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Solution for a non-Homogeneous Klein-Gordon Equation with 5th Degree Polynomial Forcing Function

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Abstract

In this paper we establish a condition for the fifth-degree polynomial forcing function $f(\phi) = \alpha\phi^2 + \beta\phi^3 + \delta\phi^4 + \epsilon\phi^5$, under which the one dimensional non homogeneous Klein-Gordon equation $\phi_{tt} - \phi_{xx} + \phi = f(\phi)$, admits a solution of the traveling wave type that is obtained by using the Omega Function Method. Once this condition is determined, some specific solutions are analyzed.

Keywords: Klein-Gordon equation, Nonlinear partial differential equations, non-Homogeneous partial differential equations, Traveling wave solutions

1 Introduction

In 1924, the French physicist Louis de Broglie see [3], proposed for the first time the idea of the matter wave for quantum particle similarly as in 1905, the physicist Albert Einstein developed the wave-particle theory of light. In 1926 Erwin Schrödinger an Austrian theoretical physicist see [9], supposed that the wave equation should be:

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = -\left(\frac{mc}{\hbar}\right)^2 \phi \tag{1.1}$$

Where the scalar field $\phi(x,t)$ represents the wave of matter or the wave function associated with each quantum particle, in the direction $x \in \mathbb{R}^n$ at time t. The resting mass is m, its speed c and $\hbar$ represents the reduced planck constant. Equation (1.1) uses the relativistic kinetic energy expression $E^2 = m^2c^4 + c^2p^2$ where p is the momentum. However, this equation presented several disadvantages as the presence of positive and negative values for energy. Even more serious, a probability distribution of ϕ with negative values. Unmotivated by these problems, Schrödinger discarted that equation. Later, he proposes a new equation using the nonrelativist kinetic energy expression $E = p^2/2m + V(x)$ where $V(x)$ is the potential energy; that is now known as the Schrödinger equation:

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V(x)\right)\phi = i\hbar\frac{\partial \phi}{\partial t} \tag{1.2}$$

finally, the physicists Oskar Klein and Walter Gordon, they were able to explain satisfactorily the inconveniences that Schrödinger found and so again equation (1.1) was accepted as the most appropriate model to describe the wave function of a neutral charge particle. For this last reason the equation (1.1) is today known as the Klein-Gordon equation, That in simplified form it can be written as,

$$\left(\Box^2 + k^2\right)\phi = 0 \tag{1.3}$$

where $k = \frac{mc}{\hbar}$ and $\Box$ represents the d'Alembert operator,

$$\Box = \frac{1}{c^2}\phi_{tt} - \nabla^2 u \tag{1.4}$$

This equation has been widely studied, see for example [1] [5] and [2]. However in the last few years there has been a lot of interest in generalizations of this equation, and in the non-homogenous case,

$$\phi_{tt} - \phi_{xx} + \phi = f(\phi) \tag{1.5}$$

Where $f(\phi)$ is some smooth function. Several forms of (1.5) are analyzed in [4], including the well-known sine-gordon equation that is obtained from (1.5) when $f(\phi) = sin(\phi)$. Our purpose is to establish some conditions under which equation (1.6) has solutions in the form of a traveling wave. To achieve that goal we use the omega function method, see [6].

$$\frac{1}{c^2}\phi_{tt} - \phi_{xx} + \phi = \alpha\phi^2 + \beta\phi^3 + \delta\phi^4 + \epsilon\phi^5 \tag{1.6}$$

with $\phi(x,t) \in \mathbb{R}$ $x \in \mathbb{R}$, $t \geq 0$ and $c \neq 0$, $\alpha, \beta, \delta, \epsilon$ real constants.

