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# Traveling Wave Solutions of a Generalized Burgers' Equation

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**Abstract**

In this paper we find a solution of the generalized Burgers' equation of the form $u(x,t) = U(\mu x - \lambda t)$ this is, from the traveling wave form. To achieve this purpose a method based on the omega function is proposed. A complete description of the method is make and finally a specific solution is shown.

**Mathematics Subject Classification:** 35C05

**Keywords:** Omega Function method, Nonlinear partial differential equations, Generalized Burgers' equation, Traveling wave solutions

## 1 Introduction

In various fields of science such as fluid mechanics, acoustics, gas dynamics, and in many important applications of aeronautics, hydraulics, traffic flow, the models given by the Navier-Stokes and Euler equations are frequently used. In 1948 the Dutch physicist Johannes Burgers [1] gets a simplification of these equations by means of a simple mathematical model of the motion of a viscous compressible fluid that we know today as the viscous Burgers' equation, (1.1) although this equation had already been studied by bateman [2]. At present it is one of the equations that has been studied more widely and it is posed as follows,

$$u_t + \left(u^2/2\right)_x = \epsilon u_{xx} \tag{1.1}$$

where $u = u(x,t)$ is the speed of the fluid in the direction $x$ at $t$ time, and $\epsilon$ the kinematic viscosity is a positive constant. The equation (1.1), a nonlinear second order parabolic partial differential equation is the one-dimensional form of a more general equation (1.2) given by Avrin [3] as,

$$u_t - \nabla u + \mathbf{div}\psi(u) = 0 \tag{1.2}$$

where $u = u(x,t)$ with $x \in \mathbb{R}^n$ and $t$ is a non-negative real number. Some generalizations and the initial-boundary value problem with this equation the have been studied by various authors in recent years, [4] [6] among many others. An interesting generalization treated in [5] is the follows,

$$\alpha u_t + (g(u))_x = \epsilon u_{xx} \tag{1.3}$$

with $x \in \mathbb{R}$, $t \geq 0$, $\alpha \in \mathbb{R}$, $\epsilon > 0$ and $g(u)$ a class $C^2$ nonlinear function.
If in equation (1.3) we do $g(u) = \beta u^3 + \delta u^2 + \theta u$, with $\beta, \delta, \theta \in \mathbb{R}$ we have:

$$\alpha u_t + (\beta u^3 + \delta u^2 + \theta u)_x = \epsilon u_{xx} \tag{1.4}$$

By means of the ingenious Cole-Hopf transformation [7], equation (1.1) can be transformed into the one-dimensional equation of heat that already has an explicit solution. However, not all generalizations given in equation (1.2) and (1.3) run with the same luck.

# 2 The Omega Function method

Let's now describe in a general way the Omega Function method for finding traveling wave solutions using the Lambert W-function, also called the omega function, for a non-linear partial differential equation.

i) Initially consider the equation we want to solve. Suppose that it can be written as,

$$P\left(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots\right) = 0 \tag{2.1}$$

where $u = u(x,t)$ is a function of real value with $(x,t) \in D$ and $D \subset \mathbb{R}^2$, and where P is a polynomial in $\{u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots\}$.

ii) Now let's make the substitution that allows the transformation of the traveling wave, which is the essence of the methods that look for soliton type solutions see [8] and [9], this substitution is $u(x,t) = U(\zeta)$, wiht $\zeta = \mu x - \lambda t$, where $\mu$ and $\lambda > 0$ are real constants. Then we have,

$$u_t = -\lambda \frac{dU}{d\zeta},\ u_x = \mu \frac{dU}{d\zeta},\ u_{xx} = \mu^2 \frac{d^2U}{d\zeta^2},\ u_{tt} = \lambda^2 \frac{d^2U}{d\zeta^2},\ u_{xt} = -\lambda\mu \frac{d^2U}{d\zeta^2}, ... \tag{2.2}$$

iii) Replacing (2.2) in the partial differential equation (2.1), this becomes the (ODE) ordinary differential equation,

$$Q\left(U, U', U'', U''', ...\right) = 0, \tag{2.3}$$

where $Q$ is a polynomial in $U$ and its derivatives $U'$, $U''$, $U'''$... and where the prime denotes the derivatives with respect to $\zeta$.

iv) Now, be $T(\zeta)$ a solution of differential equation $\frac{d\phi}{d\zeta} = \phi^3 - \phi^2$ , then as a consequence of the above we have,

$$\frac{dT}{d\zeta} = T^3 - T^2,\ \frac{d^2T}{d\zeta^2} = 3T^5 - 5T^4 + 2T^3,\ \frac{d^3T}{d\zeta^3} = ... \tag{2.4}$$

