On the Coercive Functions and Minimizers

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Abstract

N.G. Meyers significantly extended weak lower semicontinuity results for integral functionals depending on maps and their gradients available at that time. Special attention is paid to signed integrands and to applications to continuum mechanics of solids. In particular, we review existing results of the global minimizer of a coercive function and related statements about sequential weak continuity of minors [2].

Keywords: Coercive functions, sequentially lower semi-continuous, global minimizers.


1 Introduction

We now know to prove that a critical point of a function $f(x)$ is a global minimizer if the Hessian of $f(x)$ is positive semidefinite on all of $\mathbb{R}^n$ or a strict global minimizer if the Hessian is positive definite, but we need a way to establish global minimizers even if the Hessian is not necessarily positive semidefinite on all of $\mathbb{R}^n$.

Previously [1], we defined a set $D$ to be closed if its complement is open. This sets the stage for the following definitions.

**Definition 1.** We then say that a set $D \subset \mathbb{R}^n$ is bounded if there exists a constant $M > 0$ such that $\|x\| < M$ for all $x \in D$. The set $D$ is said to be compact if it is closed and bounded [2,3].

**Theorem 1.** Let $D$ be a compact subset of $\mathbb{R}^n$. If $f(x)$ is a continuous function on $D$, then $f(x)$ has a global maximizer and a global minimizer on $D$.

We now describe functions for which global minimizers can be found even on sets that are not bounded or not closed.

**Definition 2.** A continuous function $f(x)$ that is defined on all of $\mathbb{R}^n$ is coercive is

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$ 

**Definition 3.** The function $F$ is weakly sequentially lower semi-continuous (w.s.l.s.c) at $x_0 \in M$, if \( \forall (x_k)_{k \in \mathbb{N}} \subset M \) with $x_k \xrightarrow{w} x_0$ (for $x \to \infty$) there holds [3,4]

$$F(x_0) \leq \liminf_{k \to \infty} F(x_k).$$

That is, for any constant $M > 0$ there exists a constant $R_M > 0$ such that $\|f(x)\| > M$ whenever $\|x\| > R_M$.

**Theorem 2.** Let $f(x)$ be a continuous function defined on all of $\mathbb{R}^n$. If $f(x)$ is coercive, then $f(x)$ has a global minimizer. Furthermore, if the first partial derivatives of $f(x)$ exist on all of $\mathbb{R}^n$, then any global minimizers of $f(x)$ can be found among the critical points of $f(x)$ [5].

We can see that this theorem can be proved by using the fact that $f(x)$ is coercive to find a compact subset of $\mathbb{R}^n$ on which $f(x)$ must have a global minimizer, by the preceding theorem. Therefore, to find the global minimizer of a coercive function, it is sufficient to find the critical points of $f(x)$, and
then evaluate \( f(x) \) at each of these points. The critical points for which \( f(x) \) assumes the smallest values are then the global minimizers \[6\].

## 2 Results on \( w.s.l.s.c \) functions

Let \( \phi(x) : H^1_T \to \mathbb{R} \) be a function defined by

\[
\phi(x) = \int_0^T \frac{|x'(t)|^2}{2} + F(t, x(t)) \, dt.
\]

Here, \( F : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) is a function continuous such that:

- Let \( x \mapsto F(t, x) \) be continuously differentiable for all \( t \in [0, T] \); we denote
  \[
  \nabla F(t, x) = \left( \frac{\partial F}{\partial x_1}(t, x), \ldots, \frac{\partial F}{\partial x_N}(t, x) \right);
  \]

- There is \( a \in C([0, \infty[ \times [0, \infty]), b \in C[0, T] \) such that
  \[
  |F(t, x)| \leq a(|x|)b(t), \quad |\nabla F| \leq a(|x|)b(t), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^N.
  \]

We will show that the function \( \phi(x) \) is \( w.s.l.s.c \). To show that \( \phi \) is \( w.s.l.s.c \), we must prove that if \( x_n \to x \) then \( \liminf_{n \to \infty} \phi(x_n) \geq \phi(x) \). Let’s first write \( \phi(x_n) \) in the form

\[
\phi(x_n) = L(x_n) + T(x_n),
\]

then, we have that

\[
\liminf_{n \to \infty} \phi(x_n) \geq \liminf_{n \to \infty} L(x_n) + \liminf_{n \to \infty} T(x_n).
\]