2 Algorithm of the Omega Function Method

Various methods, see [7] and [8] are currently used to find solutions of the traveling wave type for nonlinear evolution equations, of the form:

$$Q\left(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{tt}, \phi_{xt}, ...\right) = 0 \tag{2.1}$$

where Q is a polynomial function, $\phi(x,t)$ is a real-valued function, $x \in \mathbb{R}$ and $t \geq 0$.
In this section we give a short description of a method to find soliton-type solutions for (2.1). The method we will use is described more broadly in [6]. This method there is called the Omega Function Method. Here we present three basic steps:

a) First, let's do $\phi(x,t) = \Phi(\zeta)$ with $\zeta = \mu x - \lambda t$, where μ and λ are real constants, and consequently we have an ordinary differential equation:

$$Q\left(\Phi, \Phi', \Phi'', \Phi''', ...\right) = 0, \tag{2.2}$$

where Q is a polynomial function in Φ and its derivatives.

b) Let's assume that $T(\zeta)$ is a solution of $\frac{d\psi}{d\zeta} = \psi^3 - \psi^2$ and that the solution of (2.1) can be written as:

$$\Phi(T) = \sum_{k=0}^{m} c_k T^k \tag{2.3}$$

c) With this it is achieved that equation (2.2) becomes,

$$\sum_{k=0}^{m} C_n T^n = 0 \tag{2.4}$$

where the C_i depend on the parameters of the original equation (2.1). From (2.4) an algebraic system of equations is generated, which when solved finally allows to find some solutions of (2.1).

3 Solution of a non-homogeneus Klein-Gordon Equation

Now, let's see what conditions the forcing function $f(\phi) = \alpha\phi^2 + \beta\phi^3 + \delta\phi^4 + \epsilon\phi^5$ must have, so that Equation (1.6) has some solutions. Making the substitutions given in the previous section, we have

$$\frac{1}{c^2}\lambda^2 \frac{d^2\Phi}{d\zeta^2} - \mu^2 \frac{d^2\Phi}{d\zeta^2} + \Phi = \alpha\Phi^2 + \beta\Phi^3 + \delta\Phi^4 + \epsilon\Phi^5 \tag{3.1}$$

From the highest derivative and the higher order non-linear term in the equation (3.1), we can determine by balancing that $5m = m+4$, this is $m = 1$ and therefore of (2.4) we have,

$$\Phi(T) = c_0 + c_1 T, \quad \frac{d\Phi}{dT} = c_1, \quad \frac{d^2\Phi}{dT^2} = 0 \tag{3.2}$$

From (3.2), and calculating the derivatives of Φ with respect to ζ we can replace this in (3.1) and obtain the equation:

$$\begin{aligned}&(\frac{3\lambda^2 c_1}{c^2} - c_1^5\epsilon - 3\mu^2 c_1)T^5 + (-\frac{5\lambda^2 c_1}{c^2} - c_1^4\delta + 5\mu^2 c_1 - 5c_0c_1^4\epsilon)T^4\\
&+(-10c_0^2c_1^3\epsilon - 2\mu^2 c_1 + \frac{2\lambda^2 c_1}{c^2} - 4c_0c_1^3\delta - c_1^3\beta)T^3 + (-10c_0^3c_1^2\epsilon - 6c_0^2c_1^2\delta - 3c_0c_1^2\beta - \alpha c_1^2)T^2\\
&+(-5c_0^4c_1\epsilon - 4c_0^3c_1\delta - 3c_0^2c_1\beta - 2c_0\alpha c_1 + c_1)T - c_0^5\epsilon - c_0^4\delta - c_0^3\beta - c_0^2\alpha + c_0 = 0\end{aligned} \tag{3.3}$$

