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# On Exact Solutions for (4+1)-Dimensional Fokas Equation with Variable Coefficients

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## Abstract

Exact solutions for a generalization of the (4+1)-dimensional Fokas equation are derived by means of the improved tanh-coth method. Variable coefficients (depending on the temporal variable) are considered, so that we can derive solutions for particular cases of the (4+1)-dimensional Fokas equation. By means of the 3D-profile of some of the solutions, we can show the variety of structures that we can obtain from the generalized model.

**Keywords:** Improved tanh-coth method, traveling wave solutions, variable coefficients, (4+1)-dimensional Fokas equation

## 1 Introduction

The objective of this paper is to construct exact traveling wave solutions for the following model

$$A(t)u_{tx} - B(t)u_{xxx} + B(t)u_{xyy} + D(t)u_xu_y + D(t)uu_{xy} + G(t)u_{zw} = 0, \quad (1)$$

where the coefficients  $A(t)$ ,  $B(t)$ ,  $D(t)$  and  $G(t)$  are functions depending only on the variable  $t$ . In the particular case  $A(t) = 4$ ,  $B(t) = 1$ ,  $D(t) = 12$  and  $G(t) = -6$ , we obtain the classical (4+1)-dimensional Fokas equation

$$4u_{tx} - u_{xxx} + u_{xyy} + 12u_xu_y + 12uu_{xy} - 6u_{zw} = 0, \quad (2)$$

recently derived by Fokas [1] after extension of the Lax pairs to some higher-dimensional nonlinear wave equation of the Kadomtsev-Petviashvili (KP) and the Davey-Stewartson (DS) equations. The importance of the (4+1)-dimensional Fokas equation have to see with the following two facts: First, the physic applications are related with the applications of those two equations previously mentioned (KP and DS equations), so that, the model is very important in the nonlinear theory of water waves [2]. Second, as was mentioned in [3], equation (2) suggest that the idea of complexifying of the time can to have relevance in the advanced of modern nonlinear theory. The equation (2) have been studied from of point of view of its traveling wave solutions by the authors of [3] using the modified simple equation method (MSEM) and the extended simplest equation method (ESEM). Others treatment have been made in the references [4][5][6][7]. The study of Eq. (1) have relevance in the sense that it is a generalized model and therefore, from the mathematical point of view, is an important fact due to that particular cases as (2) can be derived. On the other hand, el use of the variable coefficients (depending on the variable  $t$ ) give us a variety of some new traveling wave solutions with several structures which can be used by the physicist to understand in a better form the phenomenon described by the model. A variety of nonlinear partial differential equations with variable coefficients have been considered in some branch of the sciences, it can be seen, for instance in the references [8][9][10].

## 2 Traveling wave solutions for Eq. (1)

We begin assuming that (1) has solutions of the form

$$\begin{cases} u(x, y, z, w, t) = v(\xi), \\ \xi = \tau x + \rho y + \delta z + \mu w + \lambda t + \xi_0. \end{cases} \quad (3)$$

In this last expression,  $\lambda$  is considered as the speed of the wave and  $\xi_0$  arbitrary constant. Now, substituting (3) into (1) we have the following ordinary differential equation

$$\begin{cases} A(t)\lambda\tau v''(\xi) - B(t)(\tau^3\rho - \tau\rho^3)v'''(\xi) + D(t)\tau\rho(v'(\xi))^2 + \\ D(t)\tau\rho v(\xi)v''(\xi) + G(t)\delta\mu v''(\xi) = 0. \end{cases} \quad (4)$$

Using the idea of the improved tanh-coth method [11], we search solutions to (4) by using the expression

$$v(\xi) = \sum_{i=0}^M a_i(t)\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t)\phi(\xi)^{M-i}, \quad (5)$$

where  $M$  is a positive integer to be determinate latter,  $a_i$  functions depending on the variable  $t$  and  $\phi(\xi)$  satisfies the general Riccati equation

$$\phi'(\xi) = \alpha(t) + \beta(t)\phi(\xi) + \gamma(t)\phi(\xi)^2, \quad (6)$$

whose solutions are determined as [12]:

$$\phi(\xi) = \begin{cases} \frac{1}{\gamma(t)}\left(-\frac{1}{\xi+\xi_0} - \frac{\beta(t)}{2}\right), & \beta^2(t) - 4\alpha(t)\gamma(t) = 0, \\ \frac{-\sqrt{\beta^2(t)-4\alpha(t)\gamma(t)}\tanh[\frac{1}{2}\sqrt{\beta^2(t)-4\alpha(t)\gamma(t)}\xi]-\beta(t)}{2\gamma(t)}, & \beta^2(t) - 4\alpha(t)\gamma(t) \neq 0. \end{cases} \quad (7)$$

Substituting (5) into (4) and balancing  $v'''(\xi)$  with  $(v'(\xi))^2$  we have  $M + 4 = 2(M + 1)$ , so that

$$M = 2.$$

Therefore, (5) reduces to

$$v(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)\phi(\xi)^2 + a_3(t)\phi(\xi)^{-1} + a_4(t)\phi(\xi)^{-2}. \quad (8)$$

