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Traveling Wave Solutions for a Forced Macari System and Variable Coefficients

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Abstract

We consider a nonlinear partial differential system with variable coefficients and a forcing term, and we obtain exact solutions for it, by means of the improved tanh coth method. The obtained solutions can be used to derive solutions for the classical Macari system. We show that this last solutions are given in a more general form that those obtained in other works, showing the advantage of the used method.

Subject Classification: 35C05

Keywords: Improved tahn-coth method, traveling wave solutions, variable coefficients, Macari system

1 Introduction

The nonlinear partial differential equations (NLPDE's) are used to model several and important phenomena in various fields of the science. Clearly the search of the exact solutions for this models is a very important task today in the sense that its can be help us to understand in a better form the dynamics of those phenomena. Several direct and computational methods have been used to obtain exact solution for many of NLPDE's. Some of the most used and effective methods are the following: Hirota method [1], Lie groups [2], the Exp. function method [3], the Exp.(-$\phi(\xi)$) method [4], the tanh-coth method

[5], the improved tanh-coth method [6] and others. Many of the NLPDE's studied by using this methods have constant coefficients, however, recently, the use NLPDE's with variable coefficients (depending on the temporal variable) give us a more realistic approach in various phenomena of the physics that the previous. Some works in this directions are the following [7][8][9].

The main objective of this work is to show how the improved tanh-coth method [6], can be used to obtain traveling solutions for the following Macari system with variable coefficients and a forcing term

$$\begin{cases} \imath u_t + \delta(t) u_{xx} + \rho(t) uv = 0, \\ v_t + v_y + \mu(t)(|u|^2)_x = G(t), \end{cases} \tag{1}$$

where, u, v are the unknowns functions depending on x, y, t, $\delta(t)$, $\rho(t)$ and $\mu(t)$ are the coefficients depending only of the variable t and $G(t)$ is a forcing term. Can be seen that in the case $\delta(t) = \rho(t) = \mu(t) = 1$ and $G(t) = 0$, we obtain the classic Macari system [10][11][12]

$$\begin{cases} \imath u_t + u_{xx} + uv = 0, \\ v_t + v_y + (|u|^2)_x = 0. \end{cases} \tag{2}$$

Exact solutions for (2) have been obtained in[10][11][12] using the Exp. function method, the Exp (-$\phi(\xi)$) method and the extended trial equation and Kudryashov method respectively. Can be seen in the analysis given in [10], that the solutions derived in that work, are given in more general form that those obtained in [11]. In this work, we show that the method used in [10], can be considered as a particular case of the method used here. The two techniques are different, however, the solutions obtained with the improved tanh-coth method have a better structure that the those obtained with the Exp (-$\phi(\xi)$) method. Clearly, the solutions to (2) can be derived as particular cases of the solutions obtained for (1).

The paper is organized as follows: In Sec.2, we give a brief description of the improved tanh-coth method solving the Eq. (1); In Sec. 3, we discuss on the Exp (-$\phi(\xi)$) and the improved tanh-coth method. Finally, some conclusions are given.

2 Description of the method and solutions for Eq. (1)

The description of the method will be given step to step with the solution of Eq.(1): Given the system of nonlinear partial differential equations (1) in the variables x, y and t, the transformation

$$\begin{cases} u(x,y,t) = e^{\imath(x+y+r(t)\,t)}u(\xi), \\ v(x,y,t) = v(\xi) + \int\, G(t), \\ \xi = x + y + \lambda(t)\,t + \xi_0 \end{cases} \tag{3}$$

converts it to following system of ordinary differential equations in the unknowns $u(\xi)$, $v(\xi)$ (by simplicity, we have used the same variables u, v)

$$\begin{cases} \delta(t)u''(\xi) + (\rho(t)\int\, G(t)dt - \rho(t) - r(t))u(\xi) + \rho(t)u(\xi)v(\xi) = 0, \\ (1 - 2\delta(t))v'(\xi) + \mu(t)(u^2(\xi))' = 0, \end{cases} \tag{4}$$

where for sake of simplicity we have take $\lambda(t) = -2\delta(t)$. Here, "$'$" denote the ordinary derivation respect to ξ, $u'(\xi) = \frac{du}{d\xi}$. The improved tanh-coth method consider a solutions of (4) using the expansion

$$\begin{cases} u(\xi) = \sum_{i=0}^{M} a_i(t)\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t)\phi(\xi)^{M-i}, \\ \\ w(\xi) = \sum_{i=0}^{N} b_i(t)\phi(\xi)^i + \sum_{i=N+1}^{2N} b_i(t)\phi(\xi)^{N-i}, \end{cases} \tag{5}$$

where, $\phi(\xi)$ is solution of the Riccati equation [13]

$$\phi'(\xi) = \gamma(t)\phi^2(\xi) + \beta(t)\phi(\xi) + \alpha(t). \tag{6}$$

Now, substituting (5) into (4) and balancing the linear terms of highest order with the highest order nonlinear term in the first equation and in the second, we have $M + 2 = M + N$ and $N + 1 = 2M + 1$ respectively, so that

$$M = 1, \;\; N = 2. \tag{7}$$

Therefore, (5) reduces to

$$\begin{cases} u(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)\phi(\xi)^{-1}, \\ v(\xi) = b_0(t) + b_1(t)\phi(\xi) + b_2(t)\phi(\xi)^2 + b_3(t)\phi(\xi)^{-1} + b_4(t)\phi(\xi)^{-2}. \end{cases} \tag{8}$$

