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An Application of Compensated Compactness Method to a System of Balance Laws

Juan C. Hernández, Hernán Garzón and César Gómez

Department of Mathematics
Universidad Nacional de Colombia
Bogotá D.C., Colombia

Abstract

We prove the existence of weak solution for the Cauchy problem associated with the Aw-Rascle model for traffic flow with certain source terms and bounded measurable initial data. For it, we introduce a flux approximation and also an approximation of the source term in the first equation of the system by adding a small perturbation, after we obtain an $L^1(\mathbb{R})$ estimate related to one of the Riemann invariants of the approximated system, then using this estimate we apply the compensated compactness method to prove the pointwise convergence of the viscosity solutions.

Keywords: Aw-Rascle model, source terms, compensated compactness, weak solution

1 Introduction

Many of the traffic flow models are inspired by fluid mechanics models which are based on conservative or balance equations, see for example [1, 2, 3, 7, 8, 11].

The macroscopic model for traffic flow model proposed by A. Aw and M. Rascle in [1] is a 2×2 system of hyperbolic conservation laws given by

$$\begin{cases} \rho_t + \left(\rho v\right)_x = 0 \\ \left(\rho u\right)_t + \left(\rho u v\right)_x = 0, \end{cases} \tag{1.1}$$

here ρ and $v = u - p(\rho)$ denote, respectively, the density and the velocity of cars on the roadway and $p(\rho)$ is a smooth strictly increasing function. In [11] Zhang independently proposed the same model. If $p(\rho) = 0$, the system (1.1) is the pressureless gas dynamics model or transport equations.

Aw and Rascle in their paper [1] studied the Riemann problem (i.e., a Cauchy problem with piecewise constant initial data) for the system (1.1) assuming the conditions

$$p(0) = 0\,, \quad \lim_{\rho\to 0} \rho p'(\rho) = 0\,, \quad 2p'(\rho) + \rho p''(\rho) > 0 \quad \text{for} \quad \rho > 0. \tag{1.2}$$

Two results on existence of entropic weak solution relative to the Cauchy problem for (1.1) with bounded measurable initial data were established by Lu in [4], one under the same conditions given on $p(\rho)$ by Aw and Rascle and other modifying these conditions.

We shall rewrite the system (1.1) in a more convenient form by introducing the variable $m = \rho u$ as

$$\begin{cases} \rho_t + \big(m - \rho p(\rho)\big)_x = 0 \\ m_t + \left(\frac{m^2}{\rho} - m p(\rho)\right)_x = 0. \end{cases} \tag{1.3}$$

The eigenvalues of system (1.3) are given by

$$\lambda_1 = \frac{m}{\rho} - p(\rho)\,, \qquad \lambda_2 = \frac{m}{\rho} - p(\rho) - \rho p'(\rho)\,, \tag{1.4}$$

with corresponding eigenvectors to right

$$r_{\lambda_1} = \begin{pmatrix} 1 \\ \frac{m}{\rho} + \rho p'(\rho) \end{pmatrix}, \qquad r_{\lambda_2} = \begin{pmatrix} 1 \\ \frac{m}{\rho} \end{pmatrix}.$$

The functions

$$z(\rho, m) = \frac{m}{\rho}, \qquad w(\rho, m) = \frac{m}{\rho} - p(\rho). \tag{1.5}$$

are Riemann invariants for the system (1.3), associated respectively with λ_1 and λ_2.

When the model (1.1) contains source terms, the model is a system of laws of balance. We will study the Cauchy problem for the Aw-Rascle model with source

$$\begin{cases} \rho_t + \big(\rho(u - p(\rho))\big)_x + g_1(\rho, \rho u) = 0 \\ (\rho u)_t + \big(\rho u(u - p(\rho))\big)_x + g_2(\rho, \rho u) = 0, \end{cases} \tag{1.6}$$

and initial data

$$\big(\rho(x,0), u(x,0)\big) = \big(\rho_0(x), u_0(x)\big), \quad \rho_0(x) \geq 0, \tag{1.7}$$

where $\rho_0(x), u_0(x) \in L^\infty(\mathbb{R})$ and $g_i(\rho, \rho u)$, $i = 1, 2$ are locally Lipschitz continuous functions. For (1.6)-(1.7) with the conditions on the function $p(\rho)$ given in Theorem 2 of [4], we obtain a result on the existence of weak solution.

