The Principal Function for Irregular Lagrangian

Ola A. Jarab'ah

Applied Physics Department, Faculty of Science
Tafila Technical University
P.O.Box:179, Tafila 66110, Jordan

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Abstract

The equations of motion for nonconservative systems are derived from the Euler Lagrange equation. The solutions of these equations are substituted in the Lagrangian. The principal function is then derived by calculating the time integral of the Lagrangian. This function enables us to construct the wave function of these systems. Two examples are discussed to explain the formalism.

Keywords: Euler Lagrange equation, Principal Function, Wave Function, Irregular Lagrange, Nonconservative Systems

1. Introduction

In physics action is a property of the dynamics of a physical system from which the equations of motion of the system can be derived. It is a mathematical functional which takes the trajectory also called path. Action function has the dimensions of [energy, time] and its SI unit is Joule. Second.

Hamilton's principal function is thus the generator of a mechanical transformation to constant coordinates and momenta; when solving the Hamilton-Jacobi equation, we are at the same time obtaining the solution to the mechanical problem and the wave function [9, 10].

The principal function evolution has been investigated using Hamilton-Jacobi equation, the derivative of this function appears in this equation as $\frac{\partial S}{\partial t} = \frac{\partial S}{\partial q} \dot{q} + \frac{\partial S}{\partial t} [1, 8]$. 
Using separation of variables we can find a solution of the form
\[ S(q,\alpha,t) = w(q,\alpha) - \alpha t \]
and the Hamilton Jacobi equation becomes
\[ H(q, \frac{\partial S}{\partial q}) = \alpha , \]
where the quantity \( \alpha \) is the constant of integration. Since the momentum \( p = \frac{\partial S}{\partial q} \), then Hamilton's principal function can be computed from this formula
\[ \frac{dS}{dt} = p\dot{q} - H = L . \]

For every conservative mechanical systems, there exists a function of the generalized coordinates \( q_i \) and velocities \( \dot{q}_i \) called regular Lagrangian \( L_o = L_o(q, \dot{q}, t) \) [4], the integral of the Lagrangian between two instants of time \( t_1 \) and \( t_2 \) is defined as the principal function \( S \) which takes this form [3]:
\[ S[q(t)] = \int_{t_1}^{t_2} L_o(q(t), \dot{q}(t), t) dt \]

So that the motivation of this paper is furnished by the desire to find the principal function and the wave function for nonconservative systems using Euler Lagrange equation.

This paper is organized as follows. In section 2, principal function formulation for irregular Lagrangian is discussed. In section 3, the definition of wave function for irregular Lagrangian was briefly discussed. In section 4, two illustrative examples are examined. In section 5, the work closes with some concluding remarks.

### 2. Principal Function Formulation for Irregular Lagrangian

The Euler Lagrange equation for conservative systems is given by [12]
\[ \frac{d}{dt} \left( \frac{\partial L_o}{\partial \dot{q}} \right) - \frac{\partial L_o}{\partial q} = 0 \]  
(1)

In this work we would like to find the principal function for nonconservative systems using Euler Lagrange equation. We Start with the Lagrangian \( L = L_o(q, \dot{q}, t) e^{\alpha t} \) [7], where \( L_o(q, \dot{q}, t) \) stands for the Lagrangian of the corresponding conservative system. Because the Euler Lagrange equations are second order equations, we find the equations of motion from the corresponding Lagrangian in terms of the generalized coordinates and their derivatives. Then, we substitute the solutions of these equations in the given Lagrangian. Finally, the evolution of this Lagrangian between two instants of time \( t_1 \) and \( t_2 \) gives the principal function \( S \) as a function of time.

\[ S[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt \]  
(2)
3. Wave Function Definition for Irregular Lagrangian

In the semi classical expansion (WKB) of the Hamilton Jacobi function of regular systems has been studied [11]. This expansion leads to the following wave function [5]:

$$\psi(q_i, t) = \left[ \prod_{i=1}^{N} \psi_i(q_i) \right] e^{iS(q, t)/\hbar}$$

(3)

where $\psi_0(q_i)$ is the amplitude of the wave function. Which is defined as

$$\psi_0(q_i) = \frac{1}{\sqrt{p(q_i)}}.$$

In this paper the amplitude being set to unity for convenience. We use the principal function to formulate the wave function for irregular Lagrangian in the same manner as for regular Lagrangian.