Now, substitution of (8) into (4) leads us to an algebraic system in the unknowns  $a_i(t)$ , ( $i = 1, \dots, 4$ ),  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ ,  $\lambda(t)$ ,  $\tau(t)$ ,  $\delta(t)$ ,  $\rho(t)$  and  $\mu(t)$ . By using *Mathematica*, we obtain a lot of solutions of the resultant system, however, for sake of simplicity we consider only the following solution (other solutions are only particular cases):

$$\begin{cases} a_1(t) = -\frac{12B(t)\beta(t)\gamma(t)(\rho^2(t)-\tau^2(t))}{D(t)}, \quad a_2(t) = -\frac{12B(t)\gamma^2(t)(\rho^2(t)-\tau^2(t))}{D(t)}, \\ a_3(t) = a_4(t) = 0, \\ \lambda(t) = \frac{\begin{cases} -G(t)\delta(t)\mu(t) - B(t)\beta^2(t)\rho^3(t)\tau(t) - 8B(t)\alpha(t)\gamma(t)\rho^3(t)\tau(t) + \\ B(t)\beta^2(t)\rho(t)\tau^3(t) + 8B(t)\alpha(t)\gamma(t)\rho(t)\tau^3(t) - D(t)\rho(t)\tau(t)a_0 \end{cases}}{A(t)\tau(t)}. \end{cases} \quad (9)$$

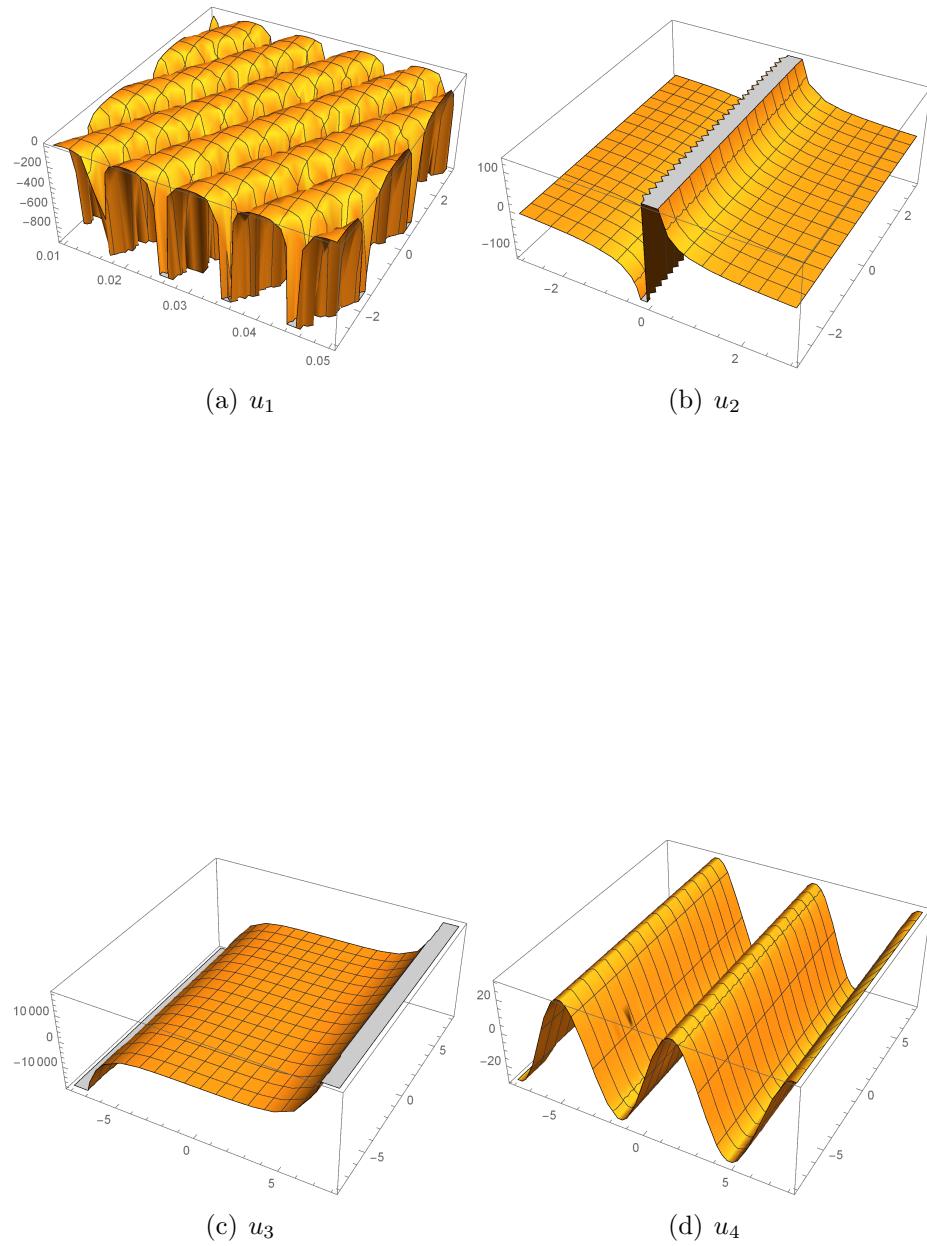
Clearly, with this values, the respective solution for (6) is given by (7), with  $\alpha(t)$ ,  $\beta t$  and  $\gamma(t)$  arbitrary functions depending on the variable  $t$ . In this order of ideas and taking into account (8) and (3) the respective solution to (1) is given by