Notice that if we make $m = \rho u$ in the system (1.6), this system becomes

$$\begin{cases} \rho_t + \big(m - \rho p(\rho)\big)_x + g_1(\rho, m) = 0 \\ m_t + \left(\frac{m^2}{\rho} - m p(\rho)\right)_x + g_2(\rho, m) = 0. \end{cases} \tag{1.8}$$

2 The Cauchy problem for the Aw-Rascle model with a source

With the conditions on $p(\rho)$ assumed in this paper an argument following the ideas of Bereux and Sainsaulieu [9] is not valid to prove the positivity of ρ^ϵ, which shows that the vacuum state is not present in the diffusion system associated with (1.8). The technique used here for a positivity proof is an adaptation of a technique due to Lu, which he first introduced in [5] to study the isentropic gas dynamics system for general pressure function. For this reason, we introduce an approximation of the flux functions and the source term $g_1(\rho, m)$ in the system (1.8).

Thus, to establish the existence of a weak solution for the Cauchy problem (1.6) - (1.7), we first consider the following approximate system

$$\begin{cases} \rho_t + \left(\frac{(\rho-\delta)m}{\rho} - (\rho - \delta)p(\rho)\right)_x + \frac{\rho-\delta}{\rho} g_1(\rho, m) = 0 \\ m_t + \left(\frac{(\rho-\delta)m^2}{\rho^2} - \frac{(\rho-\delta)m}{\rho} p(\rho)\right)_x + g_2(\rho, m) = 0, \end{cases} \tag{2.1}$$

where $\delta > 0$ is a small perturbation constant.

The matrix

$$dF_{\delta(\rho,m)} = \begin{pmatrix} \frac{\delta m}{\rho^2} - p(\rho) - (\rho - \delta)p'(\rho) & \frac{\rho-\delta}{\rho} \\ \frac{(2\delta-\rho)m^2}{\rho^3} - \frac{\delta m}{\rho^2} p(\rho) - \frac{(\rho-\delta)m}{\rho} p'(\rho) & \frac{2(\rho-\delta)m}{\rho^2} - \frac{\rho-\delta}{\rho} p(\rho) \end{pmatrix}, \tag{2.2}$$

is the Jacobian matrix of the flux functions in (2.1) with eigenvalues

$$\lambda_{\delta 1} = \frac{\rho - \delta}{\rho}\left(\frac{m}{\rho} - p(\rho)\right), \qquad \lambda_{\delta 2} = \frac{m}{\rho} - p(\rho) - (\rho - \delta)p'(\rho)\,, \tag{2.3}$$

and corresponding Riemann invariants

$$z_\delta(\rho, m) = \frac{m}{\rho} = z(\rho, m), \qquad w_\delta(\rho, m) = \frac{\rho - \delta}{\rho}\left(\frac{m}{\rho} - p(\rho)\right) = \frac{\rho - \delta}{\rho} w(\rho, m), \tag{2.4}$$

where $z(\rho, m)$ and $w(\rho, m)$ are given by (1.5).

3 Existence of weak solution

The study of the Cauchy problem (1.6) - (1.7), which we do here, is based on the ideas given by Lu in [4], paper in which he studied the Cauchy problem (1.6) - (1.7) when $g_1(\rho, \rho u) = g_2(\rho, \rho u) = 0$.

To (2.1) there is associated the diffusive system

$$\begin{cases} \rho_t^\epsilon + \left(\frac{(\rho^\epsilon-\delta)m^\epsilon}{\rho^\epsilon} - (\rho^\epsilon - \delta)p(\rho^\epsilon)\right)_x + \frac{\rho^\epsilon-\delta}{\rho^\epsilon} g_1(\rho^\epsilon, m^\epsilon) = \epsilon \rho_{xx}^\epsilon \\ m_t^\epsilon + \left(\frac{(\rho^\epsilon-\delta)(m^\epsilon)^2}{(\rho^\epsilon)^2} - \frac{(\rho^\epsilon-\delta)m^\epsilon}{\rho^\epsilon} p(\rho^\epsilon)\right)_x + g_2(\rho^\epsilon, m^\epsilon) = \epsilon m_{xx}^\epsilon. \end{cases} \tag{3.1}$$

In the following, we write the functions ρ, m with the indexes $\rho^{\epsilon,\delta}, m^{\epsilon,\delta}$, only when it avoids ambiguities.

