From the Local Observer in QM
to the Fixed-Point Observer in GR

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Abstract

After the introduction by I.E. Segal of the local observer in his chronogeometrical theory of relativity in [25], I proposed the introduction of the local observer in quantum mechanics [13]. In this paper, the notion of local observer is extended to a fixed-point observer in general relativity as a topological invariant under homeomorphisms. While Einstein’s equivalence principle allows for the passage from SR to GR through a local Lorentz frame, the topological invariant provides a unique (homeomorphic) link from the local observer in QM to the fixed-point observer in GR. Mathematical tools from algebraic topology and transfinite set theory are essentially used.

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1. Introduction

In his 1976 Mathematical Cosmology and Extragalactic Astronomy. I.E. Segal introduced the chronogeometrical local observer as the mathematical counterpart of a local Lorentz frame. Segal’s observer was located in a globally hyperbolic (causal) space as an open connected set of a topological manifold endowed with metric, homogeneous, physical (covariant) properties. The topological setting was meant to cope with causal diffeomorphisms. In Gauthier [13] Quantum Mechanics and the Local Observer, the local observer was topologically the local complement of
the Hilbert space of QM. Here we don’t need causal diffeomorphisms, mainly because we are in an acausal context and partly because of differential defects in 4-dimensional soft manifolds as locally Euclidean spaces (the Donaldson-Friedman deformation theory in the 1980s). Only the more general homeomorphisms between \( n \)-dimensional spaces and \( \mathbb{R}^n \) secure the algebraic-topological status of the local observer from QM to GR. In that context, a general theorem is demonstrated about the local observer as a topological invariant in a stochastic framework in which the local observer of QM encompasses the relativistic observers in SR and GR even in a cosmological multiverse scenario as a unifying principle from QM to GR.

2. The topological location of the local observer in QM

The usual Hilbert space formalism for QM is used here to define the topological location of the local observer. I retrieve here a theorem and its proof from my 1983 paper. Consider Hilbert space as a metric and a topological space; \( D \) is in this case the set of subspaces of the Hilbert space and \( E \) is obtained by local complementation; \( E \) is the "location" of the local observer –.

We shall see that Hilbert space can make room for a notion of local observer: the observer becomes the (local) complement of the observable, i.e. the closed linear manifolds of the Hilbert space – of course the whole Hilbert space contains all bounded linear transformations (defined on open subsets) and is therefore not orthocompleteable – but here we obtain non-orthocompleteability in a different way. (Remember that in a finite-dimensional space, every linear manifold is closed).

**THEOREM 1.** Hilbert space admits the observer as an open set through local negation (or complementation) – that is, we do not have orthocomplementation on the whole Hilbert space even in the finite-dimensional case.

**Proof.** Let \( H \) be an \( n \)-dimensional Hilbert space and let \( F^\perp \) be the set of closed linear manifolds of the Hilbert space – of course the whole Hilbert space contains all bounded linear transformations (defined on open subsets) and is therefore not orthocompleteable – but here we obtain non-orthocompleteability in a different way. (Remember that in a finite-dimensional space, every linear manifold is closed).

From the topology, we pass to the metric of \( H \); for the metric of \( H \), a subset \( A \) of \( H \) is located, if the distance

\[
\forall x \in H \left[ \rho(x, A) \equiv \inf \{ \rho(x, y) : y \in A \} \right]
\]

from \( x \) to \( A \) exists. The metric complement \( -A \) of a located subset \( A \) is the set

\[
-A \equiv \{ x : x \in H, \rho(x, A) > 0 \}
\]

which is open, since

\[
\forall x, y \in H \left[ \rho(x, A) \leq \rho(x, y) + \rho(y, A) \right].
\]
Here the observer has a topological and metrical place as the local complement of the closed set of subspaces of \( H \). Brouwer had introduced the notion of located subset (or subsequence) and E. Bishop has put it to use in his constructive analysis \[1\]. In order to further constructivize this result, the topological boundary operator \( b \) is introduced and it is to be interpreted as the boundary between the observable (or observed system \( D \)) and the observer (the observing system \( E \)) : we have the relations

\[
E = \neg D - b(E)
\]

and

\[
D = \neg E \cup b(D)
\]

thus

\[
\neg D(H) = E(H) - b(E(H)).
\]

The interior of \( E \), \textit{i.e.} \( E^\circ \), is the complement of the closure of the complement of \( E \) and is thus open; we have also

\[
E = E^\circ.
\]

For any \( x \), \( D(\neg x) \) means that \( x \in E \). So for some \( a \), we have

\[
E(a) = D(\neg a) - b(D(\neg a));
\]

On the other hand, the closure of \( D \), \textit{i.e.} \( D^\circ \) implies that

\[
B(D(\neg a)) = a^\circ \cap (D-a)^\circ.
\]

Hence

\[
A^\circ = a \cup b(D(\neg a))
\]

and

\[
a^\circ \in E(H) = a \in D(H) \cup b(a \in D(H))
\]

and

\[
a \in D(H) = a^\circ \in E(H) - b(E(H))
\]
which shows that $E$ is disjoint from its boundary, that is, it is open and consequently the whole Hilbert space $D(H) \cup E(H)$ is not orthocomplementable, since local complementation excludes $(\bar{a}) = a$.

**Remark:** The effect of abandoning orthocomplementation amounts to adopting an indefinite metric which may, in fact, be more convenient for some physical theories (e.g. quantum field theories) where the local observer has still its topological prevailing location. The finite derivation of the result in a modular polynomial calculus can be found in (Gauthier [17], chapter 5).

