_{n,1}^{p_1}(\partial_x \psi_{n,1})^{p_2} - \psi_{n,2}^{p_1}(\partial_x \psi_{n,2})^{p_2}] \, dx \, dy. \]

From the definition of \( \psi_n, i \), we have that
\[ \psi_{n,1} = \zeta_n \partial_x \varphi_n + (\varphi_n - a_n) \partial_x \zeta_n, \quad \psi_{n,2} = \eta_n \partial_x \varphi_n + (\varphi_n - b_n) \partial_x \eta_n. \]

We see directly from these estimates that
\[
\begin{align*}
\psi_n^{p_1}(\partial_x \psi_n)^{p_2} &- \psi_{n,1}^{p_1}(\partial_x \psi_{n,1})^{p_2} - \psi_{n,2}^{p_1}(\partial_x \psi_{n,2})^{p_2} \\
&= (\partial_x \varphi_n)^{p_1} (\partial_x \varphi_n)^{p_2} (1 - \zeta_n^{p+2} - \eta_n^{p+2}) \\
&+ (\partial_x \varphi_n)^{p_1} (\partial_x \varphi_n)^{p_2} (\zeta_n^{p+2} + \eta_n^{p+2}) - \psi_{n,1}^{p_1}(\partial_x \psi_{n,1})^{p_2} - \psi_{n,2}^{p_1}(\partial_x \psi_{n,2})^{p_2}.
\end{align*}
\]

First note that the first term in the right has support on \( A(R_1, 2R_n, (x_n, y_n)) \), and so it can be bound directly. For the second term, we have that
\[
\Gamma_1(n) = (\partial_x \varphi_n)^{p_1} (\partial_x \varphi_n)^{p_2} \zeta_n^{p+2} - \psi_{n,1}^{p_1}(\partial_x \psi_{n,1})^{p_2} = (\zeta_n \partial_x \varphi_n)^{p_1} (\zeta_n \partial_x \varphi_n)^{p_2} \\
- (\zeta_n \partial_x \varphi_n + (\varphi_n - a_n) \partial_x \zeta_n)^{p_1}(\zeta_n \partial_x \varphi_n + 2\partial_x \varphi_n \partial_x \zeta_n + (\varphi_n - a_n) \partial_x \zeta_n)^{p_2}.
\]

Now, for real numbers \( a, b, c, d \), we consider the function \( h : \mathbb{R} \to \mathbb{R} \) defined as
\[ h(t) = (a + tb)^{p_1} (c + td)^{p_2}. \]

Then, from the mean value Theorem, we have that
\[ ||(a + b)^{p_1} (c + d)^{p_2} - a^{p_1} c^{p_2}|| \leq C(|a|^{p_1} |c|^{p_2} + |a|^{p_1} |d|^{p_2} + |b|^{p_1} |c|^{p_2} + |b|^{p_1} |d|^{p_2}). \]
Then, using this estimate we see that

\[ |\Gamma_1(n)| \leq C \left( |\zeta_n \partial_x \varphi_n|^{p_1} |\zeta_n \partial_x^2 \varphi_n|^{p_2} + |\zeta_n \partial_x \varphi_n|^{p_1} |2 \partial_x \varphi_n \partial_x \zeta_n + (\varphi_n - a_n) \partial_x^2 \zeta_n|^{p_2} + |(\varphi_n - a_n) \partial_x \zeta_n|^{p_1} |\zeta_n \partial_x \varphi_n|^{p_2} + |(\varphi_n - a_n) \partial_x \zeta_n|^{p_1} |2 \partial_x \varphi_n \partial_x \zeta_n + (\varphi_n - a_n) \partial_x^2 \zeta_n|^{p_2} \right) \]

We also have a similar estimate for

\[ \Gamma_2(n) = (\partial_x \varphi_n)^{p_1} \left( \partial_x^2 \varphi_n \right)^{p_2} \eta_n^{p_1} - \psi_n^{p_1} (\partial_x \psi_n)^{p_2}. \]

Using that the support of the functions on the right hand side of \( \Gamma_i(n) \) are

\[ \text{centered in the annulus } A(R_1, 2R_n, (x_n, y_n)), \text{ that } \psi_n = \partial_x \varphi_n \in H^1(\mathbb{R}^2) \leftrightarrow L^q(\mathbb{R}^2) \text{ for } q \geq 1, \text{ and the Poincare inequality, we are able to use the same type of estimates to conclude that} \]

\[ |\Gamma_i(n)| \leq C \delta(\epsilon). \]

So, putting these estimates together, we are able to conclude that

\[ |K(\psi_n) - K(\psi_{n,1}) - K(\psi_{n,2})| \leq \delta(\epsilon). \]

\[
\square
\]