Lemma 3.1. *Let $\rho_0(x) \geq c_0$ and $w_0(x) = w(x, 0) \geq c_1$ for two positive constants c_0, c_1 and let $g_1(\rho, m) = \rho h(\rho, m)$ for a continuous function $h(\rho, m)$, assume that*

$$\frac{\rho - \delta}{\rho} z_{\delta\rho} g_1 + z_{\delta m} g_2 \geq c_2 z_\delta + c_3, \qquad \frac{\rho - \delta}{\rho} w_{\delta\rho} g_1 + w_{\delta m} g_2 \leq c_4 w_\delta + c_5, \tag{3.2}$$

where c_i $i = 2, \ldots, 5$ are real constants and the functions $z_\delta(\rho, m), w_\delta(\rho, m)$ are the Riemann invariants given in (2.4). If the function $\rho p(\rho)$ is strictly convex for $\rho > 0$, $\lim_{\rho\to 0} \rho p(\rho) = 0$ and $\lim_{\rho\to\infty} (\rho p(\rho))' \geq c_6$, where c_6 is a constant satisfying $\frac{1}{2}c_1 + c_6 > \sup_{x\in\mathbb{R}} \left|\frac{m_0(x)}{\rho_0(x)}\right|$. Then for any $\epsilon > 0$, we have the a-priori bounds for the Cauchy problem (3.1)-(1.7)

$$\delta \leq \rho^{\epsilon,\delta} \leq M(T), \qquad \left|m^{\epsilon,\delta}\right| \leq M(T), \quad (x, t) \in \mathbb{R} \times [0, T], \tag{3.3}$$

for a positive constant $M(T)$ independent of ϵ and δ.

Proof. We multiply the first and second equations of system (3.1) respectively by $z_{\delta\rho}$ and $z_{\delta m}$ and adding the results, we obtain

$$z_{\delta t} + \lambda_{\delta 1} z_{\delta x} + \frac{\rho - \delta}{\rho} g_1 z_{\delta\rho} + g_2 z_{\delta m} = \epsilon z_{\delta xx} + \frac{2\epsilon}{\rho} \rho_x z_{\delta x}. \tag{3.4}$$

Proceed similarly with the Riemman invariant w_δ, we find

$$w_{\delta t} + \lambda_{\delta 2} w_{\delta x} + \frac{\rho - \delta}{\rho} g_1 w_{\delta\rho} + g_2 w_{\delta m} = \epsilon w_{\delta xx} - \epsilon\left(w_{\delta\rho\rho}\rho_x^2 + 2w_{\delta\rho m}\rho_x m_x + w_{\delta mm} m_x^2\right). \tag{3.5}$$

Algebraic manipulations on the equation (3.5) yields

$$\begin{aligned} w_{\delta t} + \lambda_{\delta 2} w_{\delta x} + \frac{2\epsilon}{\rho}\left(\frac{\delta}{\rho - \delta} - 1\right)\rho_x w_{\delta x} - \frac{2\epsilon\delta^2}{\rho^2(\rho - \delta)^2}\rho_x^2 w_\delta + \frac{\rho - \delta}{\rho} g_1 w_{\delta\rho} \\ + g_2 w_{\delta m} = \epsilon w_{\delta xx} + \frac{\epsilon(\rho - \delta)}{\rho^2}\left(2p'(\rho) + \rho p''(\rho)\right)\rho_x^2. \end{aligned} \tag{3.6}$$

Using the inequalities (3.2) in the equalities (3.4)-(3.5) together with the assumption of strict convexity for $\rho p(\rho)$, we get the following inequalities

$$z_{\delta t} + \lambda_{\delta 1} z_{\delta x} + c_2 z_\delta + c_3 \le \epsilon z_{\delta xx} + \frac{2\epsilon}{\rho}\rho_x z_{\delta x}, \tag{3.7}$$

and

$$w_{\delta t} + \lambda_{\delta 2} w_{\delta x} + \frac{2\epsilon}{\rho}\left(\frac{\delta}{\rho-\delta} - 1\right)\rho_x w_{\delta x} - \frac{2\epsilon\delta^2}{\rho^2(\rho-\delta)^2}\rho_x^2 w_\delta + c_4 w_\delta + c_5 \ge \epsilon w_{\delta xx}. \tag{3.8}$$

By applying the maximum principle to the inequality (3.7) we get the estimate $z(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) = z_\delta(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) \le N(T)$. Since $\rho_0(x) \ge c_0 > 0$ and $w_0(x) \ge c_1 > 0$, this means that $w_{\delta 0}(x) = w_\delta(x,0) \ge \frac{1}{2}c_1$ for small δ, again applying the maximum principle to (3.8) we obtain the estimate $w_\delta(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) \ge \frac{1}{2}c_1$. We have from (2.4)

$$w\big(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}\big) = \frac{\rho^{\epsilon,\delta}}{\rho^{\epsilon,\delta} - \delta} w_\delta\big(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}\big),$$

this readily leads to the bounded $w\big(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}\big) \ge \frac{1}{2}c_1 > 0$. Using the first equation in (3.1), we get $\rho^{\epsilon,\delta} \ge \delta$. The region