3. **A Markovian observer**

A Markovian observer can be defined as opposed to a deterministic Laplacian observer or the observer on Sirius who can describe the initial conditions of a system in order to predict its final state. The stochastic Markov-state observer has no memory and its predictions are based only on actual observations, experiments and measurements realized in actual discrete time on a discrete state space. Probability can consequently be defined in terms of a finite probability space (E. Nelson [22]) : a finite probability space is a finite set $\Omega$ and a (strictly positive) function $pr$ on $\Omega$ such that for $\omega \in \Omega$

$$\sum pr(\omega) = 1$$

and expectation is defined

$$Ex = \sum x(\omega) pr(\omega)$$

for a random variable $x$; the probability of an event $A \subseteq \Omega$ is

$$Pr A = \sum_{\omega \in A} pr(\omega).$$

Nelson also defines the complementary event as $A^c = \Omega \setminus A$ for all $\omega \in \Omega \setminus A$. This is the Boolean complement which we replace by our local complement $(\Omega - a) + b$ or $(1 - a) + b$ for which we have a finite derivation (see again Gauthier [18], chapter 5).

What is the use of a finite probability calculus in QM? Von Neumann’s work in 1927-1932 focuses on what is called the finiteness of the eigenvalue problem. The point here is that any calculation is finite and since we have only finite results, those must be the outcomes of a finite calculation which is itself made possible only if the analytical apparatus contains the mathematical structures which enable such calculations. Such a formalism is the complex Hilbert space with

$$|\psi| \in L^2(\mu)$$
where \( \mu \) is a real positive measure on the functional space \( L^1 \) (i.e. the equivalence class of square-integrable functions). The integral

\[
\int |\psi|^2 d\mu
\]

is finite, which is equivalent to the fact that, in the theory of bounded quadratic forms, the sum

\[
K(x,x) = \sum_{p,q=1}^\infty k_p x_p x_q
\]

of all sequences \( x_1, x_2, \ldots \) (of complex numbers) is finite in an orthonormal system of vectors. That mathematical fact, which Hilbert derived in the theory of integral equations in 1907, states that a linear expression

\[
k_1 x_1 + k_2 x_2 + \ldots
\]

is a linear function, if and only if the sum of the squares of the coefficients in the linear expression \( k_1, k_2, \ldots \) is finite. The theorem, inspired by Kronecker’s result on linear forms (homogeneous polynomials), is the very basis of the Hilbert space formulation of QM. Notice that on the probabilistic or statistical interpretation, the "acausal" interaction between an observed system and an observing system takes place in a given experimental situation and produces a univocal result of finite statistics for real or realized measurements.

In order for real measurements to have real positive probability values, the analytical apparatus must satisfy certain realizability conditions, \(<\text{Realitätsbedingungen}>\), as Hilbert puts it [20]. For example, orthogonality for vectors, linearity and hermiticity for functional operators and the finiteness of the eigenvalue problem for Hermitian operators, as in von Neumann’s \textit{Mathematische Grundlagen der Quantenmechanik} [27], are such constraints of realizability.

In that connection, Born rule

\[
|\Psi \Psi^*|^2
\]

determines the probability calculus of acausal stochastic processes like atomic frequencies to which Born assigns a probability function written as \( \Phi(\alpha,\omega) \) (with a time factor) in his 1926 paper [2]). He adds in a note that it is only after a more precise calculation that he found that the probability is proportional to the square of the \( \Phi \) function making it non-linear, although he confessed in his 1954 Nobel Prize lecture that he was influenced at first by Einstein’s idea of the square of optical wave amplitudes as probability density for the occurrence of photons (see Born [3]). Born also says that he thinks that the atomic world is not determinate, but he admits that this is a philosophical question and Born adds that it is not conclusive.
However, Heisenberg’s uncertainty relations with the fundamental non-commutation relation for conjugate canonical variables, e.g. position and momentum

\[ pq - qp = i\hbar \quad \text{with} \quad \hbar = h/2\pi \]

retains some indeterminacy (Unbestimmtheit) within the measurement process itself, while Bohr’s complementarity principle attempts to recover some determinacy in terms of complementary descriptions of causal conservation laws for dynamical variables and spacetime coordinates.

In any case, it is assumed that in the context mentioned above, it is the finiteness of the measurement process of eigenvalues in an orthonormal system of vectors that is responsible for the Born rule besides the linear wave function of time evolution in Schrödinger’s partial differential equation

\[ i\hbar \frac{\partial}{\partial t} \psi(r, t) = H\psi(r, t) \]

with values in \( \mathbf{R} \) and \( \mathbf{C} \), the the non-denumerable cardinality of which is \( 2^{\aleph_0} \) – in the following non-denumerable and uncountable are taken as synonyms as are denumerable and countable –.

A memoryless Markovian observer would count only on a finite set of experiments or measurements to predict statistically the future behaviour of a quantum-mechanical system and would tend to identify QM with statistical mechanics as Nelson suggested. A Bayesian observer on the other side would use a calculus of prior probabilities to retain a causal link to the past behaviour of the observed system in a more or less deterministic spirit.