**Lemma 4.5.** *Dichotomy is not possible.*

**Proof.** Let \( \lambda_{n,1}(\epsilon) = K(\psi_{n,1}), \lambda_{n,2}(\epsilon) = K(\psi_{n,2}) \), and set

\[ \lambda_1(\epsilon) = \lim_{n \to \infty} \lambda_{n,1}(\epsilon) \text{ and } \lambda_2(\epsilon) = \lim_{n \to \infty} \lambda_{n,2}(\epsilon). \]

Assume that \( \lambda_1(\epsilon) = 0 \) and \( \lambda_2(\epsilon) = 1 \), then we have for \( n \) sufficiently large that \( \lambda_{n,2}(\epsilon) > 0 \). We define, \( w_n = (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} \psi_{n,2} \). Then \( w_n \in X \) and \( K(w_n) = 1 \). Thus, we have

\[
I_c(\psi_{n,2}) = (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} I_c(w_n) \geq (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} I_c.
\]

From Lemma (4.4) and previous inequality (45), we have that

\[
I_c(\psi_n) = I_c(\psi_{n,1}) + (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} I_c(w_{n,2}) + \delta_n
\]

\[
\geq \int_{A(R_n, (x_n, y_n))} d\nu_{n,1} + (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} I_c + \delta_n
\]

\[
\geq \int_{\mathbb{R}^2} d\nu_{n,1} + (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} I_c + \delta_n.
\]

But, we have that as \( n \to \infty \),

\[ I_c(\psi_n) \to I_c, \quad \int_{\mathbb{R}^2} d\nu_{n,1} \to \theta, \quad (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} I_c \to I_c. \]
Hence $\mathcal{I}_c \geq \frac{1}{2} \theta + \mathcal{I}_c$, but this is a contradiction. The last argument is analogous, if we suppose that $\lim_{n \to \infty} \lambda_{n,2}(\epsilon) = 0$ and $\lim_{n \to \infty} \lambda_{n,1}(\epsilon) = 1$. Now we may assume $\lim_{n \to \infty} \lambda_{n,i}(\epsilon) > 0$, $i = 1, 2$. Then for $n$ sufficiently large, $\lambda_{n,i}(\epsilon) > 0$, as in previous case, we define the functions $w_{n,1}$ and $w_{n,2}$ by

$$w_{n,1} = (\lambda_{n,1}(\epsilon))^{-\frac{1}{p+2}} \psi_{n,1}, \quad w_{n,2} = (\lambda_{n,2}(\epsilon))^{-\frac{1}{p+2}} \psi_{n,2}.$$ 

Then $w_{i,2} \in X$ and $K(w_{i,2}) = 1$. Thus,

$$I_c(\psi_n) = (\lambda_{n,1}(\epsilon))^{-\frac{2}{p+2}} I_c(w_{n,1}) + (\lambda_{n,2}(\epsilon))^{-\frac{2}{p+2}} I_c(w_{n,2}) \geq I_c\left((\lambda_{n,1}(\epsilon))^{-\frac{2}{p+2}} + (\lambda_{n,2}(\epsilon))^{-\frac{2}{p+2}}\right) + o(1).$$

Taking the limit as $n \to \infty$, we arrive to

$$1 \geq (\lambda_1(\epsilon))^{\frac{2}{p+2}} + (\lambda_2(\epsilon))^{\frac{2}{p+2}}.$$ 

But, the function $g(t) = t^{\frac{2}{p+2}}$ for $s, t > 0$ is such that

$$t^{\frac{2}{p+2}} + s^{\frac{2}{p+2}} > (t + s)^{\frac{2}{p+2}}.$$ 

Thus, we again get a contradiction since

$$1 \geq (\lambda_1(\epsilon))^{\frac{2}{p+2}} + (\lambda_2(\epsilon))^{\frac{2}{p+2}} > (\lambda_1 + \lambda_2)^{\frac{2}{p+2}} = 1.$$ 

Now, if we suppose that $\lambda_1(\epsilon) > 1$ and $\lambda_2(\epsilon) < 0$ and $w_{n,1} = (\lambda_{n,1}(\epsilon))^{-\frac{1}{p+2}} \psi_{n,1}$ then

$$I_c(\psi_n) = I_c(\psi_{n,1}) + I_c(\psi_{n,2}) + \delta_n \geq (\lambda_{n,1}(\epsilon))^{-\frac{2}{p+2}} I_c + \delta_n > I_c + \delta_n.$$ 

Thus, taking limit as $n \to \infty$, we arrive to the contradiction $I_c > I_c$. In other words, we have ruled out dichotomy.