$$\Sigma = \Big\{(\rho, m) : z(\rho, m) \le N(T),\ w(\rho, m) \ge \tfrac{1}{2}c_1,\ \rho \ge \delta\Big\}$$

is a bounded invariant region (see *figure 3.1*) for a suitable constant $N(T)$. Then for $\rho^{\epsilon,\delta}, m^{\epsilon,\delta}$ we have the bounds

$$\delta \le \rho^{\epsilon,\delta} \le M(T), \qquad \Big|m^{\epsilon,\delta}\Big| \le M(T),$$

for a suitable constant $M(T)$, which is independent of ϵ and δ. □

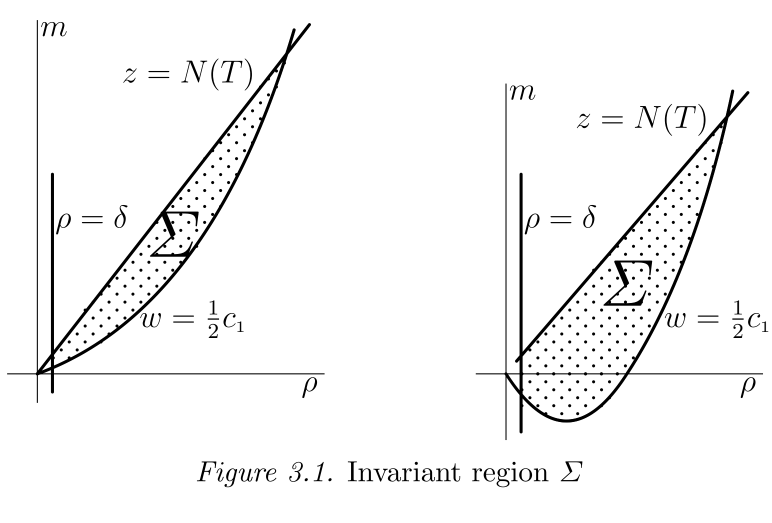

Figure 3.1. Invariant region Σ

Remark 3.2. *To understand why the condition on the constant c_6 is required in Lemma 3.1 we consider the function $p(\rho) = -\frac{1}{a}\rho^{-a} - \frac{1}{2}c_1$, $0 < a < 1$. For this function, which satisfies the hypotheses in the Lemma except the one that involves the constant c_6, the region Σ is unbounded (see figure 3.2).*

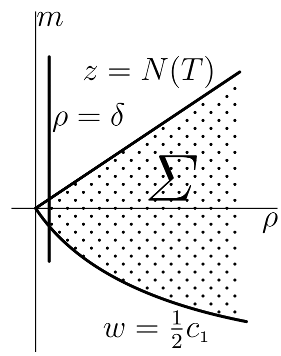

figure 3.2 Invariant region

We give below a simpler direct proof of positivity for $\rho^{\epsilon,\delta}$, proof that we adapt from Bereux and Sainsaulieu [9].

Lemma 3.3. *With assumptions given in lemma 3.1, the following a-priori bounds hold for the problem (3.1)-(1.7)*

$$\rho^{\epsilon,\delta}(x,t) \geq c(t,\epsilon,\delta) > \delta, \tag{3.9}$$

where $c(t,\epsilon,\delta)$ could tend to δ as $t \to +\infty$ or $\epsilon \to 0$.

Proof. We obtain as in the proof of Lemma 7.9.2 in [9],

$$\nu(x,t) \leq \nu_0^\epsilon(x) * k_\epsilon(x,t) + \frac{N_1}{\epsilon}t + \int_0^t \big(u - p(\rho)\big) *_x \big(k_\epsilon(x,t-s)\big)_x \, ds,$$

where $\nu = -\ln(\rho - \delta)$, $\nu_0^\epsilon(x) = -\ln\big(\rho_0(x) - \delta\big)$ and $k_\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}}\exp\left(-\frac{x^2}{4\epsilon t}\right)$. It follows from the above inequality that

$$\nu(x,t) \leq -\ln\delta + \frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}},$$

since $\rho_0(x) \geq 2\delta$. Therefore

$$\rho(x,t) \geq \delta\exp-\left(\frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\right) + \delta \geq c(t,\epsilon,\delta) > \delta > 0,$$

and thus we obtain the bounds (3.9). $\square$

From now on, $z(\cdot,t)$ denotes the function of x defined by $z(\cdot,t) = z(x,t) = z(\rho^{\epsilon,\delta}, m^{\epsilon,\delta})$.