$$u(x, y, z, w, t) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)\phi(\xi)^2, \quad (10)$$

where  $\phi(\xi)$  satisfies (7),  $a_0(t)$  arbitrary function on the variable  $t$ ,  $a_1(t)$  and  $a_2(t)$  the respective values given in (9), and  $\xi = \tau(t)x + \rho(t)y + \delta(t)z + \mu(t)w + \lambda(t)t + \xi_0$  being  $\lambda(t)$  the expression given in (9).

### 3 Discussion

In the previous solutions, it is clear that, according with the sign of  $\beta^2(t) - 4\alpha(t)\gamma(t)$  we can obtain other type of solutions [12]. We have the same consequence by varying the value of the arbitrary parameter  $\xi_0$ . Many authors, as in [3], reduces the equation (4) after integration and taking the integration constant as zero. With this, the solution given by them loss generality, in accordance with [13]. This is not the case in this work. The figure  $u_1$  is the 3D-profile of the solution (10) for the following values:  $A(t) = 4$ ,  $B(t) = 1$ ,  $D(t) = 12$ ,  $G(t) = -6$ ,  $\beta(t) = 1$ ,  $\gamma(t) = 2$ ,  $\alpha(t) = 1$ ,  $\rho(t) = 4$ ,  $\tau(t) = 1$ ,  $\delta(t) = 1$ ,  $a_0(t) = 0$ ,  $z = 1$ ,  $y = 1$ ,  $w = 1$  and  $(x, t) \in [-3, 3] \times [0.01, 0.05]$ . In the same way, the figure  $u_2$  is the 3D-profile of the solution (10) for the values:  $A(t) = 4$ ,  $B(t) = 1$ ,  $D(t) = 12t$ ,  $G(t) = -6$ ,  $\beta(t) = 6$ ,  $\gamma(t) = 2$ ,  $\alpha(t) = 1$ ,  $\rho(t) = 4$ ,  $\tau(t) = 1$ ,  $\delta(t) = 1$ ,  $a_0(t) = 0$ ,  $z = 1$ ,  $y = 1$ ,  $w = 1$ ,  $(x, t) \in [-3, 3] \times [-3, 3]$ . The graph  $u_3$  correspond to 3D-profile for the values  $A(t) = 4$ ,  $B(t) = \sinh t$ ,  $D(t) = 12$ ,  $G(t) = -6$ ,  $\beta(t) = 6$ ,  $\gamma(t) = 2$ ,  $\alpha(t) = 1$ ,  $\rho(t) = 4$ ,  $\tau(t) = 1$ ,  $\delta(t) = 1$ ,  $a_0(t) = 0$ ,  $z = 1$ ,  $y = 1$ ,  $w = 1$ ,  $(x, t) \in [-8, 8] \times [-8, 8]$  and finally,  $u_4$  is the 3D-profile corresponding to values  $A(t) = 4$ ,  $B(t) = \sin t$ ,  $D(t) = 12$ ,  $G(t) = -6$ ,  $\beta(t) = 6$ ,  $\gamma(t) = 2$ ,  $\alpha(t) = 1$ ,  $\rho(t) = 4$ ,  $\tau(t) = 1$ ,  $\delta(t) = 1$ ,  $a_0(t) = 0$ ,  $z = 1$ ,  $y = 1$ ,  $w = 1$ ,  $(x, t) \in [-8, 8] \times [-8, 8]$ .  $u_1$  is a periodic solution, and  $u_2$ ,  $u_3$  and  $u_4$  correspond to solitons with different structure.



## 4 Conclusion

We have obtained exact traveling wave solutions for the (4+1)-dimensional Fokas equation with variable coefficients. From the obtained solutions, we can derived periodic and soliton solutions for particular cases of it, such as (2) where a periodic solution is showed in figure  $u_1$ . Clearly, using several values for the coefficients we can obtain solutions with a variety of structures (figures  $u_2$ ,  $u_3$ ,  $u_4$ ), from which, we can obtain a best approach of the realistic model described by the model (2). Due to solution obtained for (1) have a variety of free parameters, this can be used in many applications of physics and engineering. Following the references [8][9][10], I thing that the use of variable coefficients is a new important line of investigation in the area of the nonlinear analysis.

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