4. The relativistic observer

The relativistic observer is definitely determinist. In order to define the local observer in relativity theory, I start with a thought experiment on a space-time vessel for the derivation of special relativity (SR). Take an inertial system, the space-time vessel at rest with \( \ell \) the length or the height of the stationary mast and set the system in uniform motion with velocity \( v \)
A light beam leaves L, is reflected by N and returns at O; in this scheme \( ct' \) is the distance traveled by the light beam and \( vt' \) the distance traveled by the system in uniform motion while \( \ell \) is simply the distance traveled by a light signal in an inertial system. We have \( t = \frac{\ell}{c} \), we have also the Pythagorean theorem for a right-angled triangle with sides \( a \) and \( b \) and hypotenuse \( c : \ c^2 = a^2 + b^2 \) which gives

\[
(ct')^2 = t^2 + vt'^2
\]

then

\[
\ell = \sqrt{(ct')^2 - (vt')^2}
\]

by \( t = \frac{\ell}{c} \), one has

\[
t = \sqrt{(ct')^2 - (vt')^2} - \frac{vt'^2}{c^2} = t'\sqrt{c^2 - v^2/c^2}
\]

which gives

\[
t' = \frac{t}{\sqrt{1 - v^2/c^2}}
\]

The derivation of the central formula in SR in six easy steps:

1. The Pythagorean theorem : \( c^2 = a^2 + b^2 \)
2. \( (ct')^2 = \ell^2 + vt'^2 \)
3. \( \ell = \sqrt{ct^2 - vt'^2} \)
4. \( t = \sqrt{ct^2 - vt'^2} - \frac{vt'^2}{c^2} = t'\sqrt{c^2 - v^2/c^2} \)
5. \( t' = \sqrt{1 - v^2/c^2} \)
6. \( t' = t / \sqrt{1 - v^2/c^2} \).

Consequences:

1. \( t' = t / \sqrt{1 - v^2/c^2} \leq 1 \) for time dilation
2. \( l' = l / \sqrt{1 - v^2/c^2} \geq 1 \) for Lorentz-Fitzgerald length contraction.

The transition from inertial systems to local Lorentz frames in accelerated systems offers a generalization from SR to GR via Einstein equivalence principle and the general covariance principle for coordinate transformations in a Minkowskian universe. Recall that I.E. Segal has defined his topological observer as a local Lorentz frame to obey the equivalence principle. Recently, S. Nomura in his paper [23] has introduced a metric (geodesic) observer in a supersymmetric Minkowskian universe with the metric invariant

\[
\text{ds}^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2
\]

for the metric element in SR and

\[
\text{ds}^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{for } \mu, \nu = 1, \ldots, n
\]
for the fundamental tensor of GR pseudo-Riemannian metric for
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \]
Einstein’s field equations of GR in the \( \mu\nu \) geodesic parameters of a Minkowskian manifold. In Nomura’s observable universe (a matter bulk with an apparent horizon) which he claims is completeley described by the universal wave function \( \psi(t) \), the single observer sits on a geodesic that I interpret as the origin \( O \) in the following Penrose diagram

\[ O \]

Let us assume that spacetime is a 4-dimensional sphere or an hypersphere in \( \mathbf{R}^4 \) and we take the 3-surface of this quadridimensional hypersphere to obtain a spherical space. One can then derive the linear element \( ds^2 \) by symmetry stipulations which lead to a closed sphere \( S^2 \). From the viewpoint of \( O \), \( S^2 \) has no privileged direction and the universe is isotropic and homogeneous according to the cosmological principle. The origin \( O \) can be considered as a fixed point or a vector of length 0. We have here Brouwer’s theorem:

**THEOREM 2.** Any closed ball is a space with a fixed point in \( \mathbf{R}^n \) provided that any homeomorphism in \( S^2 \) contains a fixed point, that is there must be a vector of length 0 in the vector field of \( S^2 \).

**Proof.** Immediate. See section 5 for details.

The vector of length zero corresponds topologically to an open set separating the local observer from the closed sets of the closed ball and the equivalence principle guarantees that the local Lorentz frame as an open set is projected onto the GR cosmic scene. This is our local observer or fixed-point observer in a spacetime continuum which is an \( n \)-dimensional manifold homeomorphic to the Euclidean space \( \mathbf{R}^n \). In that connection the continuum is a closed connected set of points and an other theorem of Brouwer (see [5, 6]) states that an \( n-1 \) dimensional continuum separates an \( n \)-dimensional Euclidean space in two regions, in our case, the region of the fixed –point observer and the region of the observable universe.
Nomura’s mega-universe of eternal inflation and infinite non-cloning universes mentioned above has fractal dimension, but its topology is embeddable in a Banach space and the contraction mapping theorem implies the existence of a fixed point in metric spaces, in such a way that Nomura’s supersymmetric Minkowski space is topologically contractible to a point as if the observable universe would collapse in Nomura’s single observer, our local fixed-point observer!

The same goes for other cosmological multiversal theories like the brane cosmology of M-supersymmetric theory in 11 dimensions which limits the number of universes around \(10^{500}\), since the D-brane or « multibrane » universe would not allow for homeomorphic images, except for homological mirror symmetry (Kontsevich). All these cosmologies must admit dimensional asymmetry, but they have to make room somehow for the local observer.