\textbf{Theorem 4.1. (Existence of travelling wave solutions).} Assume that $c > 0$, $a_{i,j} > 0$, $a_i > 0$, for $i = 0, 1$, for $j = 0, 1, 2$, and $\gamma > 0$. Given a minimizing sequence $(\psi_n)_n$ of $I_{c,\gamma}$, then there exists a subsequence (which we denote by the same symbol), a sequence of points $(x_n, y_n)_n \subset \mathbb{R}^2$, and a minimizer $\psi \in X$ such that $\psi_n(\cdot + x_n, \cdot + y_n) \to \psi$ in $X$.

\textbf{Proof.} Let $\psi \in X$ be a weak limit of the bounded minimizing sequence $(\psi_n)_n$ of $I_{c,\gamma}$ ($\psi_n \to \psi$ in $X$). From Lions' Concentration-Compactness principle, we have compactness, since we already ruled out dichotomy and vanishing. In
other words, there exists a sequence \((x_n, y_n) \subset \mathbb{R}^2\) such that for all \(\tau > 0\) there exists \(R = R(\tau) > 0\) such that
\[
\int_{B_R(x_n, y_n)} \rho(\psi_n) \, dx \, dy \geq \int_{\mathbb{R}^2} \rho(\psi_n) \, dx \, dy - \tau.
\]
The latter inequality is equivalent to
\[
\int_{B_R(0,0)} \rho(\tilde{\psi}_n) \, dx \, dy \geq \int_{\mathbb{R}^2} \rho(\tilde{\psi}_n) \, dx \, dy - \tau,
\]
when translate to origin the inequality and \(\tilde{\psi}_n(x, y) = \psi_n(x + x_n, y + y_n)\). This is,
\[
\int_{\mathbb{R}^2 \setminus B_R(0,0)} \rho(\tilde{\psi}_n) \, dx \, dy \leq \tau.
\]
From the definition of \(\rho\) in (32), we have that for all \(0 \leq j \leq 2\) that there exists \(M > 0\) such that
\[
M \int_{\mathbb{R}^2 \setminus B_R(0,0)} \left( \partial_x^j \tilde{\psi}_n \right)^2 \, dx \, dy \leq \int_{\mathbb{R}^2 \setminus B_R(0,0)} \rho(\tilde{\psi}_n) \, dx \, dy \leq \tau.
\]
Then we conclude that
\[
\int_{\mathbb{R}^2} \left( \partial_x^j \tilde{\psi}_n \right)^2 \, dx \, dy \leq \int_{B_R(0,0)} \left( \partial_x^j \tilde{\psi}_n \right)^2 \, dx \, dy + \frac{\tau}{M}.
\]
We claim that \(\partial_x^j \tilde{\psi}_n \to \partial_x^j \psi\) in \(L^2(\mathbb{R})\), when \(0 \leq j \leq 2\). Since \(\psi_n \to \psi_0\) in \(X\) and the embedding \(X \hookrightarrow L^s(\mathbb{R}^2)\) is compact for \(2 \leq s < 6\), we have that
\[
\partial_x^j \tilde{\psi}_n \to \partial_x^j \psi_0 \text{ in } L^2(\mathbb{R}^2), \quad \partial_x^j \tilde{\psi}_n \to \partial_x^j \psi_0 \text{ in } L^2_{\text{loc}}(\mathbb{R}^2), \quad \partial_x^j \tilde{\psi}_n \to \partial_x^j \psi_0 \text{ a.e. in } \mathbb{R}^2.
\]
From previous facts and using Fatou’s Theorem, we see that
\[
\left\| \partial_x^j \psi \right\|_{L^2(\mathbb{R})}^2 \leq \liminf_{n \to \infty} \left\| \partial_x^j \tilde{\psi}_n \right\|_{L^2(\mathbb{R})}^2 \leq \liminf_{n \to \infty} \int_{B_R(0,0)} \left( \partial_x^j \tilde{\psi}_n \right)^2 \, dx \, dy + \frac{\tau}{M} \leq \int_{B_R(0,0)} \left( \partial_x^j \psi \right)^2 \, dx \, dy + \frac{\tau}{M} \leq \left\| \partial_x^j \psi \right\|_{L^2(\mathbb{R})}^2 + \frac{\tau}{M}.
\]
In other words, we have shown for \(0 \leq j \leq 2\) that \(\partial_x^j \tilde{\psi}_n \to \partial_x^j \psi\) in \(L^2(\mathbb{R})\). In a similar fashion we see that
\[
\partial_x^{j-1} \left( \partial_y \psi_n \right) \to \partial_x^{j-1} \left( \partial_y \psi_0 \right), \quad \partial_x^{-1} \tilde{\psi}_n \to \partial_x^{-1} \psi_0 \text{ in } L^2(\mathbb{R}) \quad \text{for } j = 0, 1.
\]
Now, since we know that $K$ is Lipschitz (continuous) and that $K(\psi_n) = 1$ we conclude that $K(\psi_n) \to K(\psi) = 1$. So, we also have that $I_{c,\gamma} \leq I_{c,\gamma}(\psi)$. By the weak convergence of $\psi_n$ to $\psi$ in $X$ and the weak lower semicontinuity of functional $I_{c,\gamma}$, we have that