Substituting (8) into (1) and taking into account (6) we have an algebraic system in the unknowns $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $r(t)$, $a_0(t)$, $a_1(t)$, $a_2(t)$, $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$, $b_4(t)$. By space reasons, we omit here. Solving this system, we have a lot of solutions, however, we consider only the following two, which give us the most general expressions:

First Case:

$$\begin{cases} a_0(t) = -\frac{\sqrt{\delta(t)-2\delta(t)^2}\beta(t)}{\sqrt{2}\sqrt{\rho(t)}\sqrt{\mu(t)}}, \;\; a_2(t) = b_3(t) = b_4(t) = 0, \;\; b_1(t) = \frac{\sqrt{2}\delta(t)\sqrt{\mu(t)}\beta(t)a_1(t)}{\sqrt{\delta(t)-2\delta(t)^2}\sqrt{\rho(t)}}, \\ b_2(t) = \frac{\mu(t)a_1^2(t)}{2\delta(t)-1}, \;\; \gamma(t) = -\frac{\sqrt{\rho(t)}\sqrt{\mu(t)}a_1(t)}{\sqrt{2}\sqrt{\delta(t)-2\delta^2(t)}}, \\ r(t) = -\delta(t) + \rho(t)(\int G(t)dt) - \frac{\sqrt{2}\delta(t)\sqrt{\rho(t)}\sqrt{\mu(t)}\alpha(t)a_1(t)}{\sqrt{\delta(t)-2\delta^2(t)}} + \rho(t)b_0(t). \end{cases} \tag{9}$$

Now, we know that the general solution of (6) is given by [13]

$$\phi(\xi) = \frac{-\sqrt{\beta^2(t) - 4\alpha(t)\gamma(t)}\tanh[\frac{1}{2}\sqrt{\beta^2(t) - 4\alpha(t)\gamma(t)}\xi] - \beta(t)}{2\gamma(t)}. \tag{10}$$

Then, with the values given by (9) we have

$$\begin{cases} \phi(\xi) = \\ \frac{-\sqrt{\beta^2(t)-4\alpha(t)(-\frac{\sqrt{\rho(t)}\sqrt{\mu(t)}a_1(t)}{\sqrt{2}\sqrt{\delta(t)-2\delta^2(t)}})}\tanh[\frac{1}{2}\sqrt{\beta^2(t)-4\alpha(t)(-\frac{\sqrt{\rho(t)}\sqrt{\mu(t)}a_1(t)}{\sqrt{2}\sqrt{\delta(t)-2\delta^2(t)}})}\xi]-\beta(t)}{2(-\frac{\sqrt{\rho(t)}\sqrt{\mu(t)}a_1(t)}{\sqrt{2}\sqrt{\delta(t)-2\delta^2(t)}})}, \end{cases} \tag{11}$$

therefore, according with (8), (11) and (3), we have the following solution for (1)

$$\begin{cases} u(x,y,t) = e^{\imath(x+y+r(t)t)}(-\frac{\sqrt{\delta(t)-2\delta(t)^2}\beta(t)}{\sqrt{2}\sqrt{\rho(t)}\sqrt{\mu(t)}} + a_1(t)\phi(\xi)), \\ v(x,y,t) = b_0(t) + \frac{\sqrt{2}\delta(t)\sqrt{\mu(t)}\beta(t)a_1(t)}{\sqrt{\delta(t)-2\delta(t)^2}\sqrt{\rho(t)}}\phi(\xi) + \frac{\mu(t)a_1^2(t)}{2\delta(t)-1}\phi^2(\xi) + \int G(t)dt, \end{cases} \tag{12}$$

where, $r(t) = -\delta(t) + \rho(t)(\int G(t)dt) - \frac{\sqrt{2}\delta(t)\sqrt{\rho(t)}\sqrt{\mu(t)}\alpha(t)a_1(t)}{\sqrt{\delta(t)-2\delta^2(t)}} + \rho(t)b_0(t)$, $\xi = x + y - 2\delta(t)\,t + \xi_0$, $b_0(t)$, $\beta(t)$, $\alpha(t)$, $a_1(t)$ arbitrary functions depending only of variable t, ξ_0 arbitrary constant and $\phi(\xi)$ given by (11).