From the perspective of the local fixed-point observer, the cosmological holographic principle which reduces the dimensions of 3-dimensional universe to a 2-dimensional sphere is certainly compatible with our result, since everything happens outside the local observer and the physical universe appears as a homeomorphic image to the same local observer. In the more general string-theoretic landscape with conformal field theory CFT, one has the ADS/CFT (ADS for anti-de Sitter space) duality with the Maldacena-Witten equivalence principle in which a higher dimension could be holographically reduced by one dimension on such spaces as Calabi-Yau or Kähler manifolds, all complex structures defined on \(\mathbb{C}\). Here is an apparent contradiction with Brouwer homeomorphic fixed-point invariant, since duality equivalence, maybe via homology and homotopy concepts which are weaker than homeomorphism, could seem to imply homeomorphic images of worlds in different dimensions \(m \neq n\), but there is no problem anyway for the fixed –point invariant as the local observer or holographer. Put differently, the local observer is a fixed-point observer in interaction with the observable universe and even though string theory (superstring or M-theory) is a viable theory as a finitary renormalizable theory, it should be free of doppelgängers or ghost observers…

Let us remark finally that the local observer from a topological point of view is a fixed-point observer in interaction with the observable universe and as a primary topological invariant, it is gauge invariant and can trespass dimensional barriers. The local observer has nothing to do with the anthropic principle, since it is any observing system, may it be human or non-human, experimental apparatus or robot, macroscopic or microscopic, at the center of the observable universe.

5. The fixed-point observer

I recall the general theorem formulated in my 2013 paper (see Gauthier [16])

THEOREM 3. In an \(n\)-dimensional infinite universe, there cannot be an exact reproduction or homeomorphic images of items (finite or countably infinite) in any finite or countably infinite \(\aleph_0\) subuniverse of the uncountable multiverse of the power of the continuum.
Proof. I skip the proof given in the paper to adopt it to the notion of the fixed-point observer in GR in the following theorem:

**THEOREM 4.** The local observer as a fixed-point observer is the central topological invariant under homeomorphisms being located in an open set of dimension zero in any universe of finite or infinite dimension.

The topological setting: Let us consider an $n$-dimensional Riemannian (semi-Riemannian) or Lorentzian spacetime for GR. We are interested in the topological manifold, a spacetime deprived of its metric field which is locally Minkowskian. What is relevant here is that the topological manifold locally corresponds to an $n$-dimensional space, i.e. a topological space is locally Euclidean and a locally Haussdorf space as an $n$-dimensional manifold is homeomorphic to $\mathbb{R}^n$. Such a setting is also appropriate for the topological field theory of string theory in some finite dimension (10,11, 26) with closed strings, loops and knots in at least $10^{500}$ universes.

The most general setting is however set-theoretic combinatorial: an arbitrary subset $X$ of a denumerably infinite set $\mathbb{N}_0$ is included in the power set of that set, that is

$$X \subseteq P(X)$$

and by the Cantor theorem on cardinalities

$$\forall X \,(\text{card}\,X < \text{card}\,P(X)),$$

the power set $P(X) > X$, $P(X) = 2^X$ and for a denumerable set $\mathbb{N}_0$

$$P\{\mathbb{N}_0\} = 2^{\mathbb{N}_0} = c \,(\text{the continuum})$$

and $P\{\mathbb{N}_0\}$ contains $\emptyset$, $\{\text{finite subsets}\}$, $\{\text{denumerable subsets } \mathbb{N}_0\}$, $\{2^{\mathbb{N}_0}\}$. Thus the continuum has cardinality $2^{\mathbb{N}_0}$ and – independently of the Continuum Hypothesis which asserts that $2^{\mathbb{N}_0} = \mathbb{N}_1$ or the Generalized Continuum Hypothesis

$$\forall \sigma (\mathbb{N}_o \prec 2^{\mathbb{N}_o} = \mathbb{N}_{o+1}) \ldots$$

Cantor has shown that all continua in any dimension are *isomorphic* (see Cantor [7]), but Brouwer has subsequently shown that they are not *homeomorphic*, that is the bijection is not continuous, for example there is no continuous homeomorphism between points on a circle and points on a sphere.

Recall that the power set $2^n$, the set of all the subsets of a given set $n$, represents all the combinations $C$ of the elements or members of the set. There is no bijection between $\mathbb{N}_0$ and $P(\mathbb{N}_0)$ and *a fortiori* there is no continuous bijection between an arbitrary subset, finite or denumerably infinite ($\mathbb{N}_0$), and the power set of cardinality $2^{\mathbb{N}_0}$. This is the situation in metric spaces with Hausdorff dimension, finite and infinite (or transfinite). Turning to the topological dimension setting, we have topological spaces (manifolds) which are locally Euclidean $\mathbb{R}^n$. Here we deal with homeomorphic mappings and since Brouwer’s proof on the invariance of the
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dimension number works for open sets and the system of their neighbourhoods, the argument is straightforward: there is no homeomorphic image of an $n$-universe to an $m$-universe for $m \neq n$. We can still specify our argument to metric topological spaces, for example a Hilbert space which is also a Banach space where we have the open mapping theorem which sends open sets to open sets between Banach spaces of finite or infinite ($\aleph_0$) topological dimension. Moreover, the sequence of linear subspaces of a Banach space has a local open complement by the following argument:

Let $B$ be an $n$-dimensional Banach space and let $F^+$ be the sequence of closed linear manifolds or subspaces $F^+$ of $B$. Set $F^+ = F^+$, the closure of all $F^+$. The local complement $F^-$ of $F^+$ such that

$$F^+ = F^+ - F^-$$

is an open subset of $B$. If we take a locally convex space as the space $B$, the sequence $G$ of open subsets $g$ containing a neigbourhood of each of their points admits readily a local complement which is also open (by locality), since the metric complement

$$-A = \{x : x \in X, \rho(x, A) > 0\}$$

of a located subset $A$, i.e.