$$I_{c,\gamma}(\psi_0) \leq \lim \inf I_{c,\gamma}(\tilde{\psi}_n) = I_{c,\gamma},$$

so we have that $I_{c,\gamma}(\psi) = I_{c,\gamma}$, meaning that $\psi$ is in fact a minimizer for $I_{c,\gamma}$. Moreover, $\psi_n \to \psi$ in $X$.

\[\square\]

5 Analyticity of Solitons

In this section, we prove that any solitary wave solution of the equation (1) is already a smooth and analytic function in the case $p = 1, 2, 3$. To illustrate the computations, we only perform a specific case, understanding that the general case follows by doing the same type of computations. So, we consider the case

$$F(q, r) = \mu_1 q^{p+2} + \mu_2 q r^{p+1}.$$ 

According with the definition of $f$ in terms of $F$, we have that

$$f(v, v_x, v_{xx}) = (p + 2)\mu_1 v^{p+1} - \mu_2 \left((p + 1)v \partial_x (\partial_x v)^p + p (\partial_x v)^{p+1}\right).$$

Note that if $p = 1$, $\mu_1 = \frac{1}{2}$ and $\mu_2 = \frac{3}{2}$, then we obtain the nonlinear term in equation (7). In a similar way if $p = 1$, $\mu_1 = \frac{1}{3}$ and $\mu_2 = 0$, then we obtain the nonlinear term in equation (5). We will use the following result (see Proposition 16 in F. Soriano [19]).

**Proposition 5.1.**

1. If $h : \mathbb{R} \to \mathbb{R}$ is a $C^\infty(\mathbb{R})$ and $\phi \in H^k(\mathbb{R}^2)$ for all $k \geq 1$, then

$$D^\alpha (h(\phi)) = \sum_{j=1}^{[\alpha]} \frac{h^k(\phi)}{j!} \sum_{A(\alpha, j)} \frac{\alpha!}{\rho_1! \cdots \rho_j!} D^{\rho_1} \phi \cdots D^{\rho_j} \phi,$$

where $A(\alpha, j) = \{(\rho_1, \ldots, \rho_j) : \rho_1 + \cdots + \rho_j = \alpha, |\rho_i| \geq 1, 1 \leq i \leq j\}$.

2. For each $(n_1, \ldots, n_j) \in \mathbb{N}^j$ we have

$$|\alpha!| = \sum_{A(\alpha, n_1, \ldots, n_j)} \frac{\alpha! |\rho_1|! \cdots |\rho_j|!}{\rho_1! \cdots \rho_j!},$$

with $A(\alpha, n_1, \ldots, n_j) = \{(\rho_1, \ldots, \rho_j) : \rho_1 + \cdots + \rho_j = \alpha, |\rho_i| = n_i, 1 \leq i \leq j\}$. 
Theorem 5.1. Assume that $c > 0$, $a_{i,j} > 0$, $a_i > 0$, for $i = 0, 1$, for $j = 0, 1, 2$, and $\gamma > 0$ and that $p = 1, 2, 3$. Then any solution $v \in X$ of the equation (8) is already analytic.

Proof. Following the same ideas given for other models (see [8], [17]) we have that $v \in H^k(\mathbb{R}^2)$. Now, if $\zeta_0 = (x_0, y_0) \in \mathbb{R}^2$ and $v$ is a solution of equation (8), we will show that there exists $r > 0$ such that the following Taylor expansion converges absolutely in $B_r(\zeta_0)$,

$$v(x, y) = \sum_{\alpha} \frac{D^\alpha v(\zeta_0)}{\alpha!} (x - x_0, y - y_0)^\alpha.$$