Second case:

$$\begin{cases} a_0(t) = -\frac{\sqrt{\delta(t)-2\delta(t)^2}\beta(t)}{\sqrt{2}\sqrt{\rho(t)}\sqrt{\mu(t)}}, \;\; a_1(t) = b_1(t) = b_2(t) = 0, \;\; b_3(t) = \frac{\sqrt{2}\delta(t)\sqrt{\mu(t)}\beta(t)a_2(t)}{\sqrt{\delta(t)-2\delta(t)^2}\sqrt{\rho(t)}}, \\ b_4(t) = \frac{\mu(t)a_2^2(t)}{2\delta(t)-1}, \;\; \alpha(t) = -\frac{\sqrt{\rho(t)}\sqrt{\mu(t)}a_2(t)}{\sqrt{2}\sqrt{\delta(t)-2\delta^2(t)}}, \\ r(t) = -\delta(t) + \rho(t)(\int G(t)dt) - \frac{\sqrt{2}\delta(t)\sqrt{\rho(t)}\sqrt{\mu(t)}\gamma(t)a_2(t)}{\sqrt{\delta(t)-2\delta^2(t)}} + \rho(t)b_0(t). \end{cases} \tag{13}$$

Then, (11) take the form

$$\begin{cases}\phi(\xi) = \\ \frac{-\sqrt{\beta^2(t)-4\gamma(t)(-\frac{\sqrt{\rho(t)}\sqrt{\mu(t)}a_2(t)}{\sqrt{2}\sqrt{\delta(t)-2\delta^2(t)}})}\tanh[\frac{1}{2}\sqrt{\beta^2(t)-4\gamma(t)(-\frac{\sqrt{\rho(t)}\sqrt{\mu(t)}a_2(t)}{\sqrt{2}\sqrt{\delta(t)-2\delta^2(t)}})}\xi]-\beta(t)}{2\gamma(t)},\end{cases} \tag{14}$$

and as before, the solution for (1) take the form

$$\begin{cases}u(x,y,t) = e^{\imath(x+y+r(t)t)}(-\frac{\sqrt{\delta(t)-2\delta(t)^2}\beta(t)}{\sqrt{2}\sqrt{\rho(t)}\sqrt{\mu(t)}} + a_2(t)\phi^{-1}(\xi)), \\ v(x,y,t) = b_0(t) + \frac{\sqrt{2}\delta(t)\sqrt{\mu(t)}\beta(t)a_2(t)}{\sqrt{\delta(t)-2\delta(t)^2}\sqrt{\rho(t)}}\phi^{-1}(\xi) + \frac{\mu(t)a_2^2(t)}{2\delta(t)-1}\phi^{-2}(\xi) + \int G(t)dt,\end{cases} \tag{15}$$

where, $r(t) = -\delta(t) + \rho(t)(\int G(t)dt) - \frac{\sqrt{2}\delta(t)\sqrt{\rho(t)}\sqrt{\mu(t)}\gamma(t)a_2(t)}{\sqrt{\delta(t)-2\delta^2(t)}} + \rho(t)b_0(t)$, $\xi = x + y - 2\delta(t)\,t + \xi_0$, $b_0(t)$, $\beta(t)$, $\gamma(t)$, $a_2(t)$ arbitrary functions depending only of variable t, ξ_0 arbitrary constant and $\phi(\xi)$ given by (14).

3 Results and Discussion

The expressions (12) and (15) are solutions to (1). We can see that if we made $\delta(t) = \rho(t) = \mu(t) = 1$ and $G(t) = 0$, we obtain exact solution to (2). In the reference [10], the authors have obtained solutions to (2) using the Exp(-$\phi(\xi)$) method, however, our solutions have a best structure that those obtained in [10]. For instance, can be seen the expressions given for function v. On the other hand, the Exp(-$\phi(\xi)$) can be considered as a particular case of our method in the following sense: if we made $\phi(\xi) = e^{\Phi(\xi)}$ in the equation used in the Exp(-$\phi(\xi)$) method, we obtain a particular case of equation (6) and in this case, the structure of the used solution given by them, converts to special case of (5). Moreover, when we transform (1) into (4) an imaginary term appear, is not clear what happen with this term in [10][11][12]. Finally, the authors in [10][11][12] , reduces (4) to one unique equation, making as zero an integration constant. In this work, this fact is not necessary reason for which, the solutions are given in a more general form. Finally, the sign of $\beta(t)^2 - 4\alpha(t)\gamma(t)$ can be give us other type of solutions (see [13]). In the same way, varying the parameter ξ_0 in (3) we obtain other expression for the solutions.

4 Conclusion

We have used the improved tanh-coth method to solve the generalized system (1). Solutions for the classical Macari system (2) are obtained as particular case. We have solved the system using the two equations, so that the obtained solutions for (2) are more general that those obtained, for instance, in [10] and [11]. The method is a very important tool to solve many NLPDE in a satisfactory way.

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