To determine $T(\zeta)$ you just need to find any non-trivial solution of the equation $\frac{d\phi}{d\zeta} = \phi^3 - \phi^2$, let's take the solution:

$$T(\zeta) = \left[W(\exp(-1+\zeta)) + 1\right]^{-1} \tag{2.5}$$

where $W$ is the Lambert-W function or the omega function, which is defined from the relation that satisfies the equation $W(z)e^{W(z)} = z$. In this equation $z$ represents a complex variable. When restricting to a real variable to obtain a function we must take $x > -1/e$, thus we finally define the omega function $W(x)$ as the one that satisfies the relationship $W(x)e^{W(x)} = x$ for real number $x > -1/e$. Furthermore using implicit derivation we obtain an expression for the derivative as,

$$\frac{dW}{dx} = \frac{W(x)}{x(1+W(x))} \tag{2.6}$$

and so be able to verify that in effect the proposed solution (2.5) satisfies the equation $\frac{d\phi}{d\zeta} = \phi^3 - \phi^2$ .

v) Suppose now that the solution of (2.1) can be expressed by a polynomial in $T$ as follows:

$$U(\zeta) = S(T) = \sum_{n=0}^{m} a_n T^n \tag{2.7}$$

where $m$ is a positive integer in most cases. To determine the parameter $m$ we must find the balance between Higher order derivatives and the highest non-linear order terms that appear in the equation (2.3).

vi) Using the chain rule we have the following identities,

$$\frac{dU}{d\zeta} = \frac{dT}{d\zeta}\frac{dU}{dT}, \;\; \frac{d^2U}{d\zeta^2} = \frac{d^2T}{d\zeta^2}\frac{dU}{dT} + \left(\frac{dT}{d\zeta}\right)^2 \frac{d^2U}{dT^2}... \tag{2.8}$$

and so on, if necessary. Substituting (2.8), (2.7), and (2.4) into ODE (2.3) we obtain a polynomial equation in the variable $T$.

$$\sum_{n=0}^{m} A_n T^n = 0 \tag{2.9}$$

where each $A_i$ depends on the parameters that appear in the equation (2.3), $\mu$ and $\lambda$ and $a_k$ for $k = 1, ..., m$. Then to solve equation (2.9) we must solve the system of algebraic equations:

$$\{A_0 = 0, A_1 = 0, A_2 = 0, ..., A_m = 0\} \tag{2.10}$$

we must solve this system in the variables $\mu$ and $\lambda$ and $a_k$ for $k = 1, ..., m$. And so we finally using (2.7) and (2.5) find a solution $u(x, t)$ to the equation (2.1), expressed in terms of the omega function.

# 3 The Omega Function Method applied to generalized Burger's equation

Now let's use the method to solve the equation (1.4). Making the substitutions given in (2.2) to equation (1.4), we get the ordinary differential equation,

$$-\epsilon\mu^2\frac{d^2U}{d\zeta^2} - \alpha\lambda\frac{dU}{d\zeta} + 3\beta\mu U^2\frac{dU}{d\zeta} + 2\delta\mu U\frac{dU}{d\zeta} + \theta\mu\frac{dU}{d\zeta} = 0 \tag{3.1}$$

Now let's determine the value of $m$ in equation (2.7). When balancing the number of the highest derivative and the higher order non-linear term in the equation (1.4), it turns out $m+4 = 3m+2$, where we finally have that $m = 1$. In this way we have that $U(\zeta) = a_0 + a_1 T$. Now to simplify the notation and that no subindices appear, let's do $a_0 = a$ and $a_1 = b$ with what we have to,

$$U = a + bT, \;\; \frac{dU}{dT} = b, \;\; \frac{d^2U}{dT^2} = 0 \tag{3.2}$$

Now, if we replace (2.4), (2.8) and (3.2) in (3.1) and then simplify we have the polynomial equation in T given by:

$$\begin{aligned}(3b^3\beta\mu - 3b\mu^2\epsilon)T^5 + (6ab^2\beta\mu - 3b^3\beta\mu + 2b^2\delta\mu + 5b\mu^2\epsilon)T^4 \\ + (3a^2b\beta\mu - 6ab^2\beta\mu + 2ab\delta\mu - 2b^2\delta\mu - 2b\mu^2\epsilon\alpha\ b\lambda + b\mu\theta)T^3 \\ + (-3a^2b\beta\mu - 2ab\delta\mu + \alpha b\lambda - b\mu\theta)T^2 = 0\end{aligned} \tag{3.3}$$

To solve equation (3.3) we must solve the nonlinear system of equations:

$$\left.\begin{array}{r} 3b^3\beta\mu - 3b\mu^2\epsilon = 0 \\ 6ab^2\beta\mu - 3b^3\beta\mu + 2b^2\delta\mu + 5b\mu^2\epsilon = 0 \\ 3a^2b\beta\mu - 6ab^2\beta\mu + 2ab\delta\mu - 2b^2\delta\mu - 2b\mu^2\epsilon\alpha\ b\lambda + b\mu\theta = 0 \\ -3a^2b\beta\mu - 2ab\delta\mu + \alpha b\lambda - b\mu\theta = 0 \end{array}\right\} \tag{3.4}$$