Lemma 3.4. *Let z be the Riemann invariant given in (1.5). If the total variation of $z_0(x) = z(x,0)$ is bounded and there exist a function $G(s)$ satisfying*

$$G\Big(\tfrac{m}{\rho}\Big) = \frac{\rho-\delta}{\rho} z_\rho g_1 + z_m g_2, \qquad G'(s) \geq 0, \tag{3.10}$$

then $z_{\delta x}(\cdot,t)$ is bounded in $L^1(\mathbb{R})$, moreover

$$TV\big(z_\delta(\cdot,t)\big) = \int_{-\infty}^{+\infty} \big|z_{\delta x}(x,t)\big|\,dx \leq \int_{-\infty}^{+\infty} \big|z_{0x}(x)\big|\,dx = TV\big(z_0(x)\big). \tag{3.11}$$

Proof. Following the same kind of calculation as in the proof of the Section 2 in [4], from the equality (3.4) one readily checks the next inequality

$$|\theta|_t + \big(\lambda_\delta|\theta|\big)_x + \mathrm{sign}\theta\bigg(\frac{\rho-\delta}{\rho} g_1 z_{\delta\rho} + g_2 z_{\delta m}\bigg)_x \leq \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x,$$

where $\theta = z_{\delta x}$. By using the assumption (3.10) in the above inequality, we obtain

$$\begin{aligned} |\theta|_t + \big(\lambda_\delta|\theta|\big)_x &\leq |\theta|_t + \big(\lambda_\delta|\theta|\big)_x + G'(z_\delta)|\theta| \\ &\leq \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x, \end{aligned}$$

and integrate it over $\mathbb{R}\times[0,t]$, we conclude the result of the lemma. $\square$

Remark 3.5. *There are functions g_1, g_2 and $G(s)$ satisfying the assumptions of the lemmas 3.1 and 3.4, such as*

$$g_1(\rho,m) = 0, \quad g_2(\rho,m) = a\rho, \quad G(s) = a.$$

$$g_1(\rho,m) = 0, \quad g_2(\rho,m) = am, \quad G(s) = as.$$

$$g_1(\rho,m) = a\rho^2, \quad g_2(\rho,m) = a\rho m, \quad G(s) = a\delta s.$$

where $a > 0$ is a constant.

Let $F(\cdot)$ be any convex function of the Riemman invariant z given in (1.5), the pair

$$\big(\eta(\rho,m), q(\rho,m)\big) = \bigg(\rho F\Big(\frac{m}{\rho}\Big), (\rho-\delta)\Big(\frac{m}{\rho} - p(\rho)\Big)F\Big(\frac{m}{\rho}\Big)\bigg). \tag{3.12}$$

is a pair convex entropy-entropy flux for the system (2.1) (see [10]).

Lemma 3.6. *We assume the same conditions given in the lemmas 3.1 and 3.4. Let $g(\rho)$ be an arbitrary smooth function, then*

$$g\big(\rho^{\epsilon,\delta}\big)_t + \left(\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds + g\big(\rho^{\epsilon,\delta}\big)u^{\epsilon,\delta}\right)_x \tag{3.13}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, where $f(s) = -sp(s)$.

Proof. Multiplying the first equation of system (3.1) by $g'(\rho)$, we obtain

$$\begin{aligned} g(\rho)_t - g'(\rho)\big(\rho p(\rho) - \rho u\big)_x - \delta g'(\rho)u_x + \delta g'(\rho)p(\rho)_x =& \epsilon g(\rho)_{xx} - \epsilon g''(\rho)\rho_x^2 \\ &- (\rho-\delta)g'(\rho)h(\rho,\rho u), \end{aligned}$$

this equation is equivalent to

$$\begin{aligned} g(\rho)_t + \left(\int^{\rho} g'(s)f'(s)\,ds + g(\rho)u\right)_x &= \epsilon g(\rho)_{xx} - \epsilon g''(\rho)\rho_x^2 \\ - \delta\left(\int^{\rho} g'(s)p'(s)ds\right)_x &+ \big(g(\rho) - \rho g'(\rho) + \delta g'(\rho)\big)u_x - (\rho-\delta)g'(\rho)h(\rho,\rho u). \end{aligned} \tag{3.14}$$