$$\rho(x, A) = \inf \{\rho(x, y) : y \in A\}$$

(if this distance from $x$ to $A$ exists $\forall x \in X$) is open, for we have

$$\rho(x, A) \leq \rho(x, y) + \rho(x, A) \quad (\forall y \in A)$$

(as we have seen in THEOREM 1). A still more special case can be obtained, if we restrict ourselves to a fixed point and a local homeomorphism, a fiber in the language of sheaves, defined by the restriction on a function $f : X \rightarrow Y$ for topological spaces $X$ and $Y$ with

$$f|_X : X \rightarrow f(X)$$

and the inverse image $f^{-1}(y)$ for $y \in Y$. Here again we have an open set (and neighbourhoods) and it is only in spaces of the same dimension that this obtains, since as Brouwer’s theorem shows for spaces of different dimensions $n \neq m$, there is no homeomorphic image (or open map) in the neighbourhood of an open set in a Banach space, even at infinitesimal distance. For the fixed-point observer, we need Brouwer’s theorem in (Brouwer [4]):
« A one-to-one continuous transformation of an n-dimensional element in itself possesses necessarily a fixed point ».

My translation

The element in question here may be a closed ball or a closed disk; it is used in the form of a continuous function on a compact connected subset of a Euclidean space to itself. There the fixed point is an open set and it is identified with the fixed-point observer. Thus the fixed-point observers constitute a connected set or rather the universal connection of observers in a domain of invariance and this makes the fixed-point observer the fundamental topological invariant.

Fixed points are invariant under homeomorphisms and their domain of invariance (see Brouwer [5, 6]) extends to continuous mappings between subsets of $\mathbb{R}^n$. Various extensions of Brouwer’s fixed point theorem reach up to theorems on random fixed points in probability theory on so-called Polish spaces. From a logico-mathematical point of view however, there are limitations, for example in non-expansive selfmappings of complete metric spaces, but it is always possible to define an asymptotic fixed point – for this see Kohlenbach [21] and Gauthier [15]. In our context, a random fixed-point observer is indeed part and parcel of stochastic processes. We assume that our metric space is complete in the usual Hilbert space of QM and in an indeterministic universe, the random observer is a fixed-point observer determining in a Markovian state space the probability density of upcoming events in the randomly observed system.

The local fixed-point observer may then be the connecting link of observers not only in one universe, but in the multiverse transgressing (in a random walk across worlds!) different dimensions beyond the 4-dimensional spacet ime of GR, if one is willing to speculate on the possibility of a multiversal observer.

6. Realist Observers

Most physicists (and philosophers of physics) don’t seem to be very sensitive to the mathematical and logico-mathematical background of their hypotheses and speculations. In particular, they have most of them a loose informal sense of the concept and word « infinite », especially those physicists and philosophers of physics who propose realist foundations of physics. In connection with our discussion, it is transfinite set theory which appears to be the crux of the matter, since the classical example is H. Everett’s many-universe theory. Everett defended the view that the universal wave function $\psi$ in the Schrödinger wave equation ramifies in an infinite multiplicity of worlds with an « internal » observer sitting on a branch of the multiverse, the actual world of measurement experiments. Such an internal observer plays a dual role, being part of the observed system and the observing system at the same time. But the main problem concerns measurement. What does this observer really measure? From a set-theoretical point of view, the universal wave function has the cardinality of the continuum $\mathfrak{c} = 2^{\mathfrak{c}}$ having its values in $\mathbb{R}$ and $\mathbb{C}$, while measurements have at most
the cardinality \( \aleph_0 \) and there is no bijection between \( \aleph_0 \) and \( 2^{\aleph_0} \). This means that the universal wave function remains inaccessible to measurement, as is any branch of the ramification. Everett’s thesis that the formalism generates its own interpretation is at stake, since the theory is inconsistent on mathematical grounds. An other example is the recent proposal of G. ‘t Hooft.