Note also that \( \phi(x) \) we can write it as

\[
\phi(x) = \frac{1}{2} \int_0^T |x'(t)|^2 \, dt + \int_0^T F(t, x(t)) \, dt
\]

\[
= \frac{1}{2} \int_0^T |x'(t)|^2 + |x(t)|^2 \, dt + \int_0^T -|x(t)|^2 + F(t, x(t)) \, dt
\]

\[
= \frac{1}{2} \|x\|^2 + \int_0^T -|x(t)|^2 + F(t, x(t)) \, dt.
\]

Now, we know that if \( E \) is a Banach space, then if \( x_n \to x \iff \liminf_{n \to \infty} \| (x_n) \| \geq \| x \| \). Let \( \liminf_{n \to \infty} \phi(x_n) \) with \( x_n \rightharpoonup x \) in \( H^1_T \). We will proceed in stages.
1. There is a subsequence \( \{x_{n_k}\} \) such that
\[
\liminf_{n \to \infty} \phi(x_{n_k}) = \liminf_{n \to \infty} \phi(x_n),
\]
for all subsequence \( \{x_{n_k}\} \) in \( x_n \).

2. By compactness of the injection \( i : H^1([0, T], \mathbb{R}^N) \to C([0, T], \mathbb{R}^N) \) we know that there is a subsequence \( \{x_{n_{k_l}}\} \) which converges strongly in \( C([0, T]) \) since strong convergence implies weak convergence.

3. Let \( \{x_{n_{k_l}}\} \to x \) in \( H^1_0 \). As the injection \( i \) is compact, i.e., \( i \) of \( H^1_0 \) in \( C([0, T]) \) is compact, so a sequence that tends weakly in \( H^1_0 \) converges for the norm of \( C([0, T]) \), therefore \( \{x_{n_{k_l}}\} \to x \) in \( C([0, T]) \).

4. As \( \{x_{n_{k_l}}\} \to x \) in \( C([0, T]) \), then
\[
\int_0^T -|x_{n_{k_l}}(t)|^2 + F(t, x_{n_{k_l}}(t)) \, dt \to \int_0^T -|x'(t)|^2 \, dt + F(t, x(t)) \, dt.
\]

5. Now, as \( x_n \to x \implies \|x\| \leq \liminf_{n \to \infty} \|x_n\| \) then
\[
\liminf_{n \to \infty} \|x_{n_{k_l}}\|^2 \geq \|x\|.
\]

6. Finally, steps \( i, iv \) and \( v \) can be concluded that
\[
\liminf_{n \to \infty} \phi(x_n) = \liminf_{n \to \infty} \phi(x_{n_{k_l}}) \geq \phi(x),
\]
and therefore \( \phi(x) \leq \liminf_{n \to \infty} \phi(x_n) \) for all sequence \( \{x_n\} \) of \( E \) such that \( x_n \rightharpoonup x \).

Now, if we want to prove that \( \phi \) is differentiable in the sense of Fréchet, we can see that \( \phi \) is well defined for all \( x \in H^1_0 \). Then, we know that \( \phi(x) \) is given by
\[
\phi(x) = \frac{1}{2} \int_0^T |x'(t)|^2 \, dt + \int_0^T F(t, x(t)) \, dt.
\]
Therefore, for the second integral, just use the maximum of \( F \) and the fact that \( H^1_0 \subseteq C([0, T]) \), since \([0, T]\) is bounded. Let us show that \( \phi(x) \) is differentiable in the sense of Fréchet and that its differential is given by
\[
\phi'(\bar{x}) = \int_0^T \nabla x \cdot \nabla \bar{x} \, dx + \int_0^T f(x)\bar{x} \, dx,
\]
for all $\bar{x} \in H^1_0$ where $F$ is the primitive of $f$ which vanishes 0. Let’s show the differentiability of the application $x \mapsto T(x) = \int_0^T F(t, x(t)) \, dt$, then for all $x, \bar{x} \in H^1_0([0, T])$

$$T(x + \bar{x}) - T(x) - \int_0^T f(t, x(t)) \bar{x} \, dt = \int_0^T \left( \int_0^1 f(x + s\bar{x}) \, ds - f(t, x) \bar{x} \right) \, dt$$

$$= \int_0^T \left( \int_0^1 (f(x + s\bar{x}) - f(t, x)) \, ds \right) \bar{x} \, dt.$$

In fact, $\frac{d}{ds}(F(x+s\bar{x})) = f(x+s\bar{x})\bar{x}$, so by the inequality of Cauchy-Schwarz we get that

$$\left| T(x + \bar{x}) - T(x) - \int_0^T f(t, x) \bar{x} \, dt \right| \leq \left( \int_0^T (f(x + s\bar{x}) - f(t, x)) \, ds \right)^2 \leq \left( \int_0^1 (f(x + s\bar{x}) - f(t, x))^2 \, ds \right)^{1/2} \|\bar{x}\|_{L^2([0, T])}.$$