First, we establish the result under the assumption of the existence of $R > 1$ such that for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, with $|\alpha| = \alpha_1 + \alpha_2 \geq 1$,

$$\|D^\alpha v\|_X \leq C \frac{|\alpha|!}{|\alpha| + 2} R^{|\alpha|-1},$$

(46)

where $D^\alpha v = \partial_x^{\alpha_1} \partial_y^{\alpha_2} v$. In fact, If we set $\zeta = (x, y) - \zeta_0$, then by the Taylor Theorem (with remainder) we have that

$$v(x, y) = \sum_{k=0}^{N-1} \sum_{|\alpha| = k, \alpha \in \mathbb{N}^2} \frac{D^\alpha v(\zeta_0) \zeta^\alpha}{\alpha!} + \mathcal{E}_N(x, y),$$

where $\mathcal{E}_N(x, y) = \sum_{|\alpha| = N, \alpha \in \mathbb{N}^2} \frac{D^\alpha v(\zeta_0 + t\zeta) \zeta^\alpha}{\alpha!}$. Now, using (46) and the Lemma 2.2, if $|\alpha| \geq 1$ we have that

$$|D^\alpha v(x, y)| \leq ||D^\alpha v||_{L^\infty(\mathbb{R}^2)} \leq C_2 ||D^\alpha v||_X \leq C_2 C \frac{|\alpha|!}{|\alpha| + 2} R^{|\alpha|-1} \leq C_2 C \frac{|\alpha|!}{|\alpha| + 2} R^{|\alpha|}.$$ 

Taking $r > 0$ in such a way that $4r^2R < 1$ and using that $\frac{|\alpha|!}{\alpha!} \leq 2^{|\alpha|}$ we conclude for $\|\zeta\| < r$ that

$$|\mathcal{E}_N(x, y)| \leq C_2 C \sum_{|\alpha| = N, \alpha \in \mathbb{N}^2} \frac{|\alpha|!}{(|\alpha| + 2)\alpha!} \|\zeta\|^{2|\alpha|} \leq C_2 C \sum_{|\alpha| = N, \alpha \in \mathbb{N}^2} \frac{2^N R^N}{N + 2} r^{2N} \leq C_2 C (2r^2)^N \leq C_2 C 2^{-N}.$$

In other words, the Taylor series for $v$ converges in $B_R(\zeta_0)$. 
To complete the proof, we only need to prove that there exists $R > 1$ such that (46) holds for all $\alpha \in \mathbb{N}^2$, with $|\alpha| \geq 1$. We will argue by induction on $|\alpha|$. Since $v \in H^l(\mathbb{R})$, $l \geq 0$, then $D^\alpha v \in X$ for all $|\alpha| \geq 0$. Thus, inequality (46) is obvious for $|\alpha| = 1$; it is sufficient to choose $C$ large enough. Now, suppose that (46) holds for fixed $|\alpha| = 1, \ldots, n$ and $R$ (which will be chosen later). If we apply the operator $D^\alpha$ to equation (8), then multiply with $D^\alpha v$, after integration by parts, we obtain that

$$2I_c(D^\alpha v) = (p + 2)\mu_1 \langle D^\alpha (v^{p+1}), D^\alpha v \rangle_{L^2} + \mu_2 \left( (p + 1) \langle D^\alpha [v(\partial_x v)^p], D^\alpha \partial_x v \rangle_{L^2} + p \langle D^\alpha [(\partial_x v)^{p+1}], D^\alpha v \rangle_{L^2} \right).$$

By using Hölder inequality,

$$|\langle D^\alpha [v(\partial_x v)^p], D^\alpha \partial_x v \rangle_{L^2}| \leq \|D^\alpha [v(\partial_x v)^p]\|_{L^2} \|D^\alpha \partial_x v\|_{L^2} \leq \|D^\alpha [v(\partial_x v)^p]\|_{L^2} \|D^\alpha \partial_x v\|_X.$$ 

In addition, for $j = 0, 1$ we see that

$$\left| \langle D^\alpha ((\partial_x^j v)^{p+1}), D^\alpha v \rangle_{L^2} \right| \leq \|D^\alpha ((\partial_x^j v)^{p+1})\|_{L^2} \|D^\alpha v\|_X. \quad (48)$$ 

Without loss of generality in (48) we can suppose that $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 \geq 1$. In fact, if $\alpha = (0, \alpha_2)$, $\alpha' = (1, \alpha_2 - 1)$ and $w = v, \partial_x v$ then we have that

$$\left| \langle D^\alpha (w^{p+1}), D^\alpha v \rangle_{L^2} \right| = \left| \langle D^\alpha (w^{p+1}), D^\alpha \partial_x \partial_x^{-1} v \rangle_{L^2} \right| = \left| \langle D^\alpha (w^{p+1}), D^\alpha \partial_x \partial_x^{-1} v \rangle_{L^2} \right| \leq \|D^\alpha (w^{p+1})\|_{L^2} \|D^\alpha v\|_X.$$ 