G. ‘t Hooft defends a realist interpretation of QM in his long working paper *The Cellular Automaton Interpretation of Quantum Mechanics* [26]. The author advocates a deterministic or superdeterministic view of the universal wave function for the Schrödinger equation that would not ramify as in Everett’s many-universes interpretation -- they are practically uncountable, as ‘t Hooft says in his informal idiom --, but would give rise to a unique universe, a fluctuating vacuum filled with « solid » quanta or « fluid » particles that would obey a lawlike determinism evolving from a given fundamental field such as a scalar field or a quantum (true or false) vacuum. The entire universe of the wave function has a real deterministic *ontological* basis in Hilbert space following ‘t Hooft. But ‘t Hooft doesn’t say that such a basis must have a finite cardinality or at most an infinite countable cardinality \( \aleph_0 \). The universal wave function \( \psi \) with its values in \( \mathbb{R} \) and \( \mathbb{C} \) has however the uncountable cardinality \( 2^{\aleph_0} = \aleph_1 \) (beth number), the power of the continuum -- the continuum hypothesis (CH) \( c = 2^{\aleph_0} = \aleph_1 \) is not needed in our discussion. The infinite-dimensional Hilbert spaces that are used in the separable space topology are all homeomorphic in countable cardinality (of a countable orthonormal basis): a Hilbert space of uncountable cardinality \( 2^{\aleph_0} \) is not separable and the notion of homeomorphism here has the usual definition of a bicontinuous bijection, that is, if the function \( f(x) \) is continuous, its inverse \( f^{-1}(x) \) is also continuous. The same holds in the more general Fock spaces for the second quantization in quantum field theory with its many-particle systems and von Neumann \( C^* \) separable algebras, while larger spaces like infinite-dimensional Banach spaces equipped with an uncountable Hamel basis (provided by the axiom of choice) smash all dimensions in one *indistinct* all-encompassing continuum. Separable Hausdorff spaces as functional spaces have a cardinality higher than the continuum \( c \), that is \( \aleph_2 \) or \( 2^c \), but their Hausdorff dimension \( d \) for regular metric spaces like Euclidean spaces \( \mathbb{R}^n \) or \( \mathbb{R}^\omega \) (the ordinal of \( \aleph_0 \)) corresponds to the finite or countable dimensions of Hilbert spaces – the Hausdorff dimension \( d \) for irregular finite or countable metric spaces is 0 --. Hilbert spaces of finite or \( \aleph_0 \) dimensions are also Banach spaces, but they are the natural setting for self-adjoint operators and observables designed to capture the finite probability values of the Born rule \( (\psi^* \psi = |\psi|^2) \) for actual concrete measurements. There is no bijection between \( \aleph_0 \) and \( 2^{\aleph_0} \), the set of all finite and infinite subsets or combinations of \( \aleph_0 \) quantum states. So the universal wave function is inaccessible or unassailable from the \( \aleph_0 \) infinite-dimensional Hilbert space perspective. The case of finite-dimensional Hilbert space does not fare better and is still worse for ‘t Hooft cellular automata or finite lattice-theoretic devices, since Hilbert spaces are not homeomorphic over different dimensions \( m < n \) and cannot reach up to a unique infinite-\( \aleph_0 \) dimensional Hilbert space. What this means is the mathematical fact
from transfinite arithmetic that if the universal wave function $\psi$ could be realized, the one universe would be in all its states at once in contradiction to ‘t Hooft view that there is only one state of the universal wave function at any instant. In both cases though, such a universe would be indeed immeasurable – with no experiment whatsoever to measure anything and there would be no need at all for a measurement theory, for observers, no Hilbert space of observables, not a quantum bit of quantum logic and no-go theorems, as ‘t Hooft notes, and for that matter ultimately no QM and no physics at all!

A third example is S. Gao [11] tentative interpretation of QM in terms of a realist ontology of discontinuous random motions of particles accounted for by so-called «protective» measurements. In his defence of discontinuous functions, Gao refers to measure theory and the Borel-Lebesgue duo for their exploitation of the notion of discontinuity in a set-theoretic context. Let us see the definition of a discontinuous function; the function $\mathbb{R} \to \mathbb{R}$ is discontinuous on a point $a$, if it is not continuous there, that is in the real interval $I [0,1]$ one has:

$$\forall \varepsilon > 0 \exists \eta > 0 \forall x \in I (|x - a| < \eta \rightarrow |f(x) - f(a)| \geq \varepsilon)$$

(for a continuous function, we have $|f(x) - f(a)| < \varepsilon$) for arbitrarily small $\eta$ and $\varepsilon$. In other words, a function is discontinuous if it is not continuous. This situation is similar to transcendental number theory: a number is transcendental, if it is not algebraic, i.e. it is not the solution of an algebraic equation. Transcendental numbers defined by negation are the most numerous among real numbers and discontinuous functions defined similarly are more numerous than continuous functions, but like transcendental numbers they are hard to find or to describe explicitly. For the Lebesgue measure, the cardinality of the Lebesgue tribe is $2^c$ or $2_2$, that is beyond the continuum, but a Lebesgue measure is either finitely additive or $\sigma$-additive (denumerably additive) for all mathematical purposes. Again the continuum $\mathbb{R}$ or its power set are not accessible to measurement. Realist ontologies tend to fill up $\mathbb{R}$, the real continuum and its power set $P(\mathbb{R})$, the set of all real-valued functions, with real physical particles. Such views pertain to metaphysical speculations. There is no motion, no randomness in the static mathematical continuum as a completed non-denumerable infinite totality and for that matter transfinite set theory even postulates (Zermelo’s well-ordering axiom) that all sets are well-ordered leaving no room for disordered phenomena, even at infinite limit ordinals. We should remind here that from a mathematical point of view, even in a 4-dimensional spacetime continuum, non-denumerable pathologies are due to a dimensional chirurgy of deformations (the Friedman-Donaldson theory). The assumption of a full stochastic continuum distillating probabilities by the Born rule confines again to an inconsistent QM foundation. Other realist approaches, for example the de Broglie-Bohm interpretation of the wave function or the coherent histories approach share the same cardinal defects of cardinality.

In the case of the Broglie-Bohm theory, a pilot wave crosses the entire universe in a continuous trajectory as a non-local non-relativistic quantum system while consistent or rather coherent histories «decohere» into $\aleph_0$ denumerable sequences among $2^{\aleph_0}$ non-denumerable divergent subsequences without a finite
probablity calculus being able to choose among them. Infinities of different cardinality also abound in quantum field theories and Feynman path integrals are filtered by the Feynman diagrams calculus oblivious of the mathematical inconsistent background perhaps supported ultimately by an uncountable gauge integral. Renormalization is not very helpful here leaving behind still too many rampant infinities. Even modal interpretations that are only halfway realist suppose an indefinite or indeterminate dynamical state potentially becoming a real actual definite value state upon measurement.

The morale of the story is simple. Realism and determinism in physics must be grounded on mathematical realism and the mathematical set-theoretic machinery has been used extensively in our critique of realist foundational interpretations of physics Transfinite set theory is the utmost realist theory of the mathematical continuum and any theory of the real physical continuum must be aware of the benefits and also of the pitfalls of the exploration and exploitation of the continuum: The uncountable cannot be accounted for neither in mathematics, nor in physics!