In this way from (3.3) we obtain the algebraic system of equations:

$$\left.\begin{aligned}\tfrac{3\lambda^2 c_1}{c^2} - c_1^5\epsilon - 3\mu^2 c_1 &= 0\\
-\tfrac{5\lambda^2 c_1}{c^2} - c_1^4\delta + 5\mu^2 c_1 - 5c_0c_1^4\epsilon) &= 0\\
-10c_0^2c_1^3\epsilon - 2\mu^2 c_1 + \tfrac{2\lambda^2 c_1}{c^2} - 4c_0c_1^3\delta - c_1^3\beta &= 0\\
-10c_0^3c_1^2\epsilon - 6c_0^2c_1^2\delta - 3c_0c_1^2\beta - \alpha c_1^2 &= 0\\
-5c_0^4c_1\epsilon - 4c_0^3c_1\delta - 3c_0^2c_1\beta - 2c_0\alpha c_1 + c_1 &= 0\\
-c_0^5\epsilon - c_0^4\delta - c_0^3\beta - c_0^2\alpha + c_0 &= 0\end{aligned}\right\} \tag{3.4}$$

When solving the system (3.4) with the help of a computer algebra system we obtain the following results expressed in terms of the parameter α. The First;

$$\begin{aligned}&\alpha = \alpha,\ c_0 = \frac{6}{\alpha},\ c_1 = -\frac{6}{\alpha},\ \mu = \mu,\ \lambda = \sqrt{\mu^2 - 1}\,c,\\
&\beta = -\frac{\alpha^2}{3},\ \delta = \frac{5\alpha^3}{108},\ \epsilon = -\frac{\alpha^4}{432},\end{aligned} \tag{3.5}$$

and the second,

$$\begin{aligned}&\alpha = \alpha,\ c_0 = \frac{1}{\alpha},\ c_1 = -\frac{3}{2\alpha},\ \mu = \mu,\ \lambda = \sqrt{4\mu^2 + \frac{27}{2}}\,c,\\
&\beta = 3\alpha^2,\ \delta = -5\alpha^3,\ \epsilon = 2\alpha^4.\end{aligned} \tag{3.6}$$

therefore for two forms of equation we can find the form of solution we are looking for,

$$\left.\begin{aligned}\tfrac{1}{c^2}\phi_{tt} - \phi_{xx} + \phi = &\quad \alpha\phi^2 - \tfrac{\alpha^2}{3}\phi^3 + \tfrac{5\alpha^3}{108}\phi^4 - \tfrac{\alpha^4}{432}\phi^5, \text{and,}\\
\tfrac{1}{c^2}\phi_{tt} - \phi_{xx} + \phi = &\quad \alpha\phi^2 + 3\alpha^2\phi^3 - 5\alpha^3\phi^4 + 2\alpha^4\phi^5\end{aligned}\right\} \tag{3.7}$$

now, since the solutions of equation $\frac{d\psi}{d\zeta} = \psi^3 - \psi^2$, are:

$$\left.\begin{array}{rc} \psi_1(x,t) = & [W\,(k\,\exp{(-1+\zeta)})+1]^{-1}\,, \\ \psi_2(x,t) = & ln\left(\frac{-1+\zeta}{\zeta}\right)+\frac{1}{\zeta}+k \end{array}\right\} \tag{3.8}$$

where W denotes the Lambert-W function. Then we have the following families of solutions for the equations (3.7):

$$\phi_1(x,t) = \frac{6}{\alpha} - \frac{6}{\alpha}\left[W\left(k\,\exp\left(-1+\mu x-\sqrt{\mu^2-1}\;c\;t\right)\right)+1\right]^{-1} \tag{3.9}$$

$$\phi_2(x,t) = \frac{1}{\alpha} - \frac{3}{2\alpha}\left[W\left(k\,\exp\left(-1+\mu x-\sqrt{4\mu^2+\frac{27}{2}}\;c\;t\right)\right)+1\right]^{-1} \tag{3.10}$$

$$\phi_3(x,t) = \frac{6}{\alpha} - \frac{6}{\alpha}\left[ln\left(\frac{-1+\mu x-\sqrt{\mu^2-1}\;c\;t}{\mu x-\sqrt{\mu^2-1}\;c\;t}\right)+\frac{1}{\mu x-\sqrt{\mu^2-1}\;c\;t}+k\right] \tag{3.11}$$

$$\phi_4(x,t) = \frac{1}{\alpha} - \frac{3}{2\alpha}\left[ln\left(\frac{-1+\mu x-\sqrt{4\mu^2+\frac{27}{2}}\;c\;t}{\mu x-\sqrt{4\mu^2+\frac{27}{2}}\;c\;t}\right)+\frac{1}{\mu x-\sqrt{4\mu^2+\frac{27}{2}}\;c\;t}+k\right] \tag{3.12}$$