Note that $|\alpha'| = |\alpha|$, this we will use later. Then using that $\sqrt{T_{c,n}}$ is like a norm in $X$ and applying in (47) the previous estimates we obtain that

$$\|D^\alpha v\|_X \leq C_1 \left( \|D^\alpha (v^{p+1})\|_{L^2} + \|D^\alpha [v(\partial_x v)^p]\|_{L^2} + \|D^\alpha [(\partial_x v)^{p+1}]\|_{L^2} \right).$$

We want to estimate the terms of right hand side. For simplicity we consider $p = 1$. We obtain the case $p = 2, 3$ in a similar way. Note that if $u, w \in H^1$ for any $l \geq 1$, we have for $\alpha = (\alpha_1, \alpha_2)$ that

$$D^\alpha (uv) = (D^\alpha u)w + uD^\alpha w + \sum_{k=1}^{\lfloor |\alpha| \rfloor} \sum_{\beta \leq \alpha, |\beta| = k} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} (D^{\alpha - \beta} u)(D^{\beta} w).$$
Where $\beta = (\beta_1, \beta_2)$. Then we see, for example, that

$$D^\alpha [(\partial_x v)^2] = 2D^\alpha (\partial_x v)\partial_x v + \sum_{k=1}^{\vert\alpha\vert-1} \sum_{\beta \leq \alpha \atop \vert\beta\vert = k} \binom{\alpha}{\beta_1} \binom{\alpha}{\beta_2} D^{\alpha-\beta} (\partial_x v)D^\beta (\partial_x v).$$

(49)

If $\nu = (\alpha_1 - 1, \alpha_2)$, then using Lemma 2.2 and induction hypothesis we have that

$$\|D^\alpha (\partial_x v)\partial_x v\|_{L^2} \leq \|D^\nu (\partial_x^2 v)\|_{L^2} \|\partial_x v\|_{L^\infty}$$

$$\leq C_2 \|\partial_x v\|_X \|\partial_x v\|_X$$

$$\leq C_2 C^2 R^{\vert\alpha\vert-2} (\vert\alpha\vert - 1)! \vert\alpha\vert + 1$$

$$\leq \left( \frac{C (\vert\alpha\vert + 1)!}{\vert\alpha\vert + 3} R^{\vert\alpha\vert} \right) \left( C_2 C R^{-2} \frac{\vert\alpha\vert + 3}{\vert\alpha\vert (\vert\alpha\vert + 1)} \right).$$

Also we obtain that

$$\|D^{\alpha-\beta} (\partial_x v)D^\beta (\partial_x v)\|_{L^2} \leq C_2 \|D^{\nu-\beta} v\|_X \|D^\beta (\partial_x v)\|_X$$

$$\leq C_2 C^2 \left( \frac{(|\alpha| - |\beta| - 1)! (|\beta| + 1)!}{(|\alpha| - |\beta| + 1) (|\beta| + 3)} \right) R^{\vert\alpha\vert-2}.$$

Using the previous inequality we have that

$$\sum_{k=1}^{\vert\alpha\vert-1} \sum_{\beta \leq \alpha \atop \vert\beta\vert = k} \binom{\alpha}{\beta_1} \binom{\alpha}{\beta_2} \|D^{\alpha-\beta} (\partial_x v) (D^\beta (\partial_x v))\|_{L^2}$$

$$\leq C_2 C^2 \sum_{k=1}^{\vert\alpha\vert-2} \sum_{\beta \leq \alpha \atop \vert\beta\vert = k} \frac{\alpha_1! \alpha_2! (|\alpha| - k - 1)! (k + 1)! R^{\vert\alpha\vert-2}}{(\alpha_1 - \beta_1)! (\alpha_2 - \beta_2)! \beta_1! \beta_2! (|\alpha| - k + 1) (k + 3)}$$

$$\leq C_2 C^2 R^{\vert\alpha\vert-2} \sum_{k=1}^{\vert\alpha\vert-1} \frac{k + 1}{(|\alpha| - k) (|\alpha| - k + 1) (k + 3)}$$

$$\cdot \sum_{\beta \leq \alpha \atop \vert\beta\vert = k} \frac{\alpha_1! \alpha_2! (|\alpha| - k)! k!}{(\alpha_1 - \beta_1)! (\alpha_2 - \beta_2)! \beta_1! \beta_2!}.$$