4. Examples

As a first example, let us consider one-dimensional Lagrangian of a free particle of mass $m$ in the presence of damping [6].

The Lagrangian is

$$L = \frac{1}{2} mq^2 e^{\lambda t}$$

(4)

Using Euler Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

The equation of motion is

$$\ddot{q} + \lambda \dot{q} = 0$$

(5)

The solution of Eq. (5) is

$$\dot{q} + \lambda q = C$$

Where $C$ is the constant of integration. Then,

$$\dot{q} = C - \lambda q$$

(6)
Integrating Eq. (6) and choosing \( q_0 = 0 \) this gives
\[
q(t) = \frac{C}{\lambda} (1 - e^{-\lambda t})
\] (7)

Taking the first time derivative of Eq. (7)
\[
\dot{q} = C e^{-\lambda t}
\] (8)

and
\[
\dot{q}^2 = C^2 e^{-2\lambda t}
\] (9)

Substituting Eq. (9) into Eq. (4) we get
\[
L = \frac{1}{2} m C^2 e^{-\lambda t}
\] (10)

Using Eq. (2)
\[
S = \int_0^t \frac{1}{2} m C^2 e^{-\lambda t} dt
\]
The principal function is
\[
S = \frac{m C^2}{2 \lambda} [1 - e^{-\lambda t}]
\] (11)

Using Eq. (3) the wave function takes the following form
\[
\psi(q,t) = e^{i \lambda t} = e^{\frac{i m C^2}{2 \lambda} [1 - e^{-\lambda t}]}
\] (12)

- The second example is a pendulum of mass \( m \) and length \( l \) with angular displacement \( \theta \) from the vertical [2].

The Lagrangian which describes this example is given by:
\[
L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos \theta) e^{\lambda t}
\] (13)

For small \( \theta \), we have approximately \( \cos \theta = 1 - \frac{\theta^2}{2} \)

Thus, this Lagrangian reads
\[
L = \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} mgl \theta^2 e^{\lambda t}
\] (14)

Using Euler Lagrange equation
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0
\]
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The equation of motion is
\[ \ddot{\theta} + \lambda \dot{\theta} + \omega_0^2 \theta = 0 \tag{15} \]

Using that
\[ \theta(t) = e^{Zt} \tag{16} \]

The first time derivative of Eq. (16) is
\[ \dot{\theta} = Z e^{Zt} \tag{17} \]

and the second time derivative is
\[ \ddot{\theta} = Z^2 e^{Zt} \tag{18} \]

Making use of Eqs. (16, 17 and 18) then Eq. (15) can be written as
\[ Z^2 + \lambda Z + \omega_0^2 = 0 \tag{19} \]

The solution of Eq. (19) is
\[ Z_{1,2} = \frac{-\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} - \omega_0^2} \tag{20} \]

Where
\[ \frac{\lambda^2}{4} = \omega^2 \]

and \( \omega_0^2 = \frac{g}{l} \), is the angular frequency

If
\[ \alpha = \sqrt{\omega^2 - \omega_0^2} \]

Then, Eq. (20) can be written as
\[ Z_{1,2} = \frac{-\lambda}{2} \pm \alpha \tag{21} \]

Now, there are three possible cases:
1) Over damping, \( \alpha > 0 \) which means \( \omega^2 > \omega_0^2 \)

For this case the solution is
\[ \theta(t) = A_1 e^{\frac{-\lambda}{2} - \alpha} t + A_2 e^{\frac{-\lambda}{2} + \alpha} t \tag{22} \]

and
\[ \theta^2(t) = A_1^2 e^{-2\left(\frac{\lambda}{2} - \alpha\right)t} + A_2^2 e^{-2\left(\frac{\lambda}{2} + \alpha\right)t} + 2A_1A_2 e^{-\lambda t} \]  
(23)