By using the L^1 estimate (3.11) and taking a strictly convex function $g(\rho)$ into (3.14), we see that

$$\epsilon\big(\rho_x^{\epsilon,\delta}\big)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+), \tag{3.15}$$

and from here that $-\epsilon g''(\rho)\rho_x^2$ is also bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Since the two last terms in the right-hand side of (3.14) are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, we have that $-\epsilon g''(\rho)\rho_x^2 + \big(g(\rho) - \rho g'(\rho) + \delta g'(\rho)\big)u_x - (\rho-\delta)g'(\rho)h(\rho,\rho u)$ is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$(the space of Radon measures). On the other hand, again the estimate (3.15) together with the Cauchy-Schwarz inequality allows to establish that $\epsilon g(\rho)_{xx}$ and $-\delta\big(\int^{\rho} g'(s)f'(s)ds\big)_x$ are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Finally, the $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ compactness of (3.13) follows from Murat's lemma ([6]) since (3.13) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. This completes the proof. □

Corollary 3.7. *Assuming the hypotheses as in the lemmas 3.1 and 3.4. If $g(\rho)$ is an arbitrary smooth function then*

$$\left(\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds\right)_t + \left(\int^{\rho^{\epsilon,\delta}} g'(s)f'^2(s)\,ds + u^{\epsilon,\delta}\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds\right)_x, \tag{3.16}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, where $f(s) = -sp(s)$.

Lemma 3.8. *Let $g(\rho)$ be a smooth function and together with the hypotheses of Lemma 3.1 and Corollary 3.7, we have*

$$\Big(g\big(\rho^{\epsilon,\delta}\big)u^{\epsilon,\delta}\Big)_t + \left(g\big(\rho^{\epsilon,\delta}\big)\big(u^{\epsilon,\delta}\big)^2 + u^{\epsilon,\delta}\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)\,ds\right)_x \tag{3.17}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Multiplying the firts equation of the system (3.1) by η_ρ and the second equation by η_m, where η is the convex entropy given in (3.12). Adding up, we obtain

$$\eta_t + q_x = \epsilon\eta_{xx} - \epsilon F''(u)\rho u_x^2 - \left(\frac{\rho-\delta}{\rho}\eta_\rho g_1 + \eta_m g_2\right).$$

We can choose a strictly convex function $F(u)$ in the above equation to obtain that

$$\epsilon\rho^{\epsilon,\delta}\big(u_x^{\epsilon,\delta}\big)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{3.18}$$

In order to show the compactness of (3.17), we now multiply the equations (3.4) by $g(\rho)$ and (3.14) by z_δ, then adding the results and using that $z_\delta = u$ together with (3.11), we get

$$\begin{aligned}
\big(g(\rho)u\big)_t + \left(g(\rho)u^2 + u\int^\rho g'(s)f'(s)\,ds\right)_x &= \epsilon\big(g(\rho)u\big)_{xx} + \frac{2\epsilon}{\rho}\big(g(\rho) \\
&- \rho g'(\rho)\big)\rho_x u_x - \epsilon g''(\rho)\rho_x^2 u - \delta\left(u\int^\rho g'(s)p'(s)\,ds\right)_x + \big(g(\rho) \\
&- (\rho-\delta)g'(\rho)\big)u_x u - (\rho-\delta)g'(\rho)hu - G(u)g(\rho) + \bigg(g(\rho)u \\
&- \lambda_{\delta_1}g(\rho) + \int^\rho g'(s)f'(s)\,ds + \delta\int^\rho g'(s)p'(s)\,ds\bigg)z_{\delta x}.
\end{aligned} \tag{3.19}$$

The terms $\epsilon\big(g(\rho)u\big)_{xx}$ and $-\delta\big(u\int^\rho g'(s)p'(s)\,ds\big)_x$ in the above equation are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, the other terms on the right-hand side of (3.19) are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. The left-hand side of (3.19) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. The Murat's lemma implies the H^{-1}_{loc} compactness of (3.17). □

Lemma 3.9. *When the hypotheses in the lemmas 3.1 and 3.4 are satisfied, then a subsequence of $\{\rho^{\epsilon,\delta}\}$ and a subsequence of $\{u^{\epsilon,\delta}\}$ converge pointwisely.*

Proof. Let $g(\rho)$ be nonnegative strictly increasing and satisfying the conclusions in Lemmas 3.1,3.4 and Corollary 3.7. We use the div-curl lemma to the functions (3.13), (3.17) and then making $g(\rho^{\epsilon,\delta}) = \mu^{\epsilon,\delta}$ and $\int^{\rho^{\epsilon,\delta}} g'(s)f'(s)ds = F\big(\mu^{\epsilon,\delta}\big)$, this yields

$$\left(\overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\right)^2 - \overline{\mu^{\epsilon,\delta}}\;\overline{\mu^{\epsilon,\delta}\big(u^{\epsilon,\delta}\big)^2} = \overline{\mu^{\epsilon,\delta}}\;\overline{\mu^{\epsilon,\delta}F\big(\mu^{\epsilon,\delta}\big)} - \overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\;\overline{F\big(\mu^{\epsilon,\delta}\big)}. \tag{3.20}$$