But we know that for any $(n_1, n_2) \in \mathbb{N}^2$ (see Proposition 5.1),

$$\vert\alpha\vert! = \sum_{A(\alpha, n_1, n_2)} \frac{\alpha_1! \rho_1! \rho_2!}{\rho_1! \rho_2!},$$
where
\[ A(\alpha, n_1, n_2) = \{ (\rho_1, \rho_2) \in \mathbb{N}^2 \times \mathbb{N}^2 : \rho_1 + \rho_2 = \alpha, \ |\rho_i| = n_i, \text{ for } 1 \leq i \leq 2 \}. \]

Now, for given \( 1 \leq k \leq |\alpha| - 1 \) and \( \beta = (\beta_1, \beta_2) \) with \( |\beta| = k \), we define
\[ \rho_1 = (\alpha_1 - \beta_1, \alpha_2 - \beta_2), \quad \rho_2 = (\beta_1, \beta_2). \]

Then we have that
\[ \rho_1 + \rho_2 = \alpha, \quad |\rho_1| = |(\alpha_1 - \beta_1, \alpha_2 - \beta_2)| = |\alpha| - k, \quad |\rho_2| = \beta_1 + \beta_2 = k. \]

Using the above property, we obtain that
\[ \frac{\alpha! |\rho_1|! |\rho_2|!}{\rho_1! \rho_2!} = \frac{\alpha_1! \alpha_2! (|\alpha| - k)! k!}{(\alpha_1 - \beta_1)!(\alpha_2 - \beta_2)! \beta_1! \beta_2!}. \]

Thus, for \( 1 \leq k \leq |\alpha| - 1 \), we conclude that
\[ \sum_{|\beta| = k} \frac{\alpha_1! \alpha_2! (|\alpha| - k)! k!}{(\alpha_1 - \beta_1)!(\alpha_2 - \beta_2)! \beta_1! \beta_2!} \leq \sum_{A(\alpha, |\alpha| - k, k)} \frac{\alpha! |\rho_1|! |\rho_2|!}{\rho_1! \rho_2!} = |\alpha|!. \]

From these estimates, we get that
\[
\sum_{k=1}^{|\alpha| - 1} \sum_{\beta \leq \alpha, |\beta| = k} \left( \frac{\alpha_1}{\beta_1} \right) \left( \frac{\alpha_2}{\beta_2} \right) \left\| D^{\alpha - \beta}(\partial_x v)(D^\beta(\partial_x v)) \right\|_{L^2} \leq \frac{C_2 C_2^2 R^{|\alpha| - 2} |\alpha|! (k + 1)!}{|\alpha| - k! (|\alpha| - k + 1)!(k + 3)!} \leq C_2 C_2 |\alpha|! R^{|\alpha| - 2} \sum_{k=0}^{\infty} \frac{1}{k^2} \leq \left( CR^{|\alpha|} \frac{|\alpha| + 1)!}{|\alpha| + 3} \right) \left( CC_2 R^{-2} |\alpha| + 3 \sum_{k=1}^{\infty} \frac{1}{k^2} \right).
\]

Note that there exists \( M > 0 \) such that \( \frac{|\alpha| + 3}{|\alpha| + 1} < M \). So, taking \( R \) large enough such that \( CC_2 R^{-2} M (1 + \sum_{k=1}^{\infty} \frac{1}{k^2}) < 1 \), we conclude that
\[ \left\| D^\alpha [(\partial_x v)^2] \right\|_{L^2} \leq C \frac{|\alpha| + 1)!}{|\alpha| + 3} R^{|\alpha|}. \]

In a similar fashion, we see
\[ \left\| D^\alpha (v^2) \right\|_{L^2} \leq C \frac{|\alpha| + 1)!}{|\alpha| + 3} R^{|\alpha|}. \]
On the other hand, we note that
\[ \| (D^\alpha v)(\partial_x v) \|_{L^2} \leq C_2 \| D^\alpha v \|_X \| \partial_x v \|_X \leq C_2 C^2 R^{\lvert \alpha \rvert - 1} \frac{\lvert \alpha \rvert !}{\lvert \alpha \rvert + 2}, \]
and also that
\[ \| vD^\alpha (\partial_x v) \|_{L^2} \leq C_2 \| D^\alpha v \|_X \| v \|_X \leq C_2 C^2 R^{\lvert \alpha \rvert - 1} \frac{\lvert \alpha \rvert !}{\lvert \alpha \rvert + 2}. \]
Moreover, if \( \nu = (\alpha_1 - 1, \alpha_2) \) or \( \nu = (\alpha_1, \alpha_2 - 1) \) we obtain that
\[ \| D^{\alpha - \beta} v D^{\beta} (\partial_x v) \|_{L^2} \leq C_2 \| D^{\nu - \beta} v \|_X \| D^\beta (\partial_x v) \|_X \]
\[ \leq C_2 C^2 \left( \frac{(\lvert \alpha \rvert - \lvert \beta \rvert - 1)! (\lvert \beta \rvert + 1)!}{(\lvert \alpha \rvert - \lvert \beta \rvert + 1) (\lvert \beta \rvert + 3)} \right) R^{\lvert \alpha \rvert - 2}. \]
Then, taking \( R \) large enough such that \( CC_2 R^{-1} M (1 + \sum_{k=1}^{\infty} \frac{1}{k^2}) < 1 \), the same type of above arguments show that
\[ \| D^\alpha (v \partial_x v) \|_{L^2} \leq C \frac{(\lvert \alpha \rvert + 1)!}{\lvert \alpha \rvert + 3} R^{| \alpha |}. \]