The first time derivative of Eq. (22) is

\[ \dot{\theta}(t) = -A_1 \left(\frac{\lambda}{2} - \alpha\right) e^{-2\left(\frac{\lambda}{2} - \alpha\right)t} - A_2 \left(\frac{\lambda}{2} + \alpha\right) e^{-2\left(\frac{\lambda}{2} + \alpha\right)t} \]  
(24)

and

\[ \ddot{\theta}(t) = A_1^2 \left(\frac{\lambda}{2} - \alpha\right)^2 e^{-2\left(\frac{\lambda}{2} - \alpha\right)t} + A_2^2 \left(\frac{\lambda}{2} + \alpha\right)^2 e^{-2\left(\frac{\lambda}{2} + \alpha\right)t} + 2A_1A_2 \left(\frac{\lambda}{2} - \alpha\right) \left(\frac{\lambda}{2} + \alpha\right) e^{-\lambda t} \]  
(25)

Now substituting Eqs. (23 and 25) in the given Lagrangian and using Eq. (2) we obtain

\[ S_1 = \int_0^1 \frac{1}{2} ml^2 \dot{\theta}^2 - \frac{1}{2} mgl \theta^2 |e^{\lambda t} dt \]

\[ S_1 = \frac{1}{4\alpha} ml^2 \dot{\alpha}^2 \left(\frac{\lambda}{2} - \alpha\right)^2 (e^{2\alpha t} - 1) - \frac{1}{4\alpha} ml^2 \dot{\alpha}^2 \left(\frac{\lambda}{2} + \alpha\right)^2 (e^{-2\alpha t} - 1) - \frac{1}{4\alpha} mgl \dot{\alpha}^2 (e^{2\alpha t} - 1) + \frac{1}{4\alpha} mgl \dot{\alpha}^2 (e^{-2\alpha t} - 1) 

- mglA_1A_2 + ml^2 A_1A_2 (\frac{\lambda}{2} + \alpha)(\frac{\lambda}{2} - \alpha) \]
(26)

Using Eq. (26), the wave function takes the following form

\[ \psi_1(q_1, t) = e^{-\frac{iS_1(q_1,t)}{\hbar}} = \exp \left[ \frac{i}{\hbar} \frac{1}{4\alpha} ml^2 \dot{\alpha}^2 \left(\frac{\lambda}{2} - \alpha\right)^2 (e^{2\alpha t} - 1) - \frac{1}{4\alpha} ml^2 \dot{\alpha}^2 \left(\frac{\lambda}{2} + \alpha\right)^2 (e^{-2\alpha t} - 1) - \frac{1}{4\alpha} mgl \dot{\alpha}^2 (e^{2\alpha t} - 1) + \frac{1}{4\alpha} mgl \dot{\alpha}^2 (e^{-2\alpha t} - 1) 

- mglA_1A_2 + ml^2 A_1A_2 (\frac{\lambda}{2} + \alpha)(\frac{\lambda}{2} - \alpha) \right] \]  
(27)

2) Critical damping, \[ \alpha = 0 \] which means \[ \omega_c^2 = \omega^2 \]

The solution of this case is

\[ \theta(t) = Ate^{-\frac{\lambda}{2} t} + Be^{-\frac{\lambda}{2} t} \]  
(28)

Then

\[ \theta^2(t) = A^2 t^2 e^{-\lambda t} + B^2 e^{-\lambda t} + 2ABte^{-\lambda t} \]  
(29)

The first time derivative of Eq. (28) is

\[ \dot{\theta}(t) = \frac{-\lambda At}{2} e^{-\frac{\lambda}{2} t} + \frac{\lambda B}{2} e^{-\frac{\lambda}{2} t} \]  
(30)
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\[ \theta^2(t) = \frac{\dot{\theta}^2}{4} - \lambda \dot{\theta}^2 \left[ \frac{1}{4} \dot{\theta}^2 + \frac{1}{8} \dot{\theta}^2 \theta A^2 t^2 \right] - \frac{1}{2} \lambda \dot{\theta}^2 \theta A^2 t^2 - \frac{1}{2} \lambda \dot{\theta}^2 \theta A^2 t^2 + \frac{1}{2} \lambda \dot{\theta}^2 \theta A^2 t^2 + \frac{1}{2} \lambda \dot{\theta}^2 \theta A^2 t^2 \]