In the context of physics, the Schrödinger’s quantum version of the classical Hamiltonian wave equation is a deterministic configuration of the physical continuum and as such it is immune to observers and to finite measurement. But experimental physics is a finite measurement process and theoretical physics has to provide the adequate analytical apparatus for the experimental apparatus. A constructivist or « finitist » theory is needed to counteract the excesses and the inconsistencies of physical and metaphysical realism. Here Hermann Weyl, mathematician, physicist, philosopher and the pionner of modern gauge theory, could be the good measure of the constructivist approach.

7. Stochastic Observers

One should go back to Hermann Weyl for a more sober appraisal of the probabilistic approach to physics in a little-known 1920 paper (see [28]). At the time, Weyl shared with the intuitionist Brouwer the idea that the mathematical continuum was a process of becoming « ein Prozess im Werden ». For Weyl, the physical world is also in an infinite (endless) process of becoming. While the causal outlook is bound to the eternal cycle of causal chains, the statistical or probabilistic view privileges decisions that are autonomous and causally absolutely independent of each other « selbständige, kausal voneinander absolut unabhängige Entscheidungen » ([28], p. 541). Weyl adds that these « decisions » are the real ingredients of the world. The causal static view describes only the world scene « Schauplatz », not the real events (of the statistical probabilistic worldview), Weyl concludes – see Gauthier [14] for Weyl’s view on the causal universe of Relativity Theory and also Weyl [29] for his general philosophical outlook –.
7.1 Weyl’s stochastic universe

Weyl himself uses the binomial distribution ([28], 539) to stress the stochastic independence of physical events – Weyl had insisted on the one world of infinitely novel becoming. Let us start with the binomial distribution

\[ p(k) = \binom{n}{k} p^k q^{n-k} \]

and the expansion of the binomial coefficient given by the factorial

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ for } \frac{n!}{k!(n-k)!} = \binom{n}{k} \]

for the combinations \( C \) with the power set \( 2^n \) of a set of \( n \) finite elements or experiments. Finite experiments include actual experiments and thought experiments that are the free-will choices of the experimenters as accounted for in the recent Conway-Kochen proposal (see [8] and [9]). Their free-will hypothesis extends to the Weylian decisions (Entscheidungen) in a probabilistic universe. Instead of the universal free-will postulate, random observers could play the role of free-will observers in a stochastic universe without assuming individual conscious choices, simply because randomness fills the purpose of a statistical mechanics identifiable with quantum mechanics without collapse, only the determination of the indeterminate in the observed-observer in the measurement interaction, and without consciousness (the von Neumann’s cut or Schnitt) unless consciousness is conceived as a purely physical process to underscore the quantum-mechanical nature of the projection postulate which states that immediately after a measurement (that is, an interaction), the superposition \( \sum \sigma_j \alpha_j \) is transformed or reduced to \( \sigma \alpha_n \).

The reduction or collapse of the wave packet characterizes von Neumann’s theory of measurement. For instance, the superposition of states is made up of the combined system — the observer and the observed system — and for von Neumann a measurement projects the system in a determinate state while vectors \( \sigma_n \alpha_n \) have a well-defined value since projections are in bijection with the subspaces of the Hilbert space and the system is no more in a pure state, but in a mixture. Everett’s multiverse theory (or relative state theory) supposes that the superposition is not reduced or projected in a determinate state, but ramifies after an interaction in a multitude of branches each corresponding to a component of the superposition: there would be as many worlds as there are components and the result of measurement would be valid on only one world among a (non-denumerable) infinity of universes. Here is the rub, more irritating than von Neumann’s cut <Schnitt> between the observed system and the observer: the set of all values of the wave function \( \psi \) is \( \mathbb{C} \), the set of complex numbers, which has the cardinality \( 2^{\aleph_0} \); thus, the ramified \( \psi \) cannot be measured for the set of all possible measurements certainly does not exceed \( \mathbb{R}_0 \) and there is no bijection between \( \mathbb{R}_0 \) and \( 2^{\aleph_0} \). The inconsistency is fatal in view of Everett’s idea that the formalism generates its own interpretation. If the ramification of \( \psi \) must have a probabilistic objective content, one is obliged to admit that it cannot emerge from
the divergent ramification of non-denumerable probability values, a probability theory being at most $\sigma$-additive, that is denumerably convergent. Another example of an inconsistent probability theory of QM is the theory of consistent histories, first formulated by Griffiths (see [19]) and adopted since by some important physicists, Gell-Mann and Hartle, among others. The theory can be considered as a variant of Everett’s many-universe or multiverse interpretation with a historical component, since parallel universes can have different histories, that is temporal sequences of quantum events. In order for a given history to be consistent, it is granted a weakened logical status which forbids, for instance, to join two incompatible events (e.g. spin states $a$ and $b$ of an electron) in a classical conjunction $a \wedge b$. These singular histories must preserve probability measures or $\sigma$-additivity for denumerable measures with the help of elementary logical notions as Modus Ponens, conditional probabilities and counterfactuals, truth and liability. But the main question is the consistency of consistent histories. Work by Goldstein and Page (see [18]), Dowker and Kent (see [10]) tends to show that Griffith’s theory is inconsistent in its probabilistic assumptions about consistent histories. From a combinatorial point of view, denumerable or $\sigma$-additivity supposes that the decomposition of probability measures covers up inconsistent history subsequences (subsets) as well as consistent but irreconcilable subsequences in the density matrix of consistent histories; in other words, there is no bijection between the set $\mathcal{K}_0$ sequences and the set $2^{\mathcal{K}_0}$ of subsequences (the power set of all histories) and standard probabilities are lost in the multiplicity of divergent histories (and subhistories). The lesson to be drawn here is perhaps that a logic that accommodates contradictions besides tautologies can take care of a "quasi-consistency" for the "quasi-classicality" in a mixture of coherent histories in quantum systems and decoherent histories in classical (macroscopic) systems, as quantum decoherence theory seems to indicate. But the term "consistent histories" would nonetheless sound like a misnomer for a theory which makes room for too many divergent histories, as the universal ramification of the wave function would have it in Everett’s multiverse interpretation.