By Cauchy-Schwarz we have

$$\left( \int_0^T (f(x + s\bar{x}) - f(t, x)) \, ds \right)^2 \leq \int_0^1 (f(x + s\bar{x}) - f(t, x))^2 \, ds,$$

so we can get

$$\left| T(x + \bar{x}) - T(x) - \int_0^T f(t, x) \bar{x} \, dt \right| \leq \|f(x+s\bar{x})-f(t,x)\|_{L^2([0,T]\times[0,1])} \|\bar{x}\|_{L^2([0,T])}.$$

Now, to conclude just show that for any sequence $\bar{x}_n$ which tends to 0 in $H^1_0([0, T])$, we have that

$$\|f(x + s\bar{x}_n) - f(x)\|_{L^2([0,T]\times[0,1])} \|\bar{x}\|_{H^1([0,T])} \to 0.$$

Here we start by extracting a subsequence which is the upper limit of this sequence and which converges almost everywhere. So for this sequence we have

$$|f(x + s\bar{x}_n) - f(t, x)|^2 \leq \|f\|_{L^2}^2 \in H^1([0, T] \times [0, 1]).$$

It can be shown using the same type of arguments, that the application $x \mapsto \phi'(x)$ is continuous $H^1_0([0, T])$ in its dual $(H^1_0([0, T]))'$. 

On the coercive functions and minimizers 713
Now, we show that $f$ is weakly coercive. For that, we must show that $\phi \to \infty$ when $\|x\| \to \infty$. Here, we can write $F(t, x(t))$ in the form

$$F(t, x(t)) = F(t, \bar{x}) + F(t, x(t)) - F(t, \bar{x}).$$

Now, for $x \in H^1_T$ we have by (a) that $x = \bar{x} + \tilde{x}$ where $\bar{x} = (1/T) \int_0^T x(t) \, dt$, then

$$\phi(x) = \int_0^T \frac{|x'(t)|^2}{2} \, dt + \int_0^T F(t, \bar{x}) \, dt + \int_0^T F(t, x(t)) - F(t, \bar{x}) \, dt$$

$$= \int_0^T \frac{|x'(t)|^2}{2} \, dt + \int_0^T F(t, \bar{x}) \, dt + \int_0^T \int_0^1 (\nabla F(t, \bar{x} + s\tilde{x}(t)), \tilde{x}(t)) \, ds \, dt$$

$$\geq \int_0^T \frac{|x'(t)|^2}{2} \, dt + \int_0^T F(t, \bar{x}) \, dt - \left( \int_0^T c(t) \, dt \right) \|\tilde{x}\|_{\infty},$$

and by the Sobolev inequality we get

$$\geq \int_0^T \frac{|x'(t)|^2}{2} \, dt + \int_0^T F(t, \bar{x}) \, dt - k \left( \int_0^T |x'(t)|^2 \, dt \right)^{1/2},$$

therefore, we know that $\|x\| \to \infty \iff \left( |\bar{x}|^2 + \int_0^T |x'(t)|^2 \, dt \right)^{1/2} \to \infty$ and knowing that $\int_0^T F(t, x) \, dt \to \infty$ when $|x| \to \infty$ we can conclude that

$$\phi(x) \to \infty \quad \text{when} \quad \|x\| \to \infty.$$

3 Conclusion

In this paper, we prove that $\phi$ is coercive using the fact that the functional $\frac{1}{2} \int_0^T |x(t)|^2 \, dt$ is convex and more is continuous, so knowing that $\phi$ is s.l.s.c and convex, then $\phi$ is w.s.l.s.c. On the other hand, we prove that the functional $\int_0^T F(t, x(t)) \, dt$, so we know that if a sequence $x_n$ converges weakly towards $x$ in $H^1_T$, then $x_n$ converges uniformly to $x$ in $[0, T]$, i.e., $x_n \rightharpoonup x$, then tends uniformly. This shows us that $\int_0^T F(t, x_n(t)) \, dt$ converges to $\int_0^T F(t, x(t)) \, dt$, so this functional is weakly continuous and therefore w.s.l.s.c.

Acknowledgements. We would like to thank the referee for his valuable suggestions that improved the presentation of this paper and our gratitude to Grupo de Modelación Matemática en Epidemiología of the Universidad del Quindío and Department of Mathematics of the Universidad Tecnológica de Pereira (Colombia) and the group GEDNOL.
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Received: November 30, 2017; Published: December 17, 2017