In other words, we have shown for \( R \) large enough that
\[ \| D^\alpha (v^2) \|_{L^2} + \| D^\alpha [v \partial_x v] \|_{L^2} + \| D^\alpha [(\partial_x v)^2] \|_{L^2} \leq C \frac{(\lvert \alpha \rvert + 1)!}{\lvert \alpha \rvert + 3} R^{| \alpha |}. \] (52)

In a similar way we obtain the estimate in the cases \( p = 2, 3 \). Then we can see for \( R \) large enough that
\[ \| D^\alpha (v^{p+1}) \|_{L^2} + \| D^\alpha [v (\partial_x v)^p] \|_{L^2} + \| D^\alpha [(\partial_x v)^{p+1}] \|_{L^2} \leq C \frac{(\lvert \alpha \rvert + 1)!}{\lvert \alpha \rvert + 3} R^{| \alpha |}, \] (53)

Now, by using (53) we will establish that
\[ \| D^\alpha \partial_i v \|_X \leq C \frac{(\lvert \alpha \rvert + 1)!}{\lvert \alpha \rvert + 3} R^{| \alpha |}, \quad i = 1, 2. \]

To do this, we apply operator \( D^\alpha \partial_i \) to equation (8) and compute the \( L^2 \)- inner product with \( D^\alpha \partial_i v \). Thus, we have that
\[ 2I_c(D^\alpha \partial_i v) = (p + 2) \mu_1 \langle D^\alpha \partial_i (v^{p+1}), D^\alpha \partial_i v \rangle_{L^2} \]
\[ + \mu_2 \left( (p + 1) \langle D^\alpha \partial_i [v(\partial_x v)^p], D^\alpha \partial_i \partial_x v \rangle_{L^2} + p \langle D^\alpha \partial_i [(\partial_x v)^{p+1}], D^\alpha \partial_i v \rangle_{L^2} \right). \]

Next, we see that
\[ |\langle D^\alpha \partial_i [v(\partial_x v)^p], D^\alpha \partial_i (\partial_x v) \rangle_{L^2}| = |\langle D^\alpha [v(\partial_x v)^p], D^\alpha \partial^2 v \rangle_{L^2}| \]
\[ \leq \| D^\alpha [v(\partial_x v)^p] \|_{L^2} \| D^\alpha \partial^2 v \|_{L^2} \]
\[ \leq \| D^\alpha [v(\partial_x v)^p] \|_{L^2} \| D^\alpha \partial_i v \|_{X}. \]
In a similar way we have that
\[
\left| \langle D^\alpha \partial_i (v^{p+1}), D^\alpha \partial_i v \rangle \right|_{L^2} \leq \| D^\alpha (v^{p+1}) \|_{L^2} \| D^\alpha \partial_i v \|_X,
\]
and
\[
\left| \langle D^\alpha \partial_i [(\partial_x v)^{p+1}], D^\alpha \partial_i v \rangle \right|_{L^2} \leq \| D^\alpha [(\partial_x v)^{p+1}] \|_{L^2} \| D^\alpha \partial_i v \|_X.
\]
Therefore
\[
I_c(D^\alpha \partial_i v) 
\leq C_1 \left( \| D^\alpha (v^{p+1}) \|_{L^2} + \| D^\alpha [v(\partial_x v)^p] \|_{L^2} + \| D^\alpha [(\partial_x v)^{p+1}] \|_{L^2} \right) \| D^\alpha \partial_i \|_X.
\]
Then we conclude that
\[
\| D^\alpha \partial_i v \|_X \leq C_1 \left( \| D^\alpha (v^{p+1}) \|_{L^2} + \| D^\alpha [v(\partial_x v)^p] \|_{L^2} + \| D^\alpha [(\partial_x v)^{p+1}] \|_{L^2} \right)
\leq C \left( |\alpha| + 1 \right)! \| \alpha | + 3 | R^{1|\alpha|}.
\]}

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