7.2. Probabilities

Hilbert (followed in this by von Neumann) introduced the notion of analytical apparatus <der analytische Apparat> drawn from the general structure of an axiomatic system in physics and he made no mystery of his intention to provide physics with the same kind of axiomatic foundations as geometry. Physical situations must be mirrored in an analytical apparatus, physical quantities are represented by mathematical constructs which are translated back into the language of physics in order to give real meaning to empirical statements. The analytical apparatus is not subjected to change while its physical interpretation has a variable degree of freedom or arbitrariness. What this means is that the mathematical formalism of a physical theory is a syntactical structure which does not possess a canonical interpretation, the analytical apparatus does not generate a unique model. At the same time, axiomatization helps in clarifying a concept like probability which is thus rescued from its mystical state, as Hilbert says.
It is noteworthy that another pair of renowned mathematicians, Hardy and Littlewood expressed the same opinion at about the same time: "Probability is not a notion of pure mathematics, but of philosophy or physics".

Probabilities had, long before Quantum Mechanics, been knocking at the door of physics, but Laplace had entitled his work *Essai philosophique sur les probabilités* (1814) after having called it *Théorie analytique des probabilités* (1812). Statistical mechanics can certainly count as a forerunner of QM as far as the statistical behaviour of a large number of particles is an essential ingredient in the probability theory of quantum-mechanical systems. But even in the work of pioneers like Born and Pauli, probability has entered QM somehow through the backdoor and it seems that it is only reluctantly that Born, for example, has admitted the idea of probability. Later work by Kolmogorov on the axiomatic foundations of elementary probability theory or von Mises and Reichenbach on the frequentist interpretation of probability will achieve some measure of success, but it is the historical advent of a rigorous formalization of the notion of probability as it occurs in quantum physics which has not been sufficiently stressed.

A Markovian observer as described above (in Section 3) has no memory and its random observations, experiments, measurements have no past. It is a discrete observer in a finite probability space where a stochastic process corresponds to repeated independent observations of a given random variable the state space of which is either finite or denumerable (Kolmogorov’s $\sigma$-additivity) in ergodic theory and statistical mechanics. In that context, the wave function $\psi$ of the Schrödinger equation has no determinate past, it is as indeterminate as a mathematical continuum from a Brouwerian point of view and it is only when a measurement interaction occurs that a determinate state is produced with Born-ruled probability values. Quantum mechanics as the fundamental theory of the physical world is thus an indeterminist theory with a determinate outcome only in measurement theory of the local interaction between an observed system and an observer system. Kochen and Specker believe that randomness does not account for their free-will ontological notion, but they think of probability theory in retrospect as if it could recover the stochastic processes or chaotic fluctuations that we attribute to the indeterminate substratum of the universal wave function $\psi$ as if it had some determinate state *a posteriori* from the realist viewpoint they adopt. Finally, the observers in some realist deterministic interpretations of QM appear (and disappear!) as local hidden variables, while the observers in the constructivist interpretation have a definite mathematical location in the algebraic-topological space formulation of QM and GR.

8. Conclusion

The constructivist interpretation (as opposed to a realist interpretation) defended here is in agreement with the Weylian standpoint and can be viewed in the case of Quantum Mechanics as a variant of the Copenhagen Interpretation with explicit constructivist logical and mathematical foundations in the scope of the local observer
at micro- and macroscopic scales, in 2-3-4 or \( n \)-dimensional space – and for the observer in 0-dimensional space –. There the local observer appears as the open relative (local) complement of a topological space or as a fixed-point observer an \( n \)-dimensional manifold homeomorphic to an \( n \) – dimensional Euclidean space. As for logic, the negation involved in the local complement corresponds to a non-Boolean or constructive notion for which we have \( \neg \neg a \neq a \) and this leads naturally to a non-classical logic internal to the given physical theory. In QM and in Relativity Theory, the local observer is the coupling constant of the relational system observed-observer. Those ideas have been introduced early in my critical work on the foundations of physics from a physico-mathematical point of view (see [12] and [13]). The physicist C. Rovelli has recently developed loosely related ideas in his informal realist notion of a relational QM (see [24]). In cosmology, the local observer located anywhere is a fixed point as the central observation post of the cosmic panorama at equal distance from any point on the cosmic (hemispherical) horizon of the celestial sphere which is itself bounded by homeomorphic reflections of the local isotropic universe as required by the cosmological principle. For the local observer, everywhere is localized and what is beyond the horizon boundary lives in the same dimension since the visible is cobordant with the invisible. Finally, from an information-theoretic point of view, the local observer is the first informant for the information content of the observable universe and constitutes the fundamental link for a unifying theory of QM and GR.

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References


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