Substituting Eqs. (29, 31) in the given Lagrangian and using Eq. (2) the principal function is

\[
S_2 = \int_0^t \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} m g l \dot{\theta}^2 v^{\dot{\theta}} dt
\]

\[
S_2 = \frac{1}{24} m l^2 \dot{\theta}^2 \theta A^2 t^3 + \frac{1}{8} m l^2 \dot{\theta}^2 \theta A^2 t^2 + \frac{1}{6} m l^2 \dot{\theta}^2 \theta A^2 t - \frac{1}{2} m l^2 \dot{\theta}^2 \theta A^2 t
\]

The wave function is

\[
\psi_2(q, t) = e^{i \frac{5\pi}{2}} \left[ \frac{1}{24} m l^2 \dot{\theta}^2 \theta A^2 t^3 + \frac{1}{8} m l^2 \dot{\theta}^2 \theta A^2 t^2 + \frac{1}{6} m l^2 \dot{\theta}^2 \theta A^2 t - \frac{1}{2} m l^2 \dot{\theta}^2 \theta A^2 t \right]
\]

3) Under damping, \( \alpha \) is imaginary which means \( \omega^2 > \omega^2 \)

The solution takes this form

\[
\theta(t) = e^{-\frac{\dot{\theta}^2}{2}} [A \cos(\omega_d t + \phi_0)]
\]

where: \( \omega_d = \sqrt{\omega^2 - \omega^2} \).

For simplifying, the initial phase is given by \( \phi_0 = 0 \)

Then, Eq. (34) becomes

\[
\theta(t) = A e^{-\frac{\dot{\theta}^2}{2}} \cos(\omega_d t)
\]

also

\[
\dot{\theta}^2(t) = A^2 e^{-\dot{\theta}^2} \cos^2(\omega_d t)
\]

The first time derivative of Eq. (35) is
$$\dot{\theta}(t) = -\omega_d A e^{-\frac{\lambda}{2} t} \sin(\omega_d t) - \frac{\lambda}{2} A e^{-\frac{\lambda}{2} t} \cos(\omega_d t)$$

and

$$\ddot{\theta}(t) = \omega_d^2 A^2 e^{-\lambda t} \sin^2(\omega_d t) + \frac{\lambda^2}{4} A^2 e^{-\lambda t} \cos^2(\omega_d t) + \omega_d A^2 e^{-\lambda t} \sin(\omega_d t) \cos(\omega_d t)$$

Substituting Eqs. (36) and (38) in the given Lagrangian and using Eq. (2) we obtain

$$S_3 = \frac{1}{4} ml^2 \dot{\theta}^2 - \frac{1}{8} ml^2 \dot{\theta} \omega_d \sin(2\omega_d t) + \frac{1}{16} ml^2 \dot{\theta}^2 A^2 t + \frac{1}{32\omega_d} ml^2 \dot{\theta}^2 A^2 \sin(2\omega_d t)$$

$$- \frac{1}{8} ml^2 A^2 \dot{\theta} \cos(2\omega_d t) + \frac{1}{8} ml^2 A^2 - \frac{1}{16} mglA^2 t - \frac{1}{8\omega_d} mglA^2 \sin(2\omega_d t)$$

and the wave function is

$$\psi_3(q_1, t) = e^{iS_3/q_1} t = \exp\left[ \frac{i}{\hbar} \left\{ \frac{1}{4} ml^2 A^2 \dot{\theta}^2 - \frac{1}{8} ml^2 A^2 \dot{\theta} \omega_d \sin(2\omega_d t) + \frac{1}{16} ml^2 \dot{\theta}^2 A^2 t + \frac{1}{32\omega_d} ml^2 \dot{\theta}^2 A^2 \sin(2\omega_d t) \right. \right.$$

$$- \frac{1}{8} ml^2 A^2 \dot{\theta} \cos(2\omega_d t) + \frac{1}{8} ml^2 A^2 - \frac{1}{16} mglA^2 t - \frac{1}{8\omega_d} mglA^2 \sin(2\omega_d t) \right\}]$$

5. Conclusion

The principal function for irregular Lagrangian is investigated using the Euler Lagrange equation. This function is determined by substituting the solutions of equations of motion in the given Lagrangian; and by finding the time integral of this Lagrangian in the same manner as for regular systems. Then, this function enables us to formulate the wave function. We illustrated through two examples how the Euler Lagrange equation can be used to find the principal function and the corresponding wave function. The first example is the motion of a free particle in one dimension in the presence of damping and the second example is the simple pendulum.

References


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