4 Some particular cases

Four particular cases are shown here. In all four cases (3.9), (3.10), (3.11) and (3.12) be $\alpha = 1$, $\mu = 2$, and $c = k = 1$, so the solutions that we will denote as φ_1 , φ_2 , φ_3 and φ_4 are given by,

$$\varphi_1(x,t) = 6 - 6\left[W\left(\exp\left(-1+2x-\sqrt{3}\;t\right)\right)+1\right]^{-1} \tag{4.1}$$

$$\varphi_2(x,t) = 1 - \frac{3}{2}\left[W\left(\exp\left(-1+2x-\sqrt{\frac{59}{2}}\;t\right)\right)+1\right]^{-1} \tag{4.2}$$

$$\varphi_3(x,t) = 6 - 6\left[ln\left(\frac{-1+2x-\sqrt{3}\;t}{2x-\sqrt{3}\;t}\right)+\frac{1}{2x-\sqrt{3}\;t}+1\right] \tag{4.3}$$

$$\varphi_4(x,t) = 1 - \frac{3}{2}\left[ln\left(\frac{-1+2x-\sqrt{\frac{59}{2}}\;t}{2x-\sqrt{\frac{59}{2}}\;t}\right)+\frac{1}{2x-\sqrt{\frac{59}{2}}\;t}+1\right] \tag{4.4}$$