We have that (3.16) can be written as

$$F\left(\mu^{\epsilon,\delta}\right)_t + \left(\int^{\mu^{\epsilon,\delta}} F'^2(s)\,ds + u^{\epsilon,\delta} F\left(\mu^{\epsilon,\delta}\right)\right)_x , \tag{3.21}$$

since $\int^{\rho^{\epsilon,\delta}} g'(s) f'^2(s)\,ds = \int^{\mu^{\epsilon,\delta}} F'^2(s)\,ds$.
By the div-curl lemma, for the functions (3.13) and (3.21), it follows that

$$\begin{aligned}\overline{\mu^{\epsilon,\delta}\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - F^2\left(\mu^{\epsilon,\delta}\right)} \;=\; & \overline{\mu^{\epsilon,\delta}}\;\overline{\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds} \;+\; \overline{\mu^{\epsilon,\delta}}\;\overline{u^{\epsilon,\delta} F\left(\mu^{\epsilon,\delta}\right)} \\ & - \;\overline{F\left(\mu^{\epsilon,\delta}\right)}^2 \;-\; \overline{F\left(\mu^{\epsilon,\delta}\right)}\;\overline{\mu^{\epsilon,\delta} u^{\epsilon,\delta}}\,,\end{aligned} \tag{3.22}$$

where k is real constant and the overline denotes the weak-star limit (i.e. $\overline{\mu^{\epsilon,\delta}} = w^*\!-\!\lim\mu^{\epsilon,\delta}$. Let $\overline{\mu^{\epsilon,\delta}} = \mu$, so from (3.22) we obtain that

$$\begin{aligned}&\overline{\left(\mu^{\epsilon,\delta} - \mu\right)\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \left(F\left(\mu^{\epsilon,\delta}\right) - F(\mu)\right)^2} + \left(\overline{F\left(\mu^{\epsilon,\delta}\right)} - F(\mu)\right)^2 \\ &\qquad = \overline{\mu^{\epsilon,\delta}}\;\overline{u^{\epsilon,\delta} F\left(\mu^{\epsilon,\delta}\right)} \;-\; \overline{F\left(\mu^{\epsilon,\delta}\right)}\;\overline{\mu^{\epsilon,\delta} u^{\epsilon,\delta}}\,,\end{aligned} \tag{3.23}$$

where we have used that

$$\begin{aligned}\overline{\mu^{\epsilon,\delta}\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - F^2\left(\mu^{\epsilon,\delta}\right)} \;=\; & \overline{\mu^{\epsilon,\delta}\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \left(F\left(\mu^{\epsilon,\delta}\right) - F(\mu)\right)^2} \\ + \left(\overline{F\left(\mu^{\epsilon,\delta}\right)} - F(\mu)\right)^2 + \;\overline{\mu^{\epsilon,\delta}}\; & \int_k^{\mu} F'^2(s)\,ds - \;\overline{F\left(\mu^{\epsilon,\delta}\right)}^2\end{aligned}$$

and

$$\overline{\mu^{\epsilon,\delta}}\;\overline{\int_k^{\mu^{\epsilon,\delta}} F'^2(s)\,ds} \;=\; \overline{\mu^{\epsilon,\delta}}\;\overline{\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds} \;+\; \overline{\mu^{\epsilon,\delta}}\;\overline{\int_k^{\mu} F'^2(s)\,ds}\,.$$

Since the right-hand side of both equations (3.20) and (3.23) are equal, we have

$$\begin{aligned}&\left(\overline{\mu^{\epsilon,\delta} u^{\epsilon,\delta}}\right)^2 - \;\overline{\mu^{\epsilon,\delta}}\;\overline{\mu^{\epsilon,\delta}\left(u^{\epsilon,\delta}\right)^2} \;= \\ &\overline{\left(\mu^{\epsilon,\delta} - \mu\right)\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \left(F\left(\mu^{\epsilon,\delta}\right) - F(\mu)\right)^2} + \left(\overline{F\left(\mu^{\epsilon,\delta}\right)} - F(\mu)\right)^2.\end{aligned} \tag{3.24}$$