Of the three families of solutions obtained by solving the system (3.4) only one corresponds to non-trivial solutions,

$$\left\{a = a, b = \frac{-3a\beta - \delta}{\beta}, \lambda = \frac{(3a\beta+\delta)^2(3a^2\beta + 2a\delta + \theta)}{\beta\epsilon\alpha}, \mu = \frac{(3a\beta+\delta)^2}{\beta\epsilon}\right\} \tag{3.5}$$

in this way we finally have the solution:

$$u(x,t) = a + \frac{-3a\beta - \delta}{\beta} T\left(\mu x - \lambda t\right) \tag{3.6}$$

with, $T(\mu x - \lambda t) =$

$$\left[W\left(\exp\left(-1 + \frac{(3a\beta+\delta)^2}{\beta\epsilon}x - \frac{(3a\beta+\delta)^2(3a^2\beta+2a\delta+\theta)}{\beta\epsilon\alpha}t\right)\right) + 1\right]^{-1} \tag{3.7}$$

# 4 Some particular cases

Two particular cases are shown here. In both cases for equation (1.3) be $g(u) = u^3 - u^2 + u$ and for equation (3.5) be $a = 1$ and $\alpha = 8$. In the first case let's do $\epsilon = 2$ , therefore by the equations (3.5),(3.6) and (3.7) we have to $b = -2,\ \mu = 2,\ \lambda = 1/4$ , and so the solution that we denote as $u_1(x,t)$ is given by:

$$u_1(x,t) = 1 - 2\left[W\left(\exp\left(-1 + 2x - \frac{1}{4}t\right)\right) + 1\right]^{-1} \tag{4.1}$$

Now let's show a second case. Consider the same conditions as in the previous case, but let's consider now a very small viscosity coefficient, let's say $\epsilon = 1/25$. In this case, by the equations (3.5),(3.6) and (3.7) we have to $b = -2,\ \mu = 100,\ \lambda = 25/2$ , let's denote this solution as $u_2(x,t)$ and therefore:

$$u_2(x,t) = 1 - 2\left[W\left(\exp\left(-1 + 100x - \frac{25}{2}t\right)\right) + 1\right]^{-1} \tag{4.2}$$

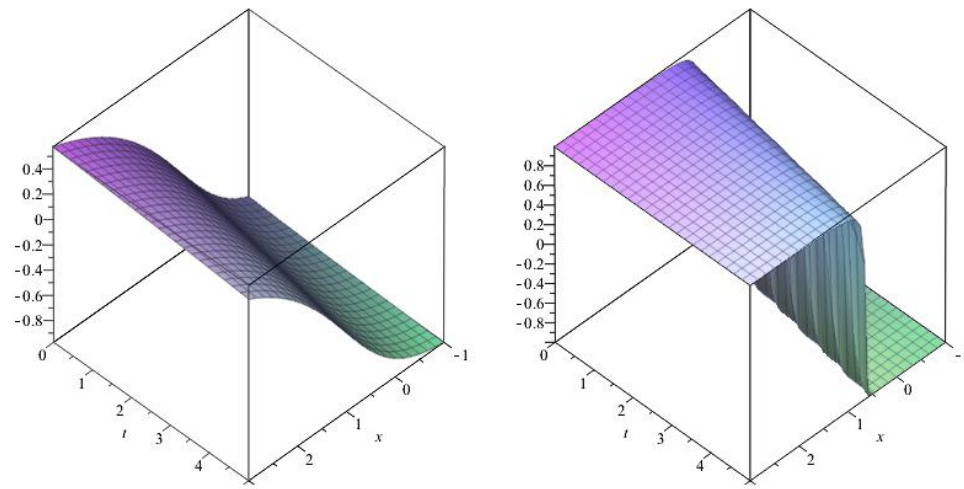


Figure 1: $u_1(x,t)$ *with* $\epsilon = 2$, $u_2(x,t)$ *with* $\epsilon = 1/25$

The graph of the solutions (4.1) and (4.2) for $-1 < x < 3$ and $0 < t < 5$ is shown in the figure (1). We can observe the effect that the value of the parameter $\epsilon$ (viscosity coefficient) has on the solutions of the equation (1.4). When $\epsilon$ is very small, there is a jump effect, as we see in the graph of $u_2$ (right side of the figure), while the graph is much smoother when $\epsilon$ is greater as can be seen in the graph of $u_1$ (left side of the figure).

# 5 Conclusions

We have proposed and described a method to solve PDEs. We have used the proposed method to solve a generalized of the Burgers' equation. The relevance of the proposed method is that it allows finding traveling wave solutions expressed in terms of the omega function, which as we know can not be expressed in terms of elementary functions.

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