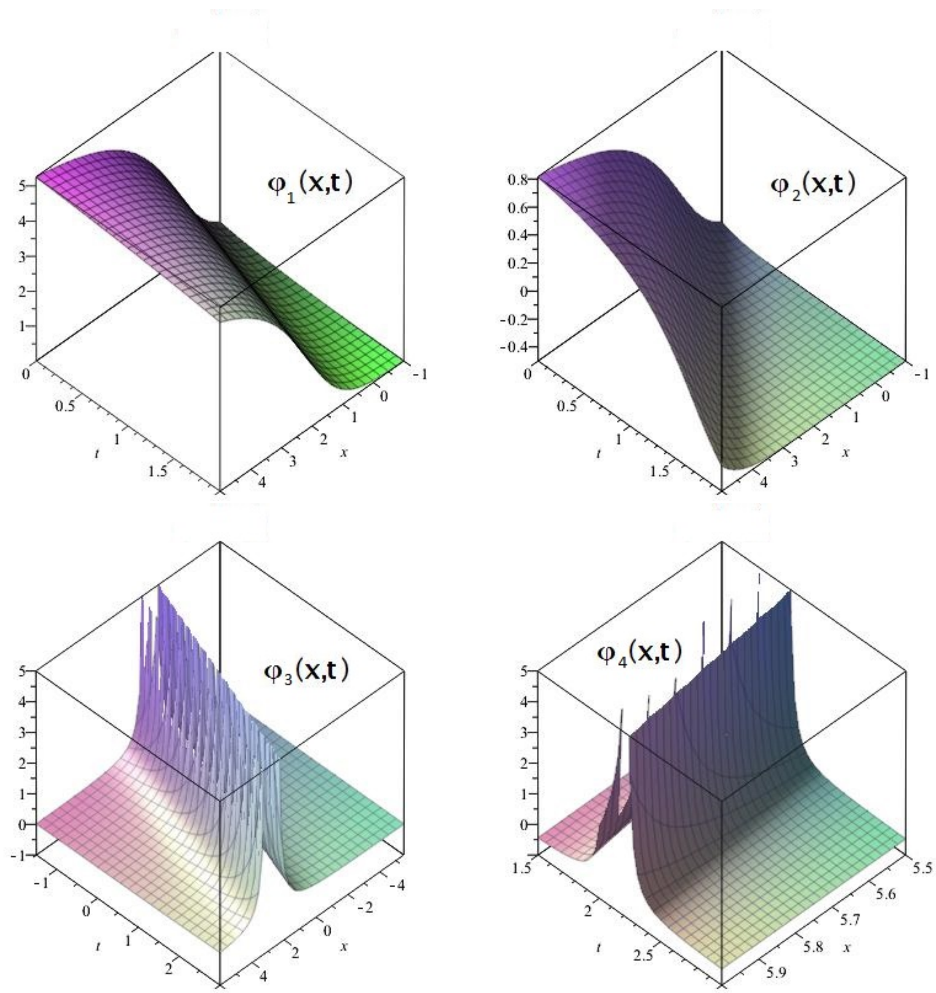

Figure 1: φ_1 ,φ_2 φ_3 ,φ_4 *with* $\alpha = 1,\ \mu = 2,\ \ c = 1,$ *and*, $k = 1$

As we can see the graphs of φ_1 and φ_2 are smooth surfaces, contrary to what appears in φ_3 and φ_4 that show a critical zone, near the strip where these solutions are not defined. this is, for φ_3 the region between the lines $x = \frac{\sqrt{3}}{2}t$ and $x = \frac{1}{2} + \frac{\sqrt{3}}{2}t$, and for φ_4 the region between the lines $x = \frac{1}{2}\sqrt{\frac{59}{2}}t$ and $x = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{59}{2}}t$.

5 Conclusions

In the present work we have found some conditions under which the non-homogeneous Klein-Gordon equation (1.6), admits solutions of the traveling wave type. We have also found quite interesting solutions, of two completely different kinds, some expressed in terms of the Omega function, which as we know can not be expressed in terms of elementary functions, and others in terms of logarithmic functions, which generate singularities near the region where the solutions are not defined.

References

[1] W. Baoxiang, On existence and scattering for critical and subcritical nonlinear Klein-Gordon equations in Hs, *Nonlinear Anal. Theory, Methods & Applications*, **31** (1998), 573-587. https://doi.org/10.1016/s0362-546x(97)00424-0

[2] A. Biswas, A. Yildirim, T. Hayat, M. Aldossary, and R. Sassaman, Soliton Perturbation Theory for the Generalized Klein-Gordon Equation with Full Nonlinearity, *Proceedings of the Romanian Academy, Series A*, **13** (2012), no. 1, 32-41.

[3] L. de Broglie, Waves and Quanta, *Nature*, **112** (1923), 540. https://doi.org/10.1038/112540a0

[4] Q. Changzheng, H. Wenli and D. Jihong, Separation of Variables and Exact Solutions of Generalized Nonlinear Klein-Gordon Equations, *Progress of Theoretical Physics*, **105** (2001), no. 3, 379-398. https://doi.org/10.1143/ptp.105.379

[5] T. D'Aprile and D. Mugnai, Non-Existence Results for the Coupled Klein-Gordon-Maxwell Equations, *Advanced Nonlinear Studies*, **4** (2004), 307-322. https://doi.org/10.1515/ans-2004-0305

[6] H. Garzon and J. Hernandez, Traveling Wave Solutions of a Generalized Burgers' Equation, *Advanced Studies in Theoretical Physics*, **11** (2017), no. 12, 621-627. https://doi.org/10.12988/astp.2017.71048

[7] W. Malflied, Solitary wave solutions of nonlinear wave equations, *American Journal of Physics*, **60** (1992), no. 7, 650-654. https://doi.org/10.1119/1.17120

[8] W. Malflied, The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations, *Journal of Computational and Applied Mathematics*, **164-165** (2004), 529-541. https://doi.org/10.1016/s0377-0427(03)00645-9

[9] E. Schrödinger, An Undulatory Theory of the Mechanics of Atoms and Molecules, *Physical Review*, **28** (1926), no. 6, 1049-1070. https://doi.org/10.1103/physrev.28.1049

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