As the left-hand side of the above equation is nonpositive and the right-hand side is nonnegative, then both sides of this equation must be zero, i.e.,

$$\overline{\left(\mu^{\epsilon,\delta} - \mu\right)\int_\mu^{\mu^{\epsilon,\delta}} F'^2(s)\,ds - \left(F\left(\mu^{\epsilon,\delta}\right) - F(\mu)\right)^2} \;+ \left(\overline{F\left(\mu^{\epsilon,\delta}\right)} - F(\mu)\right)^2 = 0, \tag{3.25}$$

and

$$\left(\overline{\mu^{\epsilon,\delta}u^{\epsilon,\delta}}\right)^2 - \overline{\mu^{\epsilon,\delta}}\ \overline{\mu^{\epsilon,\delta}\left(u^{\epsilon,\delta}\right)^2} = 0. \tag{3.26}$$

The equality (3.25) allow us to prove the pointwise convergence of $\{\mu^{\epsilon,\delta}\}$ and so we get the convergence of $\{\rho^{\epsilon,\delta}\}$ since $g(\rho)$ is a strictly increasing function. From (3.26) we get the pointwise convergence of $\{u^{\epsilon,\delta}\}$ in the region of $\rho > 0$. □

Theorem 3.10. *Let $\rho_0(x) \geq c_0$ and $w_0(x) = w(x,0) \geq c_1$ for two positive constants c_0, c_1 and let $g_1(\rho,m) = \rho h(\rho,m)$ for a continuous function $h(\rho,m)$, assume that $\frac{\rho-\delta}{\rho} z_{\delta\rho}g_1 + z_{\delta m}g_2 \geq c_2 z_\delta + c_3$, $\quad \frac{\rho-\delta}{\rho} w_{\delta\rho}g_1 + w_{\delta m}g_2 \leq c_4 w_\delta + c_5$, where c_i $i = 2,\dots,5$ are real constants and the functions $z_\delta(\rho,m), w_\delta(\rho,m)$ are the Riemann invariants given in (2.4). If the function $\rho p(\rho)$ is strictly convex for $\rho > 0$, $\lim_{\rho\to 0}\rho p(\rho) = 0$ and $\lim_{\rho\to\infty}(\rho p(\rho))' \geq c_6$, where c_6 is a constant satisfying $\frac{1}{2}c_1 + c_6 > \sup_{x\in\mathbb{R}}\left|\frac{m_0(x)}{\rho_0(x)}\right|$. Let z be the Riemann invariant given in (1.5). If the total variation of $z_0(x) = z(x,0)$ is bounded and there exist a function $G(s)$ satisfying $G\left(\frac{m}{\rho}\right) = \frac{\rho-\delta}{\rho} z_\rho g_1 + z_m g_2$, $G'(s) \geq 0$, then we have that the Cauchy problem (1.6)-(1.7) has a weak solution.*

Proof. We write the approximate system (3.1) in the weak form

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^{+\infty}\Big(\rho^{\epsilon,\delta}\varphi_t + \big(\rho^{\epsilon,\delta}-\delta\big)\Big(u^{\epsilon,\delta} - p\big(\rho^{\epsilon,\delta}\big)\Big)\varphi_x - \big(\rho^{\epsilon,\delta}-\delta\big)h\big(\rho^{\epsilon,\delta},\rho^{\epsilon,\delta}u^{\epsilon,\delta}\big)\varphi\Big)\,dt\,dx \\
&\quad + \int_{\mathbb{R}}\rho_0\varphi(x,0)\,dx + \int_{\mathbb{R}}\int_0^{+\infty}\Big(\rho^{\epsilon,\delta}u^{\epsilon,\delta}\psi_t + \big(\rho^{\epsilon,\delta}-\delta\big)u^{\epsilon,\delta}\Big(u^{\epsilon,\delta} - p\big(\rho^{\epsilon,\delta}\big)\Big)\psi_x \\
&\qquad - g_2\big(\rho^{\epsilon,\delta},\rho^{\epsilon,\delta}u^{\epsilon,\delta}\big)\psi\Big)\,dt\,dx + \int_{\mathbb{R}}\rho_0 u_0\psi(x,0)\,dx = \\
&\qquad\qquad - \epsilon\int_{\mathbb{R}}\int_0^{+\infty}\big(\rho^{\epsilon,\delta}\varphi_{xx} + \rho^{\epsilon,\delta}u^{\epsilon,\delta}\psi_{xx}\big)\,dt\,dx
\end{aligned} \tag{3.27}$$

for all functions $\varphi,\psi \in C_0^\infty\big(\mathbb{R}\times[0,\infty)\big)$. The term on the right-hand side of (3.27) goes to zero as ϵ,δ go to zero and the pointwise convergence of $\{\rho^{\epsilon,\delta}(x,t)\}$ and $\{u^{\epsilon,\delta}(x,t)\}$, ensures the existence of a weak solution of (1.6)-